Regular Representations of Time-Frequency Groups

Azita Mayeli* and Vignon Oussa**
Mathematics Department, Queensborough C. College, City University of New York, Bayside, NY 11362 U.S.A
Department of Mathematics, Bridgewater State University, Bridgewater, MA 02325 U.S.A.

** Key words ** Plancherel measure, time, frequency, Gabor, Heisenberg group, admissibility, nilpotent

2000 Mathematics Subject Classification 22E40

In this paper, we study the Plancherel measure of a class of non-connected nilpotent groups which is of special interest in Gabor theory. Let $G$ be a time-frequency group. That is $G = \langle T_k, M_l : k \in \mathbb{Z}^d, l \in B\mathbb{Z}^d \rangle$, where $T_k$, $M_l$ are translations and modulations operators acting in $L^2(\mathbb{R}^d)$, and $B$ is a non-singular matrix. We compute the Plancherel measure of the left regular representation of $G$ which is denoted by $L$. The action of $G$ on $L^2(\mathbb{R}^d)$ induces a representation which we call a Gabor representation. Motivated by the admissibility of this representation, we compute the decomposition of $L$ into direct integral of irreducible representations by providing a precise description of the unitary dual and its Plancherel measure. As a result, we generalize Hartmut Führ’s results which are only obtained for the restricted case where $d = 1$, $B = 1/L$, $L \in \mathbb{Z}$ and $L > 1$. Even in the case where $G$ is not type I, we are able to obtain a decomposition of the left regular representation of $G$ into a direct integral decomposition of irreducible representations when $d = 1$. Some interesting applications to Gabor theory are given as well. For example, when $B$ is an integral matrix, we are able to obtain a direct integral decomposition of the Gabor representation of $G$.

1 Introduction

Let $G$ be a locally compact group with type I left regular representation. The Plancherel theorem guarantees the existence of a measure $\mu$ on the unitary dual of $G$ such that once a Haar measure is fixed on the group $G$, $\mu$ is uniquely determined. Although the existence of the Plancherel measure is given; it is generally a hard problem to compute it. Let $G$ be a time-frequency group. That is $G = \langle T_k, M_l : k \in \mathbb{Z}^d, l \in B\mathbb{Z}^d \rangle$, where $T_k$, $M_l$ are translations and modulations operators acting in $L^2(\mathbb{R}^d)$, and $B$ is a non-singular matrix. We compute the Plancherel measure of the left regular representation of $G$ which is denoted by $L$. The action of $G$ on $L^2(\mathbb{R}^d)$ induces a representation which we call a Gabor representation. Motivated by the admissibility of this representation, we compute the decomposition of $L$ into direct integral of irreducible representations by providing a precise description of the unitary dual and its Plancherel measure. As a result, we generalize Hartmut Führ’s results which are only obtained for the restricted case where $d = 1$, $B = 1/L$, $L \in \mathbb{Z}$ and $L > 1$. Even in the case where $G$ is not type I, we are able to obtain a decomposition of the left regular representation of $G$ into a direct integral decomposition of irreducible representations when $d = 1$. Some interesting applications to Gabor theory are given as well. For example, when $B$ is an integral matrix, we are able to obtain a direct integral decomposition of the Gabor representation of $G$.

Copyright line will be provided by the publisher

* e-mail: amayeli@qcc.cuny.edu

** e-mail: vignon.oussa@bridgew.edu

Copyright line will be provided by the publisher
Characteristics of the closure of the linear span of orbits of the type \( G(S) \) where \( S \subset L^2(\mathbb{R}^d) \) have been studied in [2] when \( B \) only has rational entries. Also a thorough presentation of the theory of time-frequency analysis is given in [10]. In this paper, we are mainly interested in the following questions.

**Question 1.1** If \( G \) is type I, can we provide a description of the unitary dual of \( G \), and a precise formula for the Plancherel measure of \( G \)?

**Question 1.2** If \( G \) is not type I, can we obtain a decomposition of the left regular representation into unitary irreducible representations of \( G \)? Is it possible to provide a central decomposition of its left regular representation?

For obvious reasons, we term the group \( G \) a time-frequency group. There are three cases to consider. First, it is easy to see that \( G \) is a commutative discrete group if and only if \( B \) is an integral matrix. In that case, all irreducible representations of \( G \) are characters, and thanks to the Pontrjagin duality, the Plancherel measure is well-understood. In fact if \( B \) is a matrix with integral entries, the Plancherel measure of \( G \) is supported on a measurable fundamental domain of the lattice \( \mathbb{Z}^d \times (B^{-1})^\tau \mathbb{Z}^d \); namely the set \([0,1)^d \times (B^{-1})^\tau [0,1)^d \). Interestingly, it can be shown that the Gabor representation \( \mathbb{Z}^d \times B\mathbb{Z}^d \ni (k,l) \mapsto T_k \circ M_l \) is unitarily equivalent with a subrepresentation of the left regular representation if and only if \(|\det B| \leq 1 \). Otherwise, the Gabor representation is equivalent to a direct sum of regular representations. Although the previous statement is not technically new, the proof given here is based on the representation theory of the time-frequency group. Secondly, if \( B \) has some rational entries, some of them non-integer, then \( G \) is a non-commutative discrete type I group. Using well-known techniques developed by Mackey, and later on by Kleppner and Lipsman in [13], precise descriptions of the unitary dual of \( G \) and its Plancherel measure are obtained. Two main ingredients are required to compute the Plancherel measure of \( G \). Namely, a closed normal subgroup of \( N \) whose left regular representation is type I, and a family of subgroups of \( G/N \) known as the ‘little groups’. We will show that the Plancherel measure in the case where \( B \) has some non integral rational entries but no irrational entry, is a fiber measure supported on a fiber space with base space: the unitary dual of the commutator subgroup of \( G \). That is, the base space is \( \hat{G} \). Using some procedure given in [9], we can construct a non-singular matrix \( A \), such that \( AZ^d = \mathbb{Z}^d \cap (B^{-1})^\tau \mathbb{Z}^d \) and we show that each fiber can be identified with some compact set

\[
\Lambda_1 \times E_\sigma \times \{ \chi_\sigma \} \subset \mathbb{R}^d \times \mathbb{R}^d \times \{ \chi_\sigma \}
\]

where \( \chi_\sigma \in \hat{G} \), \( \Lambda_1 \) is a measurable fundamental domain of \( (A^{-1})^\tau \mathbb{Z}^d \) and \( E_\sigma \) is the cross-section of the action of a little group (which is a finite group here) in \( \mathbb{R}^d / (B^{-1})^\tau \mathbb{Z}^d \). We also show that all irreducible representations of \( G \) are monomial representations modelled as acting in some finite-dimensional Hilbert spaces with dimensions bounded above by \(|\det A|\). It is worth noticing that Hartmut Führ has already computed the Plancherel measure of the simplest example (Section 5.5 of [7]) of the class of groups considered in our paper.

In his example, \( d = 1 \) and \( B\mathbb{Z}^d = 1/L\mathbb{Z} \), where \( L \) is some positive integer greater than one. For the more general case in which we are interested, we obtain a parametrization of the unitary dual of \( G \), and we derive a precise formula for the Plancherel measure. In the case where \( G \) is not type I (\( B \) has some irrational entries); unfortunately the ‘Mackey machine’ fails. In the particular case where \( d = 1 \),

\[
G = \langle T_k, M_l \mid k, l \in \mathbb{Z}, l \in \alpha \mathbb{Z} \rangle, \alpha \in \mathbb{R} - \mathbb{Q},
\]

we are able to obtain a central decomposition of the left regular representation of \( L \) as well as a direct integral decomposition of the left regular representation of

\[
G \cong \Gamma = \left\{ \begin{bmatrix} 1 & m & l \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} : (m,k,l) \in \mathbb{Z}^3 \right\}
\]

into its irreducible components. In fact, we derive that the left regular representation of \( G \) is unitarily equivalent to

\[
\int_{[-1,1]} \int_{[0,|\lambda|]} \text{Ind}_K^\Gamma (\chi_\lambda) |\lambda| \, dl \, d\lambda
\]

(1.2)
Given a countable sequence \( \Lambda \) of distinct \( l, l' \in \Lambda \), we say \( \Lambda \) is a lattice if the following hold: 

- \( \Lambda \) is isolated and \( \Lambda = A\mathbb{Z}^d \) for some matrix \( A \).
- \( \Lambda \) is a full rank lattice if \( A \) is nonsingular, and we denote the dual of \( \Lambda \) by \( (A^{-1})^\text{tr} \Lambda \).

A fundamental domain (or transversal) \( D \) for a lattice in \( \mathbb{R}^d \) is a measurable set such that the following hold:

\[
(D + l) \cap (D + l') \neq \emptyset
\]

for distinct \( l, l' \in \Lambda \), and \( \mathbb{R}^d = \bigcup_{l \in \Lambda} (D + l) \).

Acknowledgements

Sincere thanks go to B. Currey, for providing support and suggestions during the preparation of this work. We also thank the anonymous referees for a very careful review of the paper. Their suggestions and corrections were crucial to the improvement of our work.

2 Preliminaries

Let us start by setting up some notation. Given a matrix \( A \) of order \( d \), \( A^\text{tr} \) stands for the transpose of \( A \), and \( A^{-\text{tr}} = (A^{-1})^\text{tr} \) stands for the inverse transpose of \( A \). Let \( F \) be a field or a ring. It is standard to use \( GL(d, F) \) to denote the set of invertible matrices of order \( d \) with entries in \( F \). Let \( G \) be a locally compact group. The unitary dual, which is the set of all irreducible unitary representations of \( G \) is denoted by \( \hat{G} \). Given \( x \in \mathbb{R}^d \), we define a character which is a one-dimensional unitary representation of \( \mathbb{R}^d \) into the one-dimensional torus as \( \chi_x : \mathbb{R}^d \to T \) where \( \chi_x(y) = e^{2\pi i \langle x, y \rangle} \). Let \( H \) be a subgroup of \( G \). The index of \( H \) in \( G \) is denoted by \( [G : H] = |G/H| \). We will use \( 1 \) to denote the identity operator acting in some Hilbert space. Given two isomorphic groups \( G, H \) we write \( G \cong H \).

