Admissibility for Quasiregular Representations of Exponential Solvable Lie Groups

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Abstract

Let $N$ be a simply connected, connected non-commutative nilpotent Lie group with Lie algebra $\mathfrak{n}$ of dimension $n$. Let $H$ be a subgroup of the automorphism group of $N$. Assume that $H$ is a commutative, simply connected, connected Lie group with Lie algebra $\mathfrak{h}$. Furthermore, let us assume that the linear adjoint action of $\mathfrak{h}$ on $\mathfrak{n}$ is diagonalizable with non-purely imaginary eigenvalues. Let $\tau = \text{Ind}_{H}^{N \rtimes H} 1$. We obtain an explicit direct integral decomposition for $\tau$, including a description of the spectrum as a sub-manifold of $(\mathfrak{n} + \mathfrak{h})^*$, and a formula for the multiplicity function of the unitary irreducible representations occurring in the direct integral. Finally, we completely settle the admissibility question of $\tau$. In fact, we show that if $G = N \times H$ is unimodular, then $\tau$ is never admissible, and if $G$ is nonunimodular, $\tau$ is admissible if and only if the intersection of $H$ and the center of $G$ is equal to the identity of the group. The motivation of this work is to contribute to the general theory of admissibility, and also to shed some light on the existence of continuous wavelets on non-commutative connected nilpotent Lie groups.

2010 Mathematics Subject Classification: Primary 22E27; Secondary 22E30.
Key words and phrases: admissibility, representation, solvable, Lie group.
1 Introduction

Let $\pi$ be a unitary representation of a locally compact group $X$, acting in some Hilbert space $\mathcal{H}$. We say that $\pi$ is admissible, if and only if there exists some function $\phi \in \mathcal{H}$ such that the operator $W_\phi$ defines an isometry on $\mathcal{H}$, and $W_\phi : \mathcal{H} \to L^2(X), W_\phi \psi (x) = \langle \psi, \pi (x) \phi \rangle$. For continuous wavelets on the real line, the admissibility of the quasiregular representation $\text{Ind}_{(0,\infty)}^{\mathbb{R} \times (0,\infty)}$ of the ‘ax+b’ group which is a unitary representation acting in $L^2(\mathbb{R})$ leads to the well-known Calderon condition.

Given any locally compact group, a great deal is already known about the admissibility of its left regular representation [10]. For example, it is known that the left regular representation of the ‘ax+b’ group is admissible. The left regular representation of $\mathbb{R} \times (0,\infty)$ admits a decomposition into a direct sum of two unitary irreducible representations acting in $L^2((0,\infty))$; each with infinite multiplicities. Thus, the Plancherel measure of this affine group, is supported on 2 points. It is also known that the quasiregular representation $\text{Ind}_{(0,\infty)}^{\mathbb{R} \times (0,\infty)}$ is unitarily equivalent with a subrepresentation of the left regular representation, and thus, is admissible.

Several authors have studied the admissibility of various representations; see [1], and also [13], where Guido Weiss and his collaborators obtained an almost complete characterization of groups of the type $H \leq GL(n, \mathbb{R})$ for which the quasiregular representation $\tau = \text{Ind}_H^{\mathbb{R}^n \times H}$ is admissible. It is known that if $\tau$ is admissible then the stabilizer subgroup of the action of $H$ on characters belonging to the unitary dual of $\mathbb{R}^n$ must be compact almost everywhere. However, this condition is not sufficient to guarantee the admissibility of $\tau$. In [11], a complete characterization of dilation groups $H \leq GL(n, \mathbb{R})$ is given. On non-commutative nilpotent domains, Liu and Peng answered the question for $\tau = \text{Ind}_H^{N \times H}$, where $N$ is the Heisenberg group, and $H$ is a 1-parameter dilation group. They have also constructed some explicit continuous wavelets on the Heisenberg group (see [16]). In 2007, Currey considered $\tau = \text{Ind}_H^{N \times H}$, where $N$ is a connected, simply connected non commutative nilpotent Lie group, and $H$ is a commutative, connected, simply connected Lie group such
that $G = N \rtimes H$ is completely solvable and $\mathbb{R}$-split. He settled the admissibility question for $\tau$ under the restriction that the stabilizer subgroup inside $H$ is trivial, and he also gave some explicit construction of some continuous wavelets (see [7]). However, he did not address the case, where the stabilizer of the action of $H$ on the unitary dual of $N$ is non trivial; leaving this problem open. In 2011, we provided some answers for the admissibility of monomial representations for completely solvable exponential Lie groups in [8]. We now know that when $N$ is not commutative, the stabilizer of the action of $H$ on the dual of $N$ does not have to be trivial in order for $\tau$ to be admissible. We remark that such fact is always false if $N$ is commutative. Also, we were recently informed that new results on the subject of admissibility were obtained by Cordero, and Tabacco in [3], and Filippo De Mari and Ernesto De Vito in [9] for a different class of groups.