The reader who is not familiar with the theory of direct integrals is invited to refer to [5].

Definition 2.1 Given a countable sequence \( \{f_i\}_{i \in I} \) of vectors in a Hilbert space \( \mathcal{H} \), we say \( \{f_i\}_{i \in I} \) forms a frame if and only if there exist strictly positive real numbers \( A, B \) such that for any vector \( f \in \mathcal{H} \)

\[
A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B \|f\|^2.\]

In the case where \( A = B \), the sequence of vectors \( \{f_i\}_{i \in I} \) is called a tight frame, and if \( A = B = 1 \), \( \{f_i\}_{i \in I} \) is called a Parseval frame.

Remark 2.2 If \( \{f_i\}_{i \in I} \) is a Parseval frame such that for all \( i \in I \), \( \|f_i\| = 1 \), then \( \{f_i\}_{i \in I} \) is an orthonormal basis for \( \mathcal{H} \).

Definition 2.3 A lattice \( \Lambda \) in \( \mathbb{R}^d \) is a discrete additive subgroup of \( \mathbb{R}^d \). In other words, every point in \( \Lambda \) is isolated and \( \Lambda = A\mathbb{Z}^d \) for some matrix \( A \). We say \( \Lambda \) is a full rank lattice if \( A \) is nonsingular, and we denote the dual of \( \Lambda \) by \( (A^{-1})^\text{tr} \Lambda \).
Definition 2.4 Let $\Lambda = AZ^d \times BZ^d$ be a full rank lattice in $\mathbb{R}^d$ and $g \in L^2(\mathbb{R}^d)$. The family of functions in $L^2(\mathbb{R}^d)$,
\[ \mathcal{G}(g, AZ^d \times BZ^d) = \left\{ e^{2\pi i(k \cdot x)}g(x-n) : k \in BZ^d, n \in AZ^d \right\} \]
(2.1)
is called a Gabor system.

Definition 2.5 Let $m$ be the Lebesgue measure on $\mathbb{R}^d$, and consider a full rank lattice $\Lambda = AZ^d$ inside $\mathbb{R}^d$.
1. The volume of $\Lambda$ is defined as $vol(\Lambda) = m(\mathbb{R}^d/\Lambda) = |\det A|$.

2. The density of $\Lambda$ is defined as $d(\Lambda) = \frac{1}{|\det A|}$.

Lemma 2.6 Given a separable full rank lattice $\Lambda = AZ^d \times BZ^d$ in $\mathbb{R}^d$. The following statements are equivalent
1. There exists $f \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(f, AZ^d \times BZ^d)$ is a Parseval frame in $L^2(\mathbb{R}^d)$.
2. $vol(\Lambda) = |\det A| |\det B| \leq 1$.
3. There exists $f \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(f, AZ^d \times BZ^d)$ is complete in $L^2(\mathbb{R}^d)$.

For a proof of the above lemma, we refer the reader to Theorem 3.3 in [11].

Lemma 2.7 Let $\Lambda$ be a full-rank lattice in $\mathbb{R}^d$. There exists a function $f \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(f, \Lambda)$ is an orthonormal basis if and only if $vol(\Lambda) = 1$. Also, if $\mathcal{G}(f, \Lambda)$ is a Parseval frame for $L^2(\mathbb{R}^d)$ then $\|f\|^2 = vol(\Lambda)$.

A proof of the Lemma 2.7 is given in [11]. Now, let $F = \{T_x, M_y | x \in \mathbb{R}^d, y \in \mathbb{R}^d\}$ and let $f$ be a square-integrable function over $\mathbb{R}^d$. It is easy to see that
\[ T_x M_y T_x^{-1} M_y^{-1} f = e^{-2\pi i(y \cdot x)} f, \]
(2.2)
and $e^{-2\pi i(y \cdot x)}$ is a central element of the group $F$. Thus $F$ is a non-commutative, connected, non-simply connected two-step nilpotent Lie group. In fact $F$ is isomorphic to the reduced $2d+1$ dimensional Heisenberg group. The $2d+1$-dimensional Heisenberg group has Lie algebra
\[ h = \mathbb{R}\text{-span} \{X_1, \cdots, X_d, Y_1, \cdots, Y_d, Z\} \]
with non-trivial Lie brackets $[X_i, Y_j] = Z$. Let
\[ \mathbb{H} = \exp(h) \]
(2.3)
Clearly, $\mathbb{H}$ is a simply connected, connected $2n+1$-dimensional Heisenberg group, and $\exp(ZZ)$ is a discrete central subgroup of $\mathbb{H}$. Moreover, $\mathbb{H}$ is the universal covering group of $F$. That is, $F$ is isomorphic to the group $\mathbb{H}/\exp(ZZ)$.

We will now provide a light introduction to the notion of admissibility of unitary representations. A more thorough exposure to the theory is given in [7]. However, we will discuss part of the material which is necessary to fully understand the results obtained in our work. Let $\pi$ be a unitary representation of a locally compact group $X$, acting in some Hilbert space $H$. We say that $\pi$ is admissible, if and only if there exists some vector $\phi \in H$ such that the operator $W_\phi$
\[ W_\phi : H \to L^2(X) \text{ where } W_\phi \psi(x) = \langle \psi, \pi(x) \phi \rangle \]
defines an isometry from $H$ to $L^2(X)$. Let $(\rho,K)$ be an arbitrary type I representation of the group $G = \langle T_k, M_l : k \in \mathbb{Z}^d, l \in BZ^d \rangle$. Moreover, let us suppose that $X$ is a type I unimodular group. Let us also suppose that we are able to obtain a direct integral decomposition of $\rho$ as follows
\[ \rho \cong \int_X^{\oplus} \pi \mu(\pi) \quad \text{d}\mu(\pi) \]
where $d\mu$ is a measure defined on the unitary dual of $X$. According to well-known theorems developed in [7]; $(\rho, K)$ is admissible if and only if it is unitarily equivalent with a subrepresentation of the left regular representation, and the multiplicity function is integrable over the spectrum $\rho$. That is: $\mu$ is absolutely continuous with respect to the Plancherel measure $\mu$ supported on $X$ and $\int_X n_\pi d\mu(\pi) < \infty$. If a representation $\rho$ is admissible, in theory it is known (see [7]) how to construct all admissible vectors. Let us describe such process in general terms. First, we must construct a unitary operator

$$U : K \to \int_X (\oplus_{k=1}^n H_k) \ d\mu(\pi)$$

intertwining the representation $\rho$ with $\int_X n_\pi d\mu(\pi)$. Next, we define a measurable field $(F_\lambda)_{\lambda \in X}$ of operators in $\int_X (\oplus_{k=1}^n H_k) \ d\mu(\pi)$ such that each operator $F_\lambda$ is an isometry. All admissible vectors are of the type $U^{-1}(F_\lambda)_{\lambda \in X}$.

In the remainder of the paper, we will focus on time-frequency groups. We will compute the Plancherel measure of the group whenever $G$ is type I, and we will obtain a direct integral decomposition of the left regular representation if $G$ is not type I and $d = 1$. Some application to Gabor theory will be discussed throughout the paper as well.

3 Normal Subgroups of $G$

In this section, we will study the structure of normal subgroups of the time frequency group. The reason why this section is important, is because part of what the Mackey machine [14] needs is an explicit description of the unitary dual of type I normal subgroups in order to compute the unitary dual of the whole group. In this paper, unless we state otherwise, $G$ stands for the following group:

$$\langle T_k, M_l | k \in \mathbb{Z}^d, l \in B\mathbb{Z}^d \rangle.$$

We recall that the subgroup generated by operators of the type $T_k \circ M_l \circ T_k^{-1} \circ M_l^{-1}$ is called the commutator subgroup of $G$ and is denoted $[G, G]$. From now on, to simplify the notation, we will simply omit the symbol $\circ$ whenever we are composing operators.

**Lemma 3.1** $[G, G]$ is isomorphic to a subgroup of the torus.

1. If $B \in GL(d, \mathbb{Z})$ then $G$ is commutative and isomorphic to $\mathbb{Z}^d \times B\mathbb{Z}^d$.

2. If $B$ is in $GL(d, \mathbb{Q}) - GL(d, \mathbb{Z})$ then $[G, G]$ is a central subgroup of $G$, and is a cyclic group. $G$ is not commutative but it is a type I discrete unimodular group.

3. If $B \in GL(d, \mathbb{R}) - GL(d, \mathbb{Q})$ then $G$ is a non-commutative two-step nilpotent group, and its commutator subgroup is an infinite subgroup of the circle group.

**Proof.** To show part (a), let $l \in B\mathbb{Z}^d, k \in \mathbb{Z}^d$ where $B$ is a non-invertible matrix. Let $f \in L^2(\mathbb{R}^d)$. We have

$$T_k M_l T_k^{-1} M_l^{-1} f = e^{-2\pi i \langle l, k \rangle} f = \chi_l(k) f.$$

If $B \in GL(d, \mathbb{Z})$ then the commutator subgroup of $G$ is trivial, and $G$ is an abelian group isomorphic to $\mathbb{Z}^d \times B\mathbb{Z}^d$. For part (b), let us suppose that $B \in GL(d, \mathbb{Q}) - GL(d, \mathbb{Z})$. So, $B = [p_{ij}/q_{ij}]_{1 \leq i, j \leq d}$ where $p_{ij}, q_{ij}$ are integral values, $\gcd(p_{ij}, q_{ij}) = 1$ and $q_{ij} \neq 0$ for $1 \leq i, j \leq d$. Let $m = \text{lcm}(q_{ij})_{1 \leq i, j \leq d}$. Clearly

$$\chi_l^m(k) = 1 \text{ for all } l \in B\mathbb{Z}^d, \text{ and } k \in \mathbb{Z}^d.$$

Thus, $[G, G]$ is a finite abelian proper closed subgroup of the circle group. As a result $[G, G]$ is cyclic. For part (c), if $B \in GL(d, \mathbb{R}) - GL(d, \mathbb{Q})$ then the commutator subgroup of $G$ is not isomorphic to a finite subgroup of the torus. That is, there exist $l \in B\mathbb{Z}^d, k \in \mathbb{Z}^d$ such that the set $\{\chi_l^m(k) : m \in \mathbb{Z}\}$ is not closed and is dense in the torus $T$. $\square$
Example 3.2 Let $G$ be group generated $T_k, M_l$ such that $k \in \mathbb{Z}^2, l \in B\mathbb{Z}^2$

$$B = \begin{bmatrix} 1/2 & 1/5 \\ 2/3 & -3/4 \end{bmatrix}$$

then, $[G, G]$ is isomorphic to $\mathbb{Z}_{60}$.