The purpose of this paper is to extend the results of Currey [5]. Firstly, we make no assumption that the little group inside $H$ is trivial. Secondly, the class of groups considered in this paper is larger than the class considered by Currey. This class of groups also contains exponential solvable Lie groups which are not completely solvable. We consider the situation where the action of $\mathfrak{h}$ on $\mathfrak{n}$ has roots of the type $\alpha + i\beta$, with $\alpha \neq 0$. Let us be more precise. Let $N$ be a simply connected, connected non-commutative nilpotent Lie group with real Lie algebra $\mathfrak{n}$. Let $H$ be a subgroup of the automorphism group of $N$, which we denote by $\text{Aut} (N)$. Assume that $H$ is isomorphic to $\mathbb{R}^r$ with Lie algebra $\mathfrak{h}$. Furthermore, let us assume that the linear adjoint action of $\mathfrak{h}$ on $\mathfrak{n}$ is diagonalizable with non-purely imaginary complex eigenvalues. We form the semi-direct product Lie group $G = N \rtimes H$ such that $G$ is an exponential solvable Lie group with Lie algebra $\mathfrak{g}$. More precisely, there exist basis elements such that $\mathfrak{h} = \mathbb{R}A_1 \oplus \cdots \oplus \mathbb{R}A_r$, and basis elements $Z_i$ for the complexification of $\mathfrak{n}$ such that $Z_i$ are eigenvectors for the linear operator $adA_k$, $k = 1, \ldots, r$. Furthermore, we have $adA_kZ_j = [A_k, Z_j] = \gamma_j (A_k) Z_j$ with weight $\gamma_j (A_k) = \lambda (A_k) (1 + i\alpha_j)$, $\lambda \in \mathfrak{h}^*$, a real-valued linear functional, and $\alpha_j \in \mathbb{R}$. $G$ is an exponential solvable Lie group, and is therefore type I. We define the action of $H$ on $N$ multiplicatively, and the multiplication law for $G$ is ob-
tained as follows: \((n, h) (n', h') = (nh \cdot n', hh')\). The Haar measure of \(G\) is 
\(|\det Ad(h)|^{-1} dndh\), where \(dn, dh\) are the canonical Haar measures on \(N, H\) 
respectively. We will denote by \(L\) the left regular representation of \(G\) acting 
in \(L^2(G)\). We consider the quasiregular representation \(\tau = \text{Ind}_{G}^{N \rtimes H}(1)\) acting 
in \(L^2(N)\) as follows 
\[
\tau(n, 1) f(m) = f(n^{-1}m) \\
\tau(1, h) f(m) = |\det(Ad(h))|^{-1/2} f(h^{-1}m).
\]

In this paper, mainly motivated by the admissibility question of \(\tau\), we aim 
to obtain an explicit decomposition of \(\tau\), including a precise description of 
its spectrum, an explicit formula for the multiplicity function, the measure 
occuring in the decomposition of \(\tau\), and finally, we completely settle the 
admissibility question for \(\tau\). Here is the main result of our paper.

**Theorem 1.** Let \(N\) be a simply connected, connected non commutative nilpotent Lie group with Lie algebra \(\mathfrak{n}\) of dimension \(n\). Let \(H\) be a subgroup of the 
automorphism group of \(N\). Assume that \(H\) is a commutative simply connected, 
connected Lie group with Lie algebra \(\mathfrak{h}\). Furthermore, let us assume 
that the linear adjoint action of \(\mathfrak{h}\) on \(\mathfrak{n}\) is diagonalizable with non-purely 
imaginary eigenvalues such that \(N \rtimes H\) is an exponential solvable Lie group. 
Let \(\tau = \text{Ind}_{H}^{N \rtimes H}(1)\).