Example 3.3 Let $G$ be group generated $T_k, M_l$ such that $k \in \mathbb{Z}^2, l \in B\mathbb{Z}^2$ such that

$$B = \begin{bmatrix} \sqrt{2} & 1 \\ -1 & 2 \end{bmatrix}$$

then $[G, G]$ is isomorphic to an infinite subgroup of the circle.

Assuming that $B$ is in $GL(d, \mathbb{Q})$, we will construct an abelian normal subgroup of $G$. For that purpose, we will need to define the groups

$$N_1 = \langle T_k, M_l | k \in B^{-tr} \mathbb{Z}^d, l \in B\mathbb{Z}^d \rangle,$$

and

$$N_2 = \langle T_k, M_l | k \in B^{-tr} \mathbb{Z}^d \cap \mathbb{Z}^d, l \in B\mathbb{Z}^d \rangle.$$  

Notice that in general $N_1$ is not a subgroup of $G$ because the lattice $\mathbb{Z}^d$ is not invariant under the action of $B^{-tr}$ if $B^{-tr}$ has non integral rational entries. However, the group $N_1$ will be important in constructing the unitary dual of $G$, and we will need to study some of its characteristics.

Lemma 3.4 If $B$ is an element of $GL(d, \mathbb{Q}) - GL(d, \mathbb{Z})$ then

$$N_1 = \langle T_k, M_l | k \in B^{-tr} \mathbb{Z}^d, l \in B\mathbb{Z}^d \rangle$$

is an abelian group.

Proof. Given $k \in B^{-tr} \mathbb{Z}^d$ and $l \in B\mathbb{Z}^d$, we recall that $T_k M_l T_k^{-1} M_l^{-1} f(x) = e^{-2\pi i (l,k)} f$. Since $k \in B^{-tr} \mathbb{Z}^d$, and $l \in B\mathbb{Z}^d$ then there exist $k', l' \in \mathbb{Z}^d$ such that

$$T_k M_l T_k^{-1} M_l^{-1} f = e^{-2\pi i (B'k', k')} f = e^{-2\pi i (l', k')} f(x) = f.$$  

Thus, for any given $k \in B^{-tr} \mathbb{Z}^d$, and $l \in B\mathbb{Z}^d$, $T_k M_l T_k^{-1} M_l^{-1}$ is equal to the identity operator. It follows that the commutator subgroup of $N_1$ is trivial.

We recall the following lemmas from [9].

Lemma 3.5 Given two lattices $\Gamma_1, \Gamma_2, \Gamma_1 \cap \Gamma_2$ is a lattice in $\mathbb{R}^d$ if and only if there exists a lattice $\Gamma$ that contains both $\Gamma_1$ and $\Gamma_2$.

Definition 3.6 Let $\Gamma$ be a full-rank lattice in $\mathbb{R}^d$ with generators

$$\begin{bmatrix} v_1^1 \\ \vdots \\ v_d^1 \end{bmatrix}, \begin{bmatrix} v_1^2 \\ \vdots \\ v_d^2 \end{bmatrix}, \ldots, \begin{bmatrix} v_1^d \\ \vdots \\ v_d^d \end{bmatrix} \in \mathbb{R}^d.$$  

The matrix of order $d$ below

$$\begin{bmatrix} v_1^1 & \cdots & v_1^d \\ \vdots & \cdots & \vdots \\ v_d^1 & \cdots & v_d^d \end{bmatrix}$$

is called a basis for $\Gamma$.  

Copyright line will be provided by the publisher
Lemma 3.7 Let $\Gamma_1, \Gamma_2$ be two distinct lattices with bases $J, K$ respectively. $\Gamma_1 + \Gamma_2, \Gamma_1 \cap \Gamma_2$ are two lattices in $\mathbb{R}^d$ if and only if $JK^{-1}$ is a rational matrix. Moreover

$$\dim (\Gamma_1 + \Gamma_2) + \dim (\Gamma_1 \cap \Gamma_2) = \dim \Gamma_1 + \dim \Gamma_1$$

Remark 3.8 If $\Gamma_1 \cap \Gamma_2$ is a lattice, there is a well-known technique given in [9] (see page 809) used to compute the basis of the lattice $\Gamma_1 \cap \Gamma_2$. We describe the procedure here. Let $J, K$ be bases for lattices $\Gamma_1$ and $\Gamma_2$ respectively. First we compute the $d \times 2d$ matrix $[J \mid K]$. Secondly, we evaluate the Hermite lower triangular form of $[J \mid K]$. This matrix has the structure $[L|0]$, and is obtained as $[J \mid K]E = [L|0]$, where $E$ is matrix of order $2d$ obtained by applying elementary row operations to $[J \mid K]$. In fact, $E$ is a block matrix of the type

$$E = \begin{bmatrix} R & S \\ C & D \end{bmatrix},$$

and $R, S, C, D$ are matrices of order $d$. Finally, a basis for the lattice $\Gamma_1 \cap \Gamma_2$ is then given by $KDE$. That is $\Gamma_1 \cap \Gamma_2 = (KDE)\mathbb{Z}^d$.

Corollary 3.9 If $B$ is in $GL(d, \mathbb{Q}) - GL(d, \mathbb{Z})$ then $B^{-\operatorname{tr}}\mathbb{Z}^d \cap \mathbb{Z}^d$ is a full-rank lattice subgroup of $\mathbb{R}^d$.

Proof. The fact that $B^{-\operatorname{tr}}\mathbb{Z}^d \cap \mathbb{Z}^d$ is a full-rank lattice follows directly from Lemma 3.7.

We assume that $B$ is an element of $GL(d, \mathbb{Q}) - GL(d, \mathbb{Z})$. We will prove that $N_2$ is a normal subgroup of $G$. However it is not a maximal normal subgroup of $G$ since it does not contain the center of the group. Thus, $N_2$ needs to be extended. For that purpose, we define the group

$N = \langle T_k, M_1, \tau \mid k \in B^{-\operatorname{tr}}\mathbb{Z}^d \cap \mathbb{Z}^d, l \in B\mathbb{Z}^d, \tau \in [G, G] \rangle \subset G$.

Proposition 3.10 If $B$ is in $GL(d, \mathbb{Q})$ then $N$ is an abelian normal subgroup of $G$.

Proof. From Lemma 3.4, we have already seen that $T_k$ commutes with $M_l$ for arbitrary $k \in B^{-\operatorname{tr}}\mathbb{Z}^d \cap \mathbb{Z}^d, l \in B\mathbb{Z}^d$. Since $[G, G]$ commutes with $T_k$ and $M_l$ for $k \in B^{-\operatorname{tr}}\mathbb{Z}^d \cap \mathbb{Z}^d, l \in B\mathbb{Z}^d$, then $N$ is abelian. For the second part of the proof, let $k \in B^{-\operatorname{tr}}\mathbb{Z}^d \cap \mathbb{Z}^d, l \in B\mathbb{Z}^d$ and $s \in \mathbb{Z}^d$. First, we compute the conjugation action of the translation operator on an arbitrary element of $N$. Let $s \in \mathbb{Z}^d$. Then $T_k (T_k M_1) T_k^{-1} = e^{-2\pi i (l,s)} T_k M_1$. Next, we compute the conjugation action of the modulation operator on an arbitrary element of $N$ as follows:

$M_s (T_k M_1) M_s^{-1} = \tau T_k M_1$. Clearly, $GNG^{-1} \subset N$. Thus, $N$ is a normal subgroup of $G$.

Lemma 3.11 Assuming that $B$ is in $GL(d, \mathbb{Q}) - GL(d, \mathbb{Z})$, then the following holds.

1. The quotient group $\frac{\mathbb{Z}^d}{B^{-\operatorname{tr}}\mathbb{Z}^d \cap \mathbb{Z}^d}$ is isomorphic to a finite abelian group.

2. The group $G/N$ is a finite group isomorphic to $\frac{\mathbb{Z}^d}{B^{-\operatorname{tr}}\mathbb{Z}^d \cap \mathbb{Z}^d}$.

Proof. For the first part of the proof, since $B^{-\operatorname{tr}}\mathbb{Z}^d \cap \mathbb{Z}^d$ is a full-rank lattice, there exists a non-singular matrix $A$ such that $B^{-\operatorname{tr}}\mathbb{Z}^d \cap \mathbb{Z}^d = A\mathbb{Z}^d$. Thus, referring to the discussion [9] (page 95), the index of $B^{-\operatorname{tr}}\mathbb{Z}^d \cap \mathbb{Z}^d$ in $\mathbb{Z}^d$ is $\langle \mathbb{Z}^d : B^{-\operatorname{tr}}\mathbb{Z}^d \cap \mathbb{Z}^d \rangle = |\det A|$. As a result,

$$\frac{\mathbb{Z}^d}{B^{-\operatorname{tr}}\mathbb{Z}^d \cap \mathbb{Z}^d}$$

is a finite abelian group. For the second part of the lemma, there exist $k_1, k_2, \ldots, k_m \in \mathbb{Z}^d$ such that

$$\frac{\mathbb{Z}^d}{B^{-\operatorname{tr}}\mathbb{Z}^d \cap \mathbb{Z}^d} \cong \{ 1, T_{k_1}, \ldots, T_{k_m} \}.$$
Let a choice for a transversal of $G$ representation of $Z$ where $h$ representations appearing in the decomposition of $\pi$. acting in of the group $e$ and let $e$ acting in $\Lambda$ worth noticing that there is no canonical way to choose forms a measurable partition for $R$ collection of sets $\Lambda$ which is supported on a measurable set $\Lambda$ and the Plancherel measure is up to multiplication by a constant equal to $\text{Plancherel measure of } G$. GL $\Lambda$ then $\text{Plancherel measure of } G$ is commutative and isomorphic to $Z$ where $\chi \otimes 1_{\mathbb{C}^n(\cdot)} d\mu (\cdot)$ acting in $\int_{\Lambda} ^{\oplus} C \otimes \mathbb{C}^n(\cdot) d\mu (\cdot)$. The function $n : \Lambda \to \mathbb{N} \cup \{0\}$ is the multiplicity function of the irreducible representations appearing in the decomposition of $\pi$.

Let us define $W_f : L^2 (\mathbb{R}^d) \to L^2 (\hat{G})$ where

$$W_f h (k,l) = \langle h, T_k M_l f \rangle .$$

We recall that $\pi$ is admissible if and only if $W_f$ defines an isometry on $L^2 (\mathbb{R}^d)$.

**Lemma 4.1** Let $B \in GL (d, \mathbb{Z})$. $\pi$ is admissible if and only if

$$\sum_{T_k M_l \in G} |\langle h, T_k M_l f \rangle|^2 = \|h\|^2$$

for all $h \in L^2 (\mathbb{R}^d)$. That is $f$ is a Gabor Parseval frame.

We recall that the **Zak transform** is a unitary operator

$$Z : L^2 (\mathbb{R}^d) \to L^2 ([0,1)^d \times [0,1)^d) \cong \int_{[0,1)^d \times [0,1)^d} \mathbb{C} dxdy$$

where

$$Z f (x,y) = \sum_{m \in \mathbb{Z}^d} f (x + m) \exp (2\pi i \langle m, y \rangle) .$$
It is easy to see that if $B \in \text{GL}(d, \mathbb{Z})$ then
\[
Z(T_k M_l f)(x, y) = \exp(-2\pi i \langle k, y \rangle) \exp(2\pi i \langle l, x \rangle) Zf(x, y).
\]
Thus, the Zak transform intertwines the Gabor representation with the representation
\[
\int_{[0,1)^d \times [0,1)^d} \chi(x,y)^{\oplus} dxdy.
\]
(4.2)

Now, we would like to compare the representations given in (4.1) and (4.2).

**Proposition 4.2** If $B \in \text{GL}(d, \mathbb{Z})$ and $|\det B| \neq 1$ then the Gabor representation is not admissible. That is, there is no Parseval frame of the type $\pi \left( \hat{G} \right) f$. Moreover, the Gabor representation is unitarily equivalent to the direct integral
\[
\int_{\Lambda} \chi_{\varsigma} \otimes 1_C \mu(\varsigma).
\]

**Proof.** First, let us notice that if $B \in \text{GL}(d, \mathbb{Z})$ then $|\det B - \text{tr} B| \leq 1$. As a result, there is a measurable set $\Lambda \subset \mathbb{R}^{2d}$ tiling $\mathbb{R}^{2d}$ by the lattice $\mathbb{Z}^d \times B^{-\text{tr}} \mathbb{Z}^d$ such that $\Lambda$ is contained in $[0,1)^d \times [0,1)^d$ (see [11]). Thus if $|\det B| \neq 1$, the representation $\int_{[0,1)^d \times [0,1)^d} \chi(x,y)^{\oplus} dxdy$ cannot be contained in $\int_{\Lambda} \chi(x,y)^{\oplus} |\det (B^{\text{tr}})| dxdy$. Picking $\Lambda = [0,1)^d \times E \subset \mathbb{R}^{2d}$ such that $E \subseteq [0,1)^d$, we obtain
\[
\pi \cong \int_{\Lambda} \chi_{\varsigma} \otimes 1_C \mu(\varsigma).
\]
(4.3)
We now claim that the multiplicity function is given by
\[
n(\varsigma) = \# \left( \left\{ j \in B^{-\text{tr}} \mathbb{Z}^d : \{\varsigma + j\} \cap [0,1)^d \neq \emptyset \right\} \right) = |\det B|.
\]
To show that the above holds, we partition $[0,1)^d \times [0,1)^d$ into $|\det B|$ many subsets $\Lambda^k$ such that each set $\Lambda^k$ is a fundamental domain for $\mathbb{Z}^d \times B^{-\text{tr}} \mathbb{Z}^d$. Writing
\[
[0,1)^d \times [0,1)^d = \bigcup_{k=1}^{\frac{|\det B|}{d}} \Lambda^k,
\]
we obtain
\[
\pi \cong \int_{[0,1)^d \times [0,1)^d} \chi(x,y)^{\oplus} dxdy
\cong \int_{[0,1)^d \times [0,1)^d} \chi(x,y)^{\oplus} dxdy
\cong \int_{\Lambda} \chi_{\varsigma} \otimes 1_C \mu(\varsigma).
\]

**Example 4.3** Let $G = \langle T_k, M_l | k \in \mathbb{Z}, l \in 3\mathbb{Z} \rangle$. The spectrum of the left regular representation of $G$ is given by $\Lambda = [0,1] \times [0,1/3]$ and $\pi \cong \int_{\Lambda} \chi_{\varsigma} \otimes 1_C \mu(\varsigma)$. Thus, $\pi$ is not an admissible representation since $\pi \cong L \oplus L \oplus L$.
**Example 4.4** Let $G = \langle T_k, M_l | k \in \mathbb{Z}^2, l \in B \mathbb{Z}^2 \rangle$ where

$$B = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$  

The spectrum of the left regular representation is

$$[0, 1)^2 \times \left[ \begin{array}{cc} \frac{2}{\pi} & -\frac{1}{\pi} \\ \frac{1}{2} & \frac{1}{2} \end{array} \right] [0, 1)^2.$$  

In the graph below we illustrate the idea that there exists a collection of sets $\{\Lambda_k\}_{k=1}^5$ such that each set $\Lambda_k$ is a fundamental domain of $[0, 1)^2 \times B^{-tr} [0, 1)^2$ and the spectrum of $\pi$ is given as follows

$$[0, 1)^2 \times [0, 1)^2 = \bigcup_{k=1}^5 \Lambda_k.$$  

Thus, $\pi \cong L \oplus L \oplus L \oplus L \oplus L$. As a result, the Gabor representation $\pi$ is not admissible.

**Remark 4.5** Let $\mathcal{K}$ be a $\pi$-invariant closed subspace of $L^2 (\mathbb{R}^d)$ ($\mathcal{K}$ is a shift-modulation invariant space [2]). There exists a unitary operator

$$J : \mathcal{K} \rightarrow \int_{(0,1)^d \times B^{-tr} (0,1)^d} \mathbb{C} \otimes \mathbb{C}^{\pi(\varsigma)} d\mu (\varsigma)$$

intertwining the representations $\pi|\mathcal{K}$ with $\int_{\Lambda} \chi_\varsigma \otimes 1_{\mathbb{C}^{\pi(\varsigma)}} d\mu (\varsigma)$ and $\overline{\pi}(\varsigma) \leq |\det B|$ a.e. As a result, we have a general characterization of shift-modulation invariant spaces in the specific case where $B$ has integral entries.

**Proof.** The proof follows from the fact that $(\pi|\mathcal{K}, \mathcal{K})$ is a subrepresentation of the Gabor representation of $\widetilde{G}$. 

**Definition 4.6** Two unitary representations $(\pi_1, H_1), (\pi_2, H_2)$ of a group $X$ are quasi-equivalent if there exist unitarily-equivalent representations $\rho_1, \rho_2$ such that $\rho_k$ is a multiple of $\pi_k$ for $k = 1, 2$. 

Copyright line will be provided by the publisher
Remark 4.7 If $B \in GL(d, \mathbb{Z})$ then the Gabor representation $\pi$ is quasi-equivalent to the left regular representation of $G$.

The proof of Remark 4.7 is a direct application of Proposition 4.2.

Remark 4.8 In the case where $B$ is the identity matrix, $\pi$ is an admissible representation. In fact, a well-known admissible vector is the indicator function of the cube $[0, 1)^d$.

5 The Plancherel Measure and Application: The Rational Case

In this section, we assume that the given matrix $B$ has at least one rational non-integral entry. We recall from Lemma 3.1 that

$$G = \langle T_k, M_l : k \in \mathbb{Z}^d, l \in B\mathbb{Z}^d \rangle$$

is not commutative but is a discrete type I group. Thus, its unitary dual exists, and its Plancherel measure is computable. Using Mackey’s Machine and results developed by Ronald Lipsman and Adam Kleppner in [13], we will describe the unitary dual of the group $G$, and a formula for the Plancherel measure. We recall that if $B$ is in $GL(d, \mathbb{Q}) - GL(d, \mathbb{Z})$ then $G$ contains a normal subgroup

$$N = \langle T_k, M_l, \tau | k \in B^{-tr} \mathbb{Z}^d \cap \mathbb{Z}^d, l \in B\mathbb{Z}^d, \tau \in [G, G] \rangle$$

which is isomorphic with a direct product of abelian groups. Since $N$ is isomorphic to $(B^{-tr} \mathbb{Z}^d \cap \mathbb{Z}^d) \times B\mathbb{Z}^d \times \mathbb{Z}_m$, its unitary dual is a group of characters. The underlying set for the group $G$ is $\mathbb{Z}^d \times B\mathbb{Z}^d \times \mathbb{Z}_m$, and we define the representation $\pi$ of the group $G$ as follows:

$$\pi(k, l, j) = T_k M_l e^{2\pi i j} \quad (5.1)$$

Lemma 5.1 Let $B \in GL(d, \mathbb{Q})$. If $\pi$ is admissible then there exists a function $\phi \in L^2(\mathbb{R}^d)$ such that given $h \in L^2(\mathbb{R}^d)$,

$$\sum_{T_k, M_l \in G} |\langle h, T_k M_l \phi \rangle|^2 = \| h \|^2 .$$

That is $\pi(G) \phi$ is Parseval Gabor frame.

Proof. Let $h \in L^2(\mathbb{R}^d)$. If $\pi$ is admissible, and if $f$ is an admissible vector,

$$\sum_{T_k, M_l, e^{2\pi i \theta_2} \in G} |\langle h, T_k M_l e^{2\pi i \theta_2} f \rangle|^2 = \sum_{T_k, M_l, e^{2\pi i \theta_2} \in [G, G]} |\langle h, T_k M_l f \rangle|^2$$

$$= \sum_{T_k, M_l} \# ([G, G]) \| h \|^2$$

$$= \sum_{T_k, M_l} \left| \langle h, T_k M_l f \rangle \right|^2$$

$$= \| h \|^2 .$$

Thus, the statement of the lemma holds by replacing $\# ([G, G])^{1/2} f$ with $\phi$. 