1. Assume that \(\dim(H \cap Z(G)) = 0\). \(\tau\) is admissible if and only if \(N \rtimes H\) 
is nonunimodular.

2. Assuming that \(\dim(H \cap Z(G)) \neq 0\), \(\tau\) is never admissible.

**2 Preliminaries**

We recall that the coadjoint action of \(G\) on \(\mathfrak{g}^*\) is simply the dual of the adjoint 
action, and is also defined multiplicatively as \(g \cdot l(X) = l(Ad_{g^{-1}}X), g \in G, X \in \mathfrak{g}^*\). In this paper, the group \(G\) always stands for \(N \rtimes H\) as described earlier.
**Definition 2.** Given 2 representations $\pi, \theta$ of $G$ acting in the Hilbert spaces $H_\pi, H_\theta$ respectively, if there exists a bounded linear operator $T : H_\pi \to H_\theta$ such that $\theta(x) T = T \pi(x)$ for all $x \in G$, we say $T$ intertwines $\pi$ with $\theta$. If $T$ is a unitary operator, then we say the representations are unitarily equivalent, we write $\pi \simeq \theta$, and $[\pi] = [\theta]$.

**Lemma 3.** Let $L$ be the left regular representation of $G$ acting in $L^2(G)$. $L$ is admissible if and only if $G$ is nonunimodular.

Lemma 3 was proved in more general terms by Hartmut Führ in Theorem 4.23 [10]. In fact, the general statement of his proof only assumes that $G$ is type I and connected.

**Lemma 4.** If $G$ is nonunimodular, $\tau$ is admissible if and only if $\tau$ is equivalent with a subrepresentation of $L$.

**Lemma 5.** Let $\pi, \rho$ be two type I unitary representations of $G$ with the following direct integral decomposition. $\pi \simeq \int_{\hat{G}}^\oplus \sigma \otimes 1_{C_c m_\pi} d\mu(\sigma)$, and $\rho \simeq \int_{\hat{G}}^\oplus \sigma \otimes 1_{C_c m'_\rho} d\mu'(\sigma)$. $\pi$ is equivalent with a subrepresentation of $\rho$ if and only if $\mu$ is absolutely continuous with $\mu'$ and $m_\pi \leq m'_\rho$ a.e.

A clear explanation of Lemma 4 and Lemma 5 is given on Page 126 of the Monograph [10]. The following theorem is due to Lipsman, and the proof is in Theorem 7.1 in [14].

**Lemma 6.** Let $G = N \rtimes H$ be a semi-direct product of locally compact groups, $N$ normal and type I. Let $\gamma \in \hat{N}$, $H_\gamma$ the stability group. Let $\tilde{\gamma}$ be any extension of $\gamma$ to $H_\gamma$. Suppose that $N$ is unimodular, $\hat{N}/H$ is countably separated and $\tilde{\gamma}$ is a type I representation for $\mu_N$ almost everywhere $\gamma \in \hat{N}$. Let $\tilde{\gamma} \simeq \int_{\tilde{H}_\gamma}^\oplus n_\gamma(\sigma) \sigma d\mu_\gamma(\sigma)$ be the unique direct integral decomposition of $\tilde{\gamma}$. Then

$$\text{Ind}^G_H 1 \simeq \int_{\hat{N}/H} \int_{\tilde{H}_\gamma}^\oplus \pi_{\gamma, \sigma} \otimes 1_{C_c \gamma(\sigma)} d\mu_\gamma(\sigma) d\mu_N(\gamma),$$

where $\mu_N$ is the push-forward of the Plancherel measure on $\mu_N$ on $\hat{N}$.
It is now clear that in order to settle the admissibility question, it is natural to compare both representations. Being that $G$ is a type I group, there exist unique direct integral decompositions for both $L$ and $\tau$. Since both representations use the same family of unitary irreducible representations in their direct integral decomposition, in order to compare both representations, it is important to obtain the direct integral decompositions for both $L$ and $\tau$, and to check for the containment of $\tau$ inside $L$. In order to have a complete picture of the results in Lemma 6, we will need the following.

1. A precise description of the spectrum of the quasiregular representation.
2. The multiplicity function of the irreducible representations occurring in the decomposition of the quasiregular representation.
3. A description of the push-forward of the Plancherel measure of $N$.

Our approach here, will rely on the orbit method, and we will construct a smooth orbital cross-section to parametrize the dual of the group $G$.

3 Orbital Parameters

In this section, we will introduce the reader to the theory developed by Currey, and Arnal, and Dali in [2] for the construction of cross-sections for coadjoint orbits in $g^*$, where $g$ is any $n$-dimensional real exponential solvable Lie algebra with Lie group $G$. First, we consider a complexification of the Lie algebra $g$ which we denote here by $c = g_C$. Let us be more precise. We begin by fixing an ordered basis $\{Z_1, \ldots, Z_n\}$ for the Lie algebra $c$, where $Z_i = \text{Re } Z_i + i \text{Im } Z_i$, $\text{Re } (Z_i)$, and $\text{Im } (Z_i)$ belong to $g$ such that the following conditions are satisfied:

1. For each $k \in \{1, \ldots, n\}$, $c_k = \mathbb{C}\text{-span } \{Z_1, Z_2, \cdots, Z_k\}$ is an ideal.
2. If $c_j \neq c_j$ then $c_{j+1} = \overline{c_{j+1}}$ and $Z_{j+1} = \overline{Z_j}$.
3. If \( c_j = \overline{c}_j \) and \( c_{j-1} = \overline{c}_{j-1} \) then \( Z_j \in \mathfrak{g} \).

4. For any \( A \in \log H, [A, Z_j] = \gamma_j(A)Z_j \mod c_{j-1} \) with weight

\[
\gamma_j(A_k) = \lambda(A_k)(1 + i\alpha_j).
\]

\( \lambda \in \mathfrak{h}^* \), is a real-valued linear functional, and \( \alpha_j \in \mathbb{R} \).

Such basis is called an **adaptable basis**. We recall the procedure described in [2]. For any \( l \in \mathfrak{g}^* \), we define for any subset \( s \) of \( c \), \( s^l = \{ Z \in c : l([s,Z]) = 0 \} \) and \( s(l) = s^l \cap s \). Also, we define

\[
i_1(l) = \min \{ j : c_j \not\subseteq c(l) \},
\]
\[
h_1(l) = c_{i_1}^{l} = (Z_{i_1})^l,
\]
\[
j_1(l) = \min \{ j : c_j \not\subseteq h_1(l) \}.
\]

By induction, for any \( k \in \{1, 2, \cdots, n\} \), we define

\[
i_k(l) = \min \{ j : c_j \cap h_{k-1}(l) \not\subseteq h_{k-1}(l)^l \},
\]
\[
h_k(l) = (h_{k-1}(l) \cap c_{i_k})^{l} \cap h_{k-1}(l),
\]
\[
j_k(l) = \min \{ j : c_j \cap h_{k-1}(l) \not\subseteq h_k(l) \}.
\]

Finally, put \( e(l) = i(l) \cup j(l) \), where \( i(l) = \{i_k(l) : 1 \leq k \leq d \} \), and \( j(l) = \{j_k(l) : 1 \leq k \leq d \} \). An interesting well-known fact is that \( \text{card}(e(l)) \) is always even. Also, observe the sequence \( \{i_k : 1 \leq k \leq d \} \) is an increasing sequence and, \( i_k < j_k \) for \( 1 \leq k \leq d \).

Following Definition 2 [2] , let \( \mathcal{P} \) be a partition of the linear dual of the Lie algebra \( \mathfrak{g} \).

**Definition 7.** We say \( \mathcal{P} \) is an **orbital stratification** of \( \mathfrak{g}^* \) if the following conditions are satisfied

1. Each element \( \Omega \) in \( \mathcal{P} \) is \( G \)-invariant.

2. For each \( \Omega \) in \( \mathcal{P} \), the coadjoint orbits in \( \Omega \) have the same dimension.
3. There is a linear ordering on $\mathcal{P}$ such that for each $\Omega \in \mathcal{P}$,

$$\bigcup \{ \Omega' | \Omega' \leq \Omega \}$$

is a Zariski open subset of $\mathfrak{g}^*$.