Using the procedure provided in Remark 3.8, we construct an invertible matrix $A$ with integral entries such that $B^{-tr} \mathbb{Z}^d \cap \mathbb{Z}^d = A\mathbb{Z}^d$ Thus, the unitary dual of $N$ is isomorphic to the commutative group

$$\mathbb{R}^d \mathbb{A}^{-tr} \mathbb{Z}^d \times \mathbb{R}^d B^{-tr} \mathbb{Z}^d \times \mathbb{Z}_m .$$

Copyright line will be provided by the publisher
Let $\Lambda_1 \subset \mathbb{R}^d$ be a measurable fundamental domain for $A^{-tr} \mathbb{Z}^d$ and $\Lambda_2 \subset \mathbb{R}^d$ a measurable fundamental domain for $B^{-tr} \mathbb{Z}^d$, the unitary dual of $N$ is parameterized by the set
$$\{(\gamma_1, \gamma_2, \sigma) : \gamma_1 \in \Lambda_1, \gamma_2 \in \Lambda_2, \sigma \in \mathbb{Z}_m\}.$$ 

Next, the action of a fixed character of $N$ is computed as follows.
$$\chi_{(\gamma_1, \gamma_2, \sigma)}(T_k M_l e^{2\pi i \theta}) = \exp[2\pi i (\gamma_1, k)] \exp[2\pi i (\gamma_2, l)] \exp[2\pi i \sigma \theta].$$ 

We recall one important result due to Mackey which is also presented in [14]. We will need this proposition to compute the unitary dual of the group $G$.

**Proposition 5.2** Let $N$ be a normal subgroup of $G$. Assume that $N$ is type I and is regularly embedded. Let $\pi$ be an arbitrary element of the unitary dual of $N$. $G$ acts on the unitary dual of $N$ as follows:
$$x \cdot \pi(y) = \pi(x^{-1} y x), \quad x, y \in G.$$ 

Let $G_N$ be the stabilizer group of the $G$-action on $\pi$.
$$\widehat{G_N} = \bigcup_{\pi \in \mathcal{N}} \left\{ \text{Ind}_{G_N}^G(v) : v \in \widehat{G_N} \text{ and } v | N \text{ is a multiple of } \pi \right\}.$$ 

Now, we will apply Proposition 5.2 to
$$G = \langle T_k, M_l : k \in \mathbb{Z}^d, l \in B \mathbb{Z}^d \rangle.$$ 

First, we recall from Lemma 3.11, that $G/N$ is a finite group, and thus $G$ is a compact extension of an abelian group. Referring to [13] I Chapter 4, $G$ is type I and $N$ is regularly embedded. Let $P \in G, k \in B^{-tr} \mathbb{Z}^d \cap \mathbb{Z}^d, l \in B \mathbb{Z}^d$ and $\chi_{(\gamma_1, \gamma_2, \sigma)}$ be a character of $N$. We define the action of $G$ on the unitary dual of $N$ multiplicatively such that for $P \in G$,
$$P \cdot \chi_{(\gamma_1, \gamma_2, \sigma)}(T_k M_l e^{2\pi i \theta}) = \chi_{(\gamma_1, \gamma_2, \sigma)}(P^{-1}(T_k M_l e^{2\pi i \theta})P).$$

**Definition 5.3** We define a measurable map $\rho : \mathbb{R}^d \to \Lambda_2$ such that $\rho(x) = y_x$ if and only $y_x$ is the unique element in $\Lambda_2$ such that $x = y_x + l$, where $l \in B^{-tr} \mathbb{Z}^d$.

Since the collection of sets
$$\{\Lambda_2 + j : j \in B^{-tr} \mathbb{Z}^d\}$$
is a measurable partition of $\mathbb{R}^d$ then the map $\rho$ makes sense.

**Lemma 5.4** For $s \in \mathbb{Z}^d, l \in B \mathbb{Z}^d$ and $e^{2\pi i \theta} \in [G, G]$ we have
1. $T_s \cdot \chi_{(\gamma_1, \gamma_2, \sigma)} = \chi_{(\gamma_1, \rho(\gamma_2 + s \sigma), \sigma)}$
2. $M_l \cdot \chi_{(\gamma_1, \gamma_2, \sigma)} = \chi_{(\gamma_1, \gamma_2, \sigma)}$
3. $e^{2\pi i \theta} \cdot \chi_{(\gamma_1, \gamma_2, \sigma)} = \chi_{(\gamma_1, \gamma_2, \sigma)}$

**Proof.** Since $e^{2\pi i \theta}$ is a central element of $G$ then clearly part (c) is correct. Now, for part (a), let $T_k, M_l$, such that $k \in B^{-tr} \mathbb{Z}^d \cap \mathbb{Z}^d, l \in B \mathbb{Z}^d$ and $s \in \mathbb{Z}^d$

$$T_s \cdot \chi_{(\gamma_1, \gamma_2, \sigma)}(T_k M_l e^{2\pi i \theta}) = \chi_{(\gamma_1, \gamma_2, \sigma)}(T_s^{-1}(T_k M_l e^{2\pi i \theta})T_s) = \chi_{(\gamma_1, \gamma_2, \sigma)}(T_k M_l e^{2\pi i \theta + (l, s)}) = \chi_{(\gamma_1, \gamma_2 + s \sigma, \sigma)}(T_k M_l e^{2\pi i \theta}).$$
The stabilizer group of a fixed character

Secondly, using the fact that $\Lambda$ is a fundamental domain for $\Gamma$, it is possible to extend $\chi$ to a character of the stabilizer group $G$. Thus, $\chi(TkM_l^e2\pi i\theta) = \chi(TkM_l^e2\pi i\theta)$. Thanks to (5.3), we may write $\chi(TkM_l^e2\pi i\theta) = \chi(TkM_l^e2\pi i\theta)$. Lemma 5.5 The stabilizer group of a fixed character $\chi(TkM_l^e2\pi i\theta)$ in the unitary dual of $N$ is given by

$$G(TkM_l^e2\pi i\theta) = \begin{cases} TkM_l^e2\pi i\theta & \sigma k = B^{-tr}j \text{ for some } j \in \mathbb{Z} \\ l \in B\mathbb{Z}^d, e^{2\pi i\theta} \in [G, G] \end{cases}$$

(5.3)

Thanks to (5.3), we may write $G(TkM_l^e2\pi i\theta) = A(\sigma) \mathbb{Z}^d \times B\mathbb{Z}^d \times G$, where $A(\sigma)$ is a full-rank matrix of order $d$ and $A(\sigma) \mathbb{Z}^d \geq A\mathbb{Z}^d$. For a fixed element $(\gamma_1, \gamma_2, \sigma)$ in the unitary dual of $N$, we define the set

$$U_\sigma = A(\sigma) \times A_2 \times [G, G]$$

(5.4)

where $A(\sigma)$ is a fundamental domain for $\mathbb{Z}^d$. Since we need Proposition 5.2 to compute the unitary dual of $G$, we would like to be able to assert that if $\chi(TkM_l^e2\pi i\theta)$ is a character of the group $N$, it is possible to extend $\chi(TkM_l^e2\pi i\theta)$ to a character of the stabilizer group $G(TkM_l^e2\pi i\theta)$. However, we will need a few lemmas first.

Lemma 5.6 Let $\lambda = (\eta, \gamma_2, \sigma) \in U_\sigma$ such that $\eta = \gamma_1 + A^{-tr}j$ for some $j \in \mathbb{Z}^d$. If $\chi(TkM_l^e2\pi i\theta)$ is a character of $G(TkM_l^e2\pi i\theta)$, then

$$\chi(\gamma_1 + A^{-tr}j, \gamma_2, \sigma) = \chi(\gamma_1, \gamma_2, \sigma)$$

$$\lambda = (\eta, \gamma_2, \sigma)$$

Proof. Let $TkM_l^e2\pi i\theta \in N$. Since $k \in A\mathbb{Z}^d$, there exists $k' \in \mathbb{Z}^d$ such that $k = Ak'$. Then

$$\chi(Ak') = \exp(2\pi i \langle \gamma_1, k' \rangle) = \exp(2\pi i \langle \gamma_1, k \rangle) \exp(2\pi i \langle \gamma_2, l \rangle) e^{2\pi i\sigma\theta}$$

$$= \exp(2\pi i \langle \gamma_1, k \rangle) \exp(2\pi i \langle \gamma_2, l \rangle) e^{2\pi i\sigma\theta}$$

$$= \chi(TkM_l^e2\pi i\theta).$$

Lemma 5.7 For a fixed $(\gamma_1, \gamma_2, \sigma) \in \Lambda_1 \times \Lambda_2 \times \mathbb{Z}_m$ in the unitary dual of $N$, if $\lambda = (\eta, \gamma_2, \sigma) \in U_\sigma, \eta = \gamma_1 + A^{-tr}j$ for some $j \in \mathbb{Z}^d$ then $\lambda = G(TkM_l^e2\pi i\theta)$. Therefore, $\lambda = G(TkM_l^e2\pi i\theta)$. $\lambda = G(TkM_l^e2\pi i\theta)$. It suffices to check that

$$\chi(TkM_l^e2\pi i\theta) = 1$$

where $\tau \in [G, G]$. First, we observe that

$$\chi(TkM_l^e2\pi i\theta) = \chi(TkM_l^e2\pi i\theta) = \exp(-2\pi i \langle \sigma k, l \rangle).$$

Applying Lemma 5.5, since $TkM_l^e2\pi i\theta$ there exists some $p \in \mathbb{Z}^d$ such that

$$\lambda = G(TkM_l^e2\pi i\theta) = \exp(2\pi i \langle B^{-tr}p, l \rangle).$$

Secondly, using the fact that $l \in B\mathbb{Z}^d$ there exists $l' \in \mathbb{Z}^d$ such that

$$\chi(TkM_l^e2\pi i\theta) = \exp(2\pi i \langle B^{-tr}p, Bl' \rangle) = 1.$$

Copyright line will be provided by the publisher
The following lemma allows us to establish the extension of characters from the normal subgroup $N$ to the stabilizer groups.