The elements $\Omega$ belonging to a stratification are called **layers** of the dual space $\mathfrak{g}^*$.

**Definition 8.** Given any subset of $e$ of $\{1, 2, \cdots, n\}$, we define the set

$$\Omega_e = \{ l \in \mathfrak{g}^* | e(l) = e \}$$

which is $G$-invariant. The collection of non-empty $\Omega_e$ forms a partition of $\mathfrak{g}^*$. Such partition is called a **coarse stratification** of $\mathfrak{g}^*$. Given $e(l) = \{i_1, \cdots, i_d\} \cup \{j_1, \cdots, j_d\}$, we define

$$\Omega_{e,j} = \{ l \in \mathfrak{g}^* | e(l) = e \text{ and } j(l) = j \}.$$  

The collection of non-empty $\Omega_{e,j}$ forms a partition of $\mathfrak{g}^*$ called the **fine stratification** of $\mathfrak{g}^*$, and the elements $\Omega_{e,j}$ are called **fine layers**.

We keep the notations used in [2].

1. We fix an adaptable basis, an open dense layer $\Omega_{e,j}$. We let $c_0 = \{0\}$, and we define the following sets:

$$I = \{0 \leq j \leq n + r : c_j = c_j \},$$

$$j' = \max(\{0, 1, \cdots, j - 1\} \cap I),$$

$$j'' = \min(\{j, j + 1, \cdots, n + r\} \cap I),$$

$$K_0 = \{1 \leq k \leq d : i_k^{j''} - i_k^{j'} = 1\},$$

$$K_1 = \{1 \leq k \leq d : i_k \notin I \text{ and } i_k + 1 \notin e\},$$

$$K_2 = \{1 \leq k \leq d : i_k - 1 \notin j \setminus I\},$$

$$K_3 = \{1 \leq k \leq d : i_k \notin I \text{ and } i_k + 1 \in j\},$$

$$K_4 = \{1 \leq k \leq d : i_k \notin I \text{ and } i_k + 1 \in i \setminus j\},$$

$$K_5 = \{1 \leq k \leq d : i_k - 1 \in i \setminus j\}.$$
We remark here that

\[ i = \bigcup_{j=0}^{5} \{i_k : k \in K_j\}. \]

2. We gather some data corresponding to the fixed fine layer \( \Omega_{e,j} \). For each \( j \in e \), we define recursively the rational function \( Z_j : \Omega \to \mathcal{C} \) such that for \( k \in \{1, 2, \ldots, d\} \),

\[ V_i(l) = Z_{i_1}(l), U_i(l) = Z_{j_1}(l), \]
\[ V_k(l) = \rho_{k-1}(Z_{i_k}(l), l), U_k(l) = \rho_{k-1}(Z_{j_k}(l), l), \]
\[ Z_{i_k}(l) = \beta_{1,k}(l) \text{ Re } Z_{i_k} + \beta_{2,k}(l) \text{ Im } Z_{i_k}, \]
\[ Z_{j_k}(l) = \alpha_{1,k}(l) \text{ Re } Z_{j_k} + \alpha_{2,k}(l) \text{ Im } Z_{j_k}, \]
\[ \alpha_{1,k} = l[\text{ Re } Z_{j_k}, V_k(l)], \alpha_{2,k} = l[\text{ Im } Z_{j_k}, V_k(l)]. \]  

And \( \rho_{k}(\cdot, l) \) is the identity map.

(a) If \( k \in K_0, \beta_{1,k}(l) = 1 \), and \( \beta_{2,k}(l) = 0 \).

(b) If \( k \in K_1, \beta_{1,k}(l) = l([\rho_{k-1}(Z_{j_k}, l), \text{ Re } Z_{i_k}]), \) and

\[ \beta_{2,k}(l) = l([\rho_{k-1}(Z_{j_k}, l), \text{ Im } Z_{i_k}]). \]

(c) If \( k \in K_2, i_k - 1 = j_k, \beta_{1,k}(l) = -\alpha_{2,k}(l) \) and \( \beta_{2,k}(l) = -\alpha_{1,k}(l) \).