**Lemma 5.8** Fix $(\gamma_1, \gamma_2, \sigma)$ in the unitary dual of $N$. Let $\lambda = (\eta, \gamma_2, \sigma) \in U_\sigma$. Then $\chi_\lambda$ defines a character on $G_{(\gamma_1, \gamma_2, \sigma)}$.

**Proof.** Given $T_{k_1}M_{i_1}e^{2\pi i \theta_1}$, and $T_{k_2}M_{i_2}e^{2\pi i \theta_2} \in G_{(\gamma_1, \gamma_2, \sigma)}$, it is easy to see that

\[
\begin{align*}
(T_{k_1}M_{i_1}e^{2\pi i \theta_1})(T_{k_2}M_{i_2}e^{2\pi i \theta_2}) &= T_{k_1+k_2}M_{i_1+i_2}e^{2\pi i (\theta_1+\theta_2)}e^{2\pi i (l_1,k_2)}
\end{align*}
\]

where $e^{2\pi i (l_1,k_2)} \in [G_{(\gamma_1, \gamma_2, \sigma)}, G_{(\gamma_1, \gamma_2, \sigma)}]$. We want to show that $\chi_\lambda$ defines a homomorphism from $G_{(\gamma_1, \gamma_2, \sigma)}$ into the circle group. Since $\chi_\lambda \left[G_{(\gamma_1, \gamma_2, \sigma)}, G_{(\gamma_1, \gamma_2, \sigma)}\right] = 1$ then

\[
\begin{align*}
\chi_\lambda \left((T_{k_1}M_{i_1}e^{2\pi i \theta_1})(T_{k_2}M_{i_2}e^{2\pi i \theta_2})\right) &= \chi_\lambda \left(T_{k_1}T_{k_2}M_{i_1}M_{i_2}e^{2\pi i \theta_1}e^{2\pi i \theta_2}e^{2\pi i (l_1,k_2)}\right) \\
&= \chi_\lambda \left(T_{k_1+k_2}M_{i_1+i_2}e^{2\pi i (\theta_1+\theta_2+(l_1,k_2))}\right) \\
&= \exp (2\pi i (\eta, k_1 + k_2)) \exp (2\pi i (\gamma_2, l_1 + l_2)) e^{2\pi i \sigma (\theta_1+\theta_2)}e^{2\pi i (\sigma (l_1,k_2))}
\end{align*}
\]

From Lemma 5.7, $e^{2\pi i (\sigma (l_1,k_2))} = 1$ and

\[
\begin{align*}
\chi_\lambda \left((T_{k_1}M_{i_1}e^{2\pi i \theta_1})(T_{k_2}M_{i_2}e^{2\pi i \theta_2})\right) &= \exp (2\pi i (\eta, k_1 + k_2)) \exp (2\pi i (\gamma_2, l_1 + l_2)) e^{2\pi i \sigma (\theta_1+\theta_2)}
\end{align*}
\]

Now, using Lemma 5.6

\[
\chi_\lambda \left((T_{k_1}M_{i_1}e^{2\pi i \theta_1})(T_{k_2}M_{i_2}e^{2\pi i \theta_2})\right) = \chi_\lambda \left(T_{k_1}M_{i_1}e^{2\pi i \theta_1}\right) \chi_\lambda \left(T_{k_2}M_{i_2}e^{2\pi i \theta_2}\right).
\]

Thus $\chi_\lambda$ defines a character on $G_{(\gamma_1, \gamma_2, \sigma)}$. □

**Remark 5.9** Fix $(\gamma_1, \gamma_2, \sigma)$ in the unitary dual of $N$. Let $\eta = \gamma_1 + A^{-1}r^j \in U_\sigma$ where $j \in \mathbb{Z}^d$. The character $\chi(\eta, \gamma_2, \sigma)$ is called an extension of $\chi(\gamma_1, \gamma_2, \sigma)$ from $N$ to the stabilizer group $G_{(\gamma_1, \gamma_2, \sigma)}$.

**Proposition 5.10** The unitary dual of $G$ is parameterized by the set

\[
\Sigma = \bigcup_{\lambda \in \Omega} \hat{G}_\lambda \text{ where } \Omega = \bigcup_{\sigma \in [\hat{G}, G]} (\Lambda_2 \times \mathbb{E}_\sigma \times \{\sigma\})
\]

and $\mathbb{E}_\sigma$ is a measurable subset of $\Lambda_2$ satisfying the condition

\[
\bigcup_{s \in \mathbb{Z}^d} \rho(\mathbb{E}_\sigma + s) = \Lambda_2.
\]

**Proof.** Fixing $\sigma \in [\hat{G}, G]$, recall that

\[
G \cdot (\gamma_1, \gamma_2, \sigma) = \{(\gamma_1, \rho(\gamma_2 + \sigma), \sigma) : s \in \mathbb{Z}^d\}.
\]

For a fixed $\sigma \in [\hat{G}, G]$, we pick a measurable set $\mathbb{E}_\sigma \subset \Lambda_2$ such that

\[
\bigcup_{s \in \mathbb{Z}^d} \rho(\mathbb{E}_\sigma + s) = \Lambda_2.
\]

The set

\[
\Omega = \bigcup_{\sigma \in [\hat{G}, G]} (\Lambda_2 \times \mathbb{E}_\sigma \times \{\sigma\})
\]

Copyright line will be provided by the publisher
parameterizes the orbit space $\tilde{N}/G$. Thus, following Mackey’s result (see Proposition 5.2), the unitary dual of $G$ is parametrized by the set
\[
\Sigma = \bigcup_{(\gamma_1, \gamma_2, \sigma) \in \Omega} G_{(\gamma_1, \gamma_2, \sigma)}
\]

Following the description given in [13], Section 4, let $\chi_{(\gamma_1, \gamma_2, \sigma)} \in \tilde{N}$ and let $\chi_{(\gamma_1, \gamma_2, \sigma)}^j = \chi_{(\gamma_1 + A \cdot r, \gamma_2, \sigma)}$ be its extended representation from $N$ to $G_{(\gamma_1, \gamma_2, \sigma)}$. Let $\zeta$ be an irreducible representation of $G_{(\gamma_1, \gamma_2, \sigma)}/N$ and define its lift to $G_{(\gamma_1, \gamma_2, \sigma)}$ which we denote by $\zeta$. For a fixed $(\gamma_1, \gamma_2, \sigma, \zeta)$ we define the representation
\[
\rho_{(\gamma_1, \gamma_2, \sigma, \zeta)} = \text{Ind}_{G_{(\gamma_1, \gamma_2, \sigma)}}^G \left( \chi_{(\gamma_1, \gamma_2, \sigma)}^j \otimes \zeta \right)
\]
acting in the Hilbert space
\[
\mathcal{H}_{(\gamma_1, \gamma_2, \sigma, \zeta)} = \left\{ u(T_k P) = \left( \left( \chi_{(\gamma_1, \gamma_2, \sigma)}^j \otimes \zeta \right)(P) \right)^{-1} u(T_k) \right\}
\]
which is naturally identified with
\[
L^2 \left( G/G_{(\gamma_1, \gamma_2, \sigma)} \right) \cong C^{\text{card}(G_{(\gamma_1, \gamma_2, \sigma)})}.
\]
The inner product on $\mathcal{H}_{(\gamma_1, \gamma_2, \sigma, \zeta)}$ is given by
\[
\langle u, v \rangle_{\mathcal{H}_{(\gamma_1, \gamma_2, \sigma, \zeta)}} = \sum_{P \in G_{(\gamma_1, \gamma_2, \sigma)} \in G/G_{(\gamma_1, \gamma_2, \sigma)}} u(P) \overline{v(P)}
\]
Notice that the number of elements in $G/G_{(\gamma_1, \gamma_2, \sigma)}$ is bounded above by the order of the finite group
\[
G/N \cong \frac{\mathbb{Z}^d}{AZ^d}
\]
which has precisely $|\text{det } A|$ elements.

**Remark 5.11** Every irreducible representation of $G$ is monomial. That is, every irreducible representation of $G$ is induced by a one-dimensional representation of some subgroup of $G$. Given $u \in \mathcal{H}_{(\gamma_1, \gamma_2, \sigma, \zeta)}$,
\[
\rho_{(\gamma_1, \gamma_2, \sigma, \zeta)} \left( T_k M_l e^{2\pi i \theta} \right) u(T_s) = u \left( T_k M_l e^{2\pi i \theta} \right)^{-1} T_s
\]
and $u \left( (T_k M_l e^{2\pi i \theta})^{-1} T_s \right)$ is computed by following the rules defined in (5.6) where
\[
T_s \in \{T_{k_1}, \ldots, T_{k_{|\text{det } A|}}\}
\]
and
\[
\{k_1 + AZ^d, \ldots, k_{|\text{det } A|} + AZ^d\}
\]
is a set of representative elements of the quotient group $\frac{\mathbb{Z}^d}{AZ^d}$.

The lemma above is a standard computation of an induced representation. The interested reader is referred to [5]

Next, we define the set
\[
\Sigma_\sigma = \left\{ \text{Ind}_{G_{(\gamma_1, \gamma_2, \sigma)}}^G \left( \chi_{(\gamma_1, \gamma_2, \sigma)}^j \otimes \zeta \right) : \zeta \in G_{(\gamma_1, \gamma_2, \sigma)}/N \right\}
\]
A more convenient description of the unitary dual of $G$ which will be helpful when we compute the Plancherel measure is

$$\Sigma = \bigcup_{\sigma \in [G, G]} \Sigma_{\sigma}. \quad (5.7)$$

Now that we have a complete description of the unitary dual of the group $G$, we will provide a computation of its Plancherel measure.

**Theorem 5.12** The Plancherel measure is a fiber measure which is given by

$$d\mu \left( \rho(\gamma_1, \gamma_2, \sigma, \zeta) \right) = \frac{dm_1(\gamma_1) \, dm_2(\gamma_2) \, dm_3(\sigma) \, dm_4(\zeta)}{|\det A|^{-1} \, m_2(E_{\sigma})}. \quad (5.7)$$

The measures $dm_1$, $dm_2$ are the canonical Lebesgue measures defined on $\Lambda_1$ and $\Lambda_2$ respectively. The measure $dm_3$ is the counting measure on the dual of the commutator $[G, G]$ and $dm_4$ is the counting measure on the dual of the little group $G(\gamma_1, \gamma_2, \sigma)/N$ with weight $(G(\gamma_1, \gamma_2, \sigma); N)$. Moreover if $L$ is the left regular representation of $G$, the direct integral decomposition of $L$ into irreducible representations of $G$ is

$$\int_{[G, G]}^{\oplus} \rho(\gamma_1, \gamma_2, \sigma, \zeta) \otimes 1_{C^m(\gamma_1, \gamma_2, \sigma, \zeta)} \, d\mu \left( \rho(\gamma_1, \gamma_2, \sigma, \zeta) \right)$$

acting in

$$\int_{[G, G]}^{\oplus} C^n(\gamma_1, \gamma_2, \sigma, \zeta) \otimes C^n(\gamma_1, \gamma_2, \sigma, \zeta) \, d\mu \left( \rho(\gamma_1, \gamma_2, \sigma, \zeta) \right)$$

and

$$m(\gamma_1, \gamma_2, \sigma, \zeta) = \text{card} \left( G/G(\gamma_1, \gamma_2, \sigma) \right).$$

**Proof.** The theorem above is an application of the abstract case given in [13] II (Theorem 2.3), and the precise weight of the Plancherel measure is obtained by some normalization.

Let us suppose that $B$ is in $GL(d, Q) - GL(d, Z)$. Let $\varphi$ be any type I representation of $G$. Then, there is a unique measure, $\mu$ defined on the spectral set $\Sigma$ such that $\varphi$ is unitarily equivalent to

$$\int_{[G, G]}^{\oplus} \rho(\gamma_1, \gamma_2, \sigma, \zeta) \otimes 1_{C^m(\gamma_1, \gamma_2, \sigma, \zeta)} \, d\mu \left( \rho(\gamma_1, \gamma_2, \sigma, \zeta) \right) \quad (5.7)$$

acting in

$$\int_{[G, G]}^{\oplus} C^n(\gamma_1, \gamma_2, \sigma, \zeta) \otimes C^n(\gamma_1, \gamma_2, \sigma, \zeta) \, d\mu \left( \rho(\gamma_1, \gamma_2, \sigma, \zeta) \right)$$

where $m$ is the multiplicity function of the irreducible representations occurring in the decomposition of $\varphi$.

**Proposition 5.13** The representation $\varphi$ is admissible if and only if

1. $d\mu \left( \rho(\gamma_1, \gamma_2, \sigma, \zeta) \right)$ is absolutely continuous with respect to the Plancherel measure of $G$.
2. $m(\gamma_1, \gamma_2, \sigma, \zeta) \leq \text{card} \left( G/G(\gamma_1, \gamma_2, \sigma) \right).$

**Proof.** See [7] page 126

**Remark 5.14** If $B$ is in $GL(d, Q) - GL(d, Z)$ and $|\det B| \leq 1$ and the Gabor representation $\pi (5.1)$ is unitarily equivalent to $5.8$ then $\pi$ is admissible and the Proposition above is applicable.

**Proof.** This remark is just an application of the density condition given in Lemma 2.6.
6 Non-Type I Groups

In general if \( B \) has at least one non-rational entry, then the commutator subgroup of \( G \) is an infinite abelian group. In this section, we will consider the case where \( d = 1 \), and

\[
G = \langle T_k, M_l | k \in \mathbb{Z}, l \in \alpha \mathbb{Z}\rangle
\]

where \( \alpha \) is irrational positive number. Unfortunately, the Mackey machine will not be applicable here, and we will have to rely on different techniques to obtain a decomposition of the left regular representation in this case.

Let \( H \) be the three-dimensional connected, simply connected Heisenberg group. We define

\[
\Gamma = \left\{ P_{l,m,k} = \begin{bmatrix} 1 & m & l \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} : (m,k,l) \in \mathbb{Z}^3 \right\}
\]

**Lemma 6.1** There is a faithful representation \( \Theta^\alpha \) of \( \Gamma \) such that

\[
\Theta^\alpha (P_{0,0,0}) = \chi^\alpha (l), \quad \Theta^\alpha (P_{0,m,0}) = T_m, \quad \text{and} \quad \Theta^\alpha (P_{0,0,k}) = M_k \alpha.
\]

**Proof.** \( \Theta^\alpha \) is the restriction of an irreducible infinite-dimensional representation of the Heisenberg group [3] to the lattice \( \Gamma \). Since

\[
\ker \Theta^\alpha = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}
\]

the representation \( \Theta^\alpha \) is clearly faithful.

Thus, \( G \cong \Gamma \) via \( \Theta^\alpha \) and for our purpose, it is more convenient to work with \( \Gamma \). We define

\[
\Gamma_1 = \{ P_{0,m,k} : (m,k) \in \mathbb{Z}^2 \}.
\]

Let \( L_H \) be the left regular representation of the simply connected, connected Heisenberg group

\[
\mathbb{H} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : (z,y,x) \in \mathbb{R}^3 \right\}.
\]

In fact, it is not too hard to show that \( \Gamma \) is a lattice subgroup of \( \mathbb{H} \). Let \( P \) be the Plancherel transform of the Heisenberg group. We recall that

\[
P : L^2(\mathbb{H}) \to \int_{\mathbb{R}^*} L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) |\lambda| \, d\lambda
\]

where the Fourier transform is defined on \( L^2(\mathbb{H}) \cap L^1(\mathbb{H}) \) by

\[
P (f) (\lambda) = \int_{\mathbb{R}^*} \pi_\lambda (n) f (n) \, dn
\]

where

\[
\pi_\lambda (n) f (t) = \pi_\lambda (P_{z,y,x}) f (t) = e^{2\pi i x \lambda} e^{-2\pi i y \lambda} f (t - x),
\]

and the Plancherel transform is the extension of the Fourier transform to \( L^2(\mathbb{H}) \) inducing the equality

\[
\|f\|^2_{L^2(\mathbb{H})} = \int_{\mathbb{R}^*} \|P (f) (\lambda)\|^2_{HS} |\lambda| \, d\lambda.
\]
In fact, $||\cdot||_{HS}$ denotes the Hilbert Schmidt norm on $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$. Let $u \otimes v$ and $w \otimes y$ be rank-one operators in $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$. We have

$$\langle u \otimes v, w \otimes y \rangle_{HS} = \langle u, w \rangle_{L^2(\mathbb{R})} \langle v, y \rangle_{L^2(\mathbb{R})}.$$  

It is well-known that

$$L_\mathbb{H} \cong \mathcal{P} \circ L_\mathbb{H} \circ \mathcal{P}^{-1} = \int_{\mathbb{R}^m} \pi_\lambda \circ 1_{L^2(\mathbb{R})} |\lambda| \, d\lambda,$$

where $1_{L^2(\mathbb{R})}$ is the identity operator on $L^2(\mathbb{R})$, and for a.e. $\lambda \in \mathbb{R}^+$, $
\mathcal{P}(L_\mathbb{H}(x)\phi)(\lambda) = \pi_\lambda(x) \circ \mathcal{P}\phi(\lambda).$

Let $(u_\lambda)_{\lambda \in [-1,0) \cup (0,1]}$ be a measurable field of unit vectors in $L^2(\mathbb{R})$. We define two left-invariant multiplicity-free subspaces of $L^2(\mathbb{H})$ such that

$$H^+ = \mathcal{P}^{-1}\left(\int_{(0,1)} L^2(\mathbb{R}) \otimes u_\lambda |\lambda| \, d\lambda\right),$$

$$H^- = \mathcal{P}^{-1}\left(\int_{[-1,0)} L^2(\mathbb{R}) \otimes u_\lambda |\lambda| \, d\lambda\right)$$

and $H^+, H^-$ are mutually orthogonal. The following lemma has also been proved in more general terms in [15]. However the proof is short enough to be presented again in this section.

**Lemma 6.2** The representation $(L_\mathbb{H}|\Gamma, H^+)$ is cyclic and $H^+$ admits a Parseval frame of the type $L_\mathbb{H}(\Gamma) f$ with $\|f\|_{L^2(\mathbb{H})}^2 = \frac{1}{2}$.

**Proof.** Let $f, \phi \in L^2(\mathbb{H})$.

$$\sum_{\gamma \in \Gamma} \left| \langle \phi, L_\mathbb{H}(\gamma) f \rangle_{L^2(\mathbb{H})} \right|^2$$

$$= \sum_{\gamma \in \Gamma} \left| \int_{(0,1]} \langle \mathcal{P}\phi(\lambda), \pi_\lambda(\gamma) \mathcal{P}f(\lambda) \rangle_{HS} |\lambda| \, d\lambda \right|^2$$

$$= \sum_{\gamma_1 \in \Gamma_1} \sum_{j \in \mathbb{Z}} \left| \int_{(0,1]} e^{2\pi ij\lambda} \langle \mathcal{P}\phi(\lambda), \pi_\lambda(\gamma_1) \mathcal{P}f(\lambda) \rangle_{HS} |\lambda| \, d\lambda \right|^2.$$  

Since $\{e^{2\pi ij\lambda} : j \in \mathbb{Z}\}$ forms a Parseval frame for $L^2((0,1])$, we have

$$\sum_{\gamma \in \Gamma} \left| \langle \phi, L_\mathbb{H}(\gamma) f \rangle_{L^2(\mathbb{H})} \right|^2$$

$$= \sum_{\gamma_1 \in \Gamma_1} \int_{(0,1]} \left| \langle \mathcal{P}\phi(\lambda), \pi_\lambda(\gamma_1) \mathcal{P}f(\lambda) |\lambda|^{1/2} \rangle_{HS} \right|^2 |\lambda| \, d\lambda.$$  