(d) If \( k \in K_3, \beta_{1,k}(l) = 0, \beta_{2,k}(l) = 1 \).

(e) If \( k \in K_4 (K_5 \text{ is covered here too}) \) and if \( Z_{j_{k+1}} = Z_{j_k} \), then \( \beta_{1,k}(l) = 1, \beta_{2,k}(l) = 0 \), and

\[ Z_{i_{k+1}}(l) = -\left( \frac{l[U_k(l), \text{ Im } Z_{i_k}]}{l[U_k(l), \text{ Re } Z_{i_k}]} \right) \text{ Re } Z_{i_{k+1}} - \text{ Im } Z_{i_{k+1}}. \]
3. Let $C_j = \ker \gamma_j \cap g$, $a_j(l) = (g^{l_j}_j \cap C_j) / (g^{l_{j'}}_j \cap C_j)$, we define the set $\varphi(l) \subset i$ such that $\varphi(l) = \{j \in e| a_j(l) = \{0\}\}$, and 

$$b_j(l) = \frac{\gamma_j(U_k(l))}{l[Z_j,U_k(l)]}.$$ 

The collection of sets $\Omega_{e,j,\varphi} = \{l \in \Omega_{e,j}| \varphi(l) = \varphi\}$ forms a partition of $g^*$, refining the fine stratification which, we call the \textit{ultrafine stratification} of $g^*$.

4. Letting $\Omega_{e,j,\varphi}$ be a layer obtained by refining the fixed fine layer $\Omega_{e,j}$, and gathering the data $Z_j(l), e, \varphi(l), b_j(l)$, the cross-section for the coadjoint orbits of $\Omega$ is given by the set 

$$\Sigma = \{l \in \Omega : l(Z_j(l)) = 0, j \in e \setminus \varphi \text{ and } |b_j(l)| = 1, j \in \varphi \}.$$ (3.6)

Let us now offer some concrete examples.

\textbf{Example 9.} Let $g$ be a Lie algebra spanned by $\{Z, Y, X, A\}$ with the following non-trivial Lie brackets:


An adaptable basis is $\{Z, X+iY, X-iY, A\}$ and an arbitrary linear functional is written as $l = (z, x + iy, x - iy, a)$. Here $I = \{0, 1, 3, 4\}$, $1' = 0, 2' = 1, 3' = 1, 4' = 3, 4'' = 1, 2'' = 3, 3'' = 3$, and $4'' = 4$. Put $e = \{1, 2, 3, 4\}$, and $j = \{3, 4\}$. Next, it is easy to see that $1 \in K_0$ and $2 \in K_3$. Moreover, we have 

$$Z_{ii}(l) = V_1(l) = Z, Z_{ji}(l) = U_1(l) = A, Z_{iz}(l) = Y, V_2(l) = \rho_1(Y,l) = Y - \frac{x+y}{2z}Z,$$

and 

$$Z_{jz}(l) = X, U_2(l) = \rho_1(X,l) = X - \frac{x-y}{2z}Z.$$ 

Then $\varphi = \{1\}$ and $\Omega_{e,j} = \{(z, x + iy, x - iy, a) : z \neq 0\}$ and 

$$\Sigma = \{(z, x + iy, x - iy, a) \in \Omega : |z| = 1, a = x = y = 0\}.$$
Example 10. Let $g$ be a Lie algebra spanned by 
\[
\{Z_1, Z_2, Y, X_1, X_2, A\}
\]
with the following non-trivial Lie brackets:
\[
[X_j, Y] = Z_j, \quad [A, X_1+iX_2] = (1+i)(X_1+iX_2), \quad [A, Z_1+iZ_2] = (1+i)(Z_1+iZ_2).
\]
We choose an adaptable basis 
\[
\{Z_1+iZ_2, Z_1-iZ_2, Y, X_1+iX_2, X_1-iX_2, A\}
\]
for $c$. We compute here that $I = \{0, 2, 3, 5, 6\}$, and $1' = 0, 2' = 0, 3' = 2, 4' = 3, 5' = 3, 6' = 5, 1'' = 2, 2'' = 2, 3'' = 3, 4'' = 5, 5'' = 5, 6'' = 6$. Pick $e = \{1, 3, 4, 6\}$, and $j = \{6, 4\}$. In this example, the set $K_1$ contains 1, $K_0$ contains 2. Next, with some simple computations, we obtain 
\[
Z_{i_1}(l) = (z_1 - z_2)Z_1 + (z_1 + z_2)Z_2, \quad Z_{j_1} = A, \quad Z_{j_2} = Y, \quad Z_{j_2} = z_1X_1 + z_2X_2.
\]
Clearly $\varphi = \{1\}$, the corresponding layer is $\Omega_{e,j} = \{(z, z, y, x, x) : z \neq 0\}$ and the corresponding cross-section is 
\[
\Sigma = \{(z, z, y, x, x) : |z| = 1, a = y = 0, \text{Re}(zx) = 0\}.
\]

Now, that we are introduced to the general construction, we will focus our attention to $N$ which is the Lie group of the nilradical of $g$. $N$ being an exponential solvable Lie group also, Formula 3.6 is valid. Let us recall the following well-known facts. The first one is due to Kirillov, and the second one is an application of the ‘Mackey Machine’ (see [17]).

**Lemma 11.** Let $f \in n^*$, and $\widehat{N}$ the set of unitary irreducible representations of $N$ up to equivalence. Let $n^*/N = \{N \cdot f : f \in n^*\}$ be the set of coadjoint orbits. There exists a unique bijection between $n^*/N$ and $\widehat{N}$ via Kirillov map. Thus, the construction of a measurable cross-section for the coadjoint orbits is a natural way to parametrize $\widehat{N}$. 

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Lemma 12. The set of unitary irreducible representations of $G$, $\hat{G}$ is a fiber set with $\hat{N}/H$ as base, and fibers $\hat{H}_\lambda$, where $H_\lambda$ is a closed subgroup of $H$ stabilizing the coadjoint action of $H$ on the linear functional $\lambda$.

We aim here to construct an $H$-invariant cross-section for the the coadjoint orbits of $N$ in $n^\ast$. We consider the nilradical $n$ of $g$ instead of $g$, and we go through the procedure described earlier. We first obtain an adaptable basis $\{Z_1, \cdots, Z_n\}$ for the complexification of the Lie algebra $n$ which we denote by $m$. Notice that, $\{Z_1, \cdots, Z_n, A_1, \cdots, A_{\dim(h)}\}$ is then an adaptable basis for $g$. First, fixing a dense open layer $\Omega \subset g^\ast$ and $f \in \Omega$, we obtain the jump indices corresponding to the generic layer of $g^\ast$. 

\[
i^\circ(f) = \{i_1, \cdots, i_{d^\circ}\} \\
j^\circ(f) = \{j_1, \cdots, j_{d^\circ}\} \\
e^\circ(f) = \{i_1, \cdots, i_{d^\circ}\} \cup \{j_1, \cdots, j_{d^\circ}\}.
\]

Second, let $\Omega_{e^\circ j^\circ}$ be a fixed fine layer obtained by refining $\Omega$. Given any subset $e^\circ \subseteq \{1, \cdots, n\}$, the non-empty sets $\Omega_{e^\circ j^\circ}$ are characterized by the Pfaffian of the skew-symmetric matrix $M_{e^\circ}(f) = [f[Z_i, Z_j]]_{i,j \in e^\circ}$. Referring to the procedure described in (3.4) and (3.5), we obtain 

\[Z_{i^\circ_i}(f), Z_{j^\circ_j}(f), \cdots Z_{i_{d^\circ}}(f), Z_{j_{d^\circ}}(f),\]

and we have the polarizing sequence $m = h_0(l) \supseteq h_1(l) \supseteq \cdots \supseteq h_{d^\circ}(l)$. Thirdly, we compute the following data: 

\[I, j', j'', K_0, K_1, K_2, K_3, K_4, K_5, V_1(f), \]
\[\cdots, V_{d^\circ}(f), U_1(f), \cdots, U_{d^\circ}(f), \varphi(f), b_j(f)\]

corresponding to our fine layer $\Omega_{e^\circ j^\circ}$ as described in (3.4) and (3.5). Finally, gathering all the data, we first notice that $\varphi(f) = \emptyset$, since according to Proposition 4.1 in [2], $a_j(l) = 0$ if and only if $\gamma_j(U_k(l)) \neq 0$ for $j = i_k$. As shown in [2], an $H$-invariant cross-section for the coadjoint $N$ orbits for $\Omega_{e^\circ}$ is given by 

\[\Lambda = \{f \in \Omega_{e^\circ j^\circ} : f(Z_j(f)) = 0, j \in e^\circ\}.\] (3.7)