Let $\mathcal{P}(f)(\lambda) |\lambda|^{1/2} = |\lambda|^{1/2} w_\lambda \otimes u_\lambda \in L^2(\mathbb{R}) \otimes u_\lambda$ a.e. Then

$$\sum_{\gamma \in \Gamma} \left| \langle \phi, L_\mathbb{H}(\gamma) f \rangle_{L^2(\mathbb{H})} \right|^2 = \int_{(0,1]} \sum_{\gamma_1 \in \Gamma_1} \left| \langle \mathcal{P}\phi(\lambda), \pi_\lambda(\gamma_1) \mathcal{P}f(\lambda) |\lambda|^{1/2} w_\lambda \rangle_{L^2(\mathbb{R})} \right|^2 |\lambda| \, d\lambda \quad (6.1)$$

where $\mathcal{P}\phi(\lambda) = s_\lambda \otimes u_\lambda$. Notice that by definition $\pi_\lambda(\gamma_1) f = \exp(2\pi i\lambda j t) f(t - k)$ where $(\lambda j, k) \in \lambda \mathbb{Z} \times \mathbb{Z}$ with $\lambda \in (0, 1]$. By the density condition given in Lemma 2.6, it is possible to find $v_\lambda$ such that the system

$$\{\pi_\lambda(\gamma_1) (v_\lambda) : \gamma_1 \in \Gamma_1\}$$
is a Parseval frame in $L^2(\mathbb{R})$ for a.e. $\lambda \in (0, 1]$. So let us suppose that we pick $f \in H^+$ so that
\[ P(f)(\lambda) = |\lambda|^{-1/2} v_\lambda \otimes u_\lambda. \]
We will later see that $f$ is indeed square-integrable. Coming back to (6.1),
\[ \sum_{\gamma \in \Gamma} \left| \langle \phi, L_{H}(\gamma) f \rangle_{L^2(\mathbb{H})} \right|^2 = \int_{(0,1]} \left| \langle \delta_{\lambda}, \pi_{\lambda}(\gamma_1) (v_\lambda) \rangle_{L^2(\mathbb{R})} \right|^2 |\lambda| d\lambda \]
\[ = \int_{(0,1]} \|P(\lambda)\|_{HS}^2 |\lambda| d\lambda \]
\[ = \|\phi\|^2_{L^2(\mathbb{H})}. \]

To prove that $\|f\|^2_{L^2(\mathbb{H})} = \frac{1}{2}$, we appeal to Lemma 2.7 and we obtain
\[ \left\| |\lambda|^{-1/2} v_\lambda \right\|^2_{L^2(\mathbb{R})} = |\lambda|^{-1} \|v_\lambda\|^2_{L^2(\mathbb{R})} \]
\[ = |\lambda|^{-1} \text{vol}(\lambda \mathbb{Z} \times \mathbb{Z}) \]
\[ = 1. \]

The above holds for almost every $\lambda \in (0, 1]$. Next, since $\|Pf(\lambda)\|^2_{HS} = 1$ we obtain
\[ \|f\|^2_{L^2(\mathbb{H})} = \int_{0}^{1} |\lambda| d\lambda = \frac{1}{2}. \]

Similarly, we also have the following lemma

**Lemma 6.3** The representation $(L_{\mathbb{H}}|\Gamma, H^-)$ is cyclic. Moreover $H^-$ admits a Parseval frame of the type $L_{\mathbb{H}}(\Gamma) h$ and
\[ \|h\|^2_{L^2(\mathbb{H})} = \frac{1}{2}. \]

**Lemma 6.4** Let $H = H^+ \oplus H^-$. Then there exists an orthonormal basis of the type $L_{\mathbb{H}}(\Gamma) f$ for $H$.

We remark that in general the direct sum of two Parseval frames in $H$, and $K$ is not an even a Parseval frame for the space $H \oplus K$, unless the ranges of the coefficients operators are orthogonal to each other. A proof of Lemma 6.4 is given by Currey and Mayeli in [4], where they show how to put together $f$ and $h$ in order to obtain an orthonormal basis for $H$. Now, we are in a good position to obtain a decomposition of the left regular representation of the time-frequency group. First, let us define
\[ K = \left\{ \begin{bmatrix} 1 & 0 & l \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} : (k, l) \in \mathbb{Z}^2 \right\}. \]

It is easy to see that $K$ is an abelian subgroup of $\Gamma$. Let $L$ be the left regular representation of $\Gamma$.

**Theorem 6.5** A direct integral decomposition of $L$ is obtained as follows
\[ \int_{[-1,1]} \pi_{\lambda}|\Gamma| |\lambda| d\lambda \cong \int_{[-1,1]} \int_{[0,|\lambda|]} \text{Ind}_{K}^{\Gamma} (\chi_{(\lambda, t)}) |\lambda| dt d\lambda \]
acting in the Hilbert space
\[ \int_{[-1,1]} \int_{[0,|\lambda|]} l^2(\mathbb{Z}) |\lambda| dt d\lambda \]
Proof. First, let us define $R_f : H \rightarrow l^2 (\Gamma)$ such that $R_f g (\gamma) = (g, L (\gamma) f)$. Using Lemma 6.4 (see [4] also), we construct a vector $f \in H$ such that $R_f$ is an unitary. As a result, the operator $R_f \circ P^{-1}$ is unitary and

$$(R_f \circ P^{-1}) \circ \left( \int_{[-1,1]}^{\oplus} \pi_\lambda |\lambda| \, d\lambda \right) (\cdot) \circ \left( P \circ R_f^{-1} \right) = L (\cdot) \quad (6.3)$$

where

$$\pi_\lambda \left( \begin{bmatrix} 1 & m & l \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \right) f (t) = e^{2\pi i \lambda t} e^{2\pi i k \lambda t} f (t - m).$$

Thus,

$$\int_{[-1,1]}^{\oplus} \pi_\lambda |\lambda| \, d\lambda \cong L.$$

Notice that $(\pi_\lambda |\lambda|, L^2 (\mathbb{R}))$ is not an irreducible representation. However, we may use Baggett’s decomposition given in [1]. For each $\lambda \in [-1,1]$, the representation $\pi_\lambda$ is decomposed into its irreducible components as follows:

$$\pi_\lambda |\lambda| \cong \int_{[0,|\lambda|]}^{\oplus} \operatorname{Ind}_{K}^{G} (\chi (|\lambda|, t)) \, dt \quad (6.4)$$

where

$$\chi (|\lambda|, t) \left( \begin{bmatrix} 1 & 0 & l \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \right) = \exp (2\pi i (|\lambda| l + tk)).$$

Combining (6.3) with (6.4), we obtain the decomposition given in (6.2). □

It is worth noticing that in the case where $\alpha \in \mathbb{R} - \mathbb{Q}$, that the group $G$ is a non-type I group. Moreover, it is well-known that the left regular representation of $G$ is a non-type I representation. In fact, (see [12]) the left regular representation of this group is type II. In this case, in order to obtain a useful decomposition of the left regular representation, it is better to consider a new type of dual. Let us recall the following well-known facts (see Section 3.4.2 [7]). Let $G$ be a locally compact group. Let $\tilde{G}$ be the collection of all quasi-equivalence classes of primary representations of $G$, and let $\pi$ be a unitary representation of $G$ acting in a Hilbert space $\mathcal{H}_\pi$. Essentially, there exists a unique way to decompose the representation $\pi$ into primary representations such that the center of the commuting algebra of the representation is decomposed as well. This decomposition is known as the central decomposition. More precisely, there exist

1. A standard measure $\nu_\pi$ on the quasi-dual of $G : \tilde{G}$
2. A $\nu_\pi$-measurable field of representations $(\rho_t)_{t \in \tilde{G}}$
3. A unitary operator

$$\mathfrak{R} : \mathcal{H}_\pi \rightarrow \int_{[0,|\lambda|]}^{\oplus} \mathcal{H}_{\pi, t} \, d\nu_\pi (t)$$

such that

$$\mathfrak{R} \pi (\cdot) \mathfrak{R}^{-1} = \int_{[0,|\lambda|]}^{\oplus} \rho_t (\cdot) \, d\nu_\pi (t).$$

Moreover, letting $Z (\mathcal{C} (\pi))$ be the center of the commuting algebra of the representation $\pi$, then

$$\mathfrak{R} (Z (\mathcal{C} (\pi))) \mathfrak{R}^{-1} = \int_{[0,|\lambda|]}^{\oplus} \mathbb{C} \cdot 1_{\mathcal{H}_{\pi, t}} \, d\nu_\pi (t).$$
The importance of the central decomposition is illustrated through the following facts. Let \( \pi, \theta \) be representations of a locally compact group and let \( \nu_\pi \) and \( \nu_\theta \) be the measures underlying the respective central decompositions. Then \( \pi \) is quasi-equivalent to a subrepresentation of \( \theta \) if and only if \( \nu_\pi \) is absolutely continuous with respect to \( \nu_\theta \). In particular, \( \pi \) is quasi-equivalent to \( \theta \) if and only if the measures \( \nu_\pi \) and \( \nu_\theta \) are equivalent. Furthermore, the representations \( \nu_\pi \) and \( \nu_\theta \) are disjoint if and only if the measures \( \nu_\pi \) and \( \nu_\theta \) are disjoint measures. Since the central decomposition provides information concerning the commuting algebra of the left regular representation \( L \), and because such information is crucial in the classification of admissible representations of \( \Gamma \), then it is important to mention the following.

**Theorem 6.6** \( L \cong \int_{[0,1]} \pi_\lambda |\Gamma| \otimes 1_{C^2} |\lambda| d\lambda \) and this decomposition is the central decomposition of the left regular representation of \( G \).

**Proof.** We have already seen that

\[
L \cong \int_{[-1,1]} \pi_\lambda |\Gamma| |\lambda| d\lambda
\]

and

\[
\pi_\lambda |\Gamma| \cong \int_{[0,1]} \text{Ind}_K^\Gamma (\chi_{(\lambda, t)}) dt.
\]

Therefore given \( \ell_1, \ell_2 \in [-1, 1] - \{0\} \), if \( |\ell_1| = |\ell_2| \) then \( \pi_{\ell_1} |\Gamma| \cong \pi_{\ell_2} |\Gamma| \). As a result, the left regular representation of \( G \) is equivalent to

\[
\int_{[0,1]} \pi_\lambda |\Gamma| \otimes 1_{C^2} |\lambda| d\lambda \quad (6.5)
\]

Also, it is well-known that the representation \( \pi_\lambda |\Gamma| \) is a primary or a factor representation of \( G \) whenever \( \lambda \) is irrational (see Page 127 of [5]). So, the decomposition given in (6.5) is indeed a central decomposition of \( L \). This completes the proof.

**References**


Copyright line will be provided by the publisher