Following the proof of Theorem 4.2 in [2], we have three separate cases
Case 1 If $j \in I$ or if $j \not\in I$ and $j + 1 \in e^{o}$ then $f(Z_j(f)) = 0$ is equivalent to $f(Z_j) = 0$.

Case 2 If $j \not\in I$, $j + 1 \not\in e^{o}$, and $j = i_k$ then

$$f(Z_j(f)) = f([\rho_{k-1}(Z_{jk}, f), \text{Re} Z_j]) \text{Re} f(Z_j) + f[\rho_{k-1}(Z_{jk}, f), \text{Im} Z_j] \text{Im} f(Z_j).$$

Case 3 If $j \not\in I$, $j + 1 \not\in e^{o}$, and $j = j_k$ then the equation $f(Z_j(f)) = 0$ is equivalent to

$$\text{Re}(f[\rho_{k-1}(Z_j, f), \text{Re} Z_{i_k}]f(Z_j)) = \text{Re}(f[\rho_{k-1}(Z_j, f), \text{Im} Z_{i_k}]f(Z_j)) = 0.$$

Remark 13. If the assumptions of Case 1 hold for all elements of $e^{o}$ then

$$\Lambda = \{f \in \Omega_{e^{o},j^{o}} : f(Z_j) = 0, j \in e^{o}\}.$$

Example 14. Let $g$ be a nilpotent Lie algebra spanned by $\{Z_1, Z_2, Y_1, Y_2, X_1, X_2\}$ with the following non-trivial Lie brackets: $[X_j, Y_j] = Z_j$. Choosing the following adaptable basis

$$\{Z_1 + iZ_2, Z_1 - iZ_2, Y_1 + iY_2, Y_1 - iY_2, X_1 + iX_2, X_1 - iX_2\},$$

letting $e^{o} = \{3, 4, 5, 6\}$, and $j^{o} = \{5, 6\}$ then

$$\Omega_{e^{o},j^{o}} = \{(z, \overline{z}, y, \overline{y}, x, \overline{x}) : z \neq 0\}$$

and

$$\Lambda = \{(z, \overline{z}, y, \overline{y}, x, \overline{x}) \in \Omega_{e^{o},j^{o}} : x = y = 0\}.$$

Now, we will compute a general formula a smooth cross-section for the $G$-orbits in some open dense set in $g^{*}$. Let $\lambda : \Omega_{e^{o},j^{o}} \to \Lambda$ be the cross-section mapping, for each $f \in n^{*}$, we define $\nu(f) = \{1 \leq j \leq n : f(Z_j) \neq 0\}$. Put

$$\mathfrak{h}(f) = \bigcap_{j \in \nu(f)} \ker \gamma_j,$$
and let \( \Lambda_\nu = \{ f \in \Lambda : \nu(f) = \nu \} \). Observe that \( \mathfrak{h}(f) \) is the Lie algebra of the stabilizer subgroup (a subgroup of \( H \)) of the linear functional \( f \). For any \( f \in \Lambda_\nu \), since we have a diagonal action, then \( \mathfrak{h}(f) \) is independent of \( f \) and is equal to some constant subalgebra \( \mathfrak{k} \subset \mathfrak{h} \).

**Lemma 15.** There exists \( \nu \subseteq \{1, \ldots, n\} \) such that \( \Lambda_\nu \) is dense and Zariski open in \( \Lambda \), and letting \( \pi \) be the projection or restriction mapping from \( g^* \) onto \( n^* \), and \( \Omega_\nu = \pi^{-1} \circ \lambda^{-1} (\Lambda_\nu) \), then \( \Omega_\nu \) is Zariski open in \( g^* \).

**Proof.** It suffices to let \( \nu = \{1, \ldots, n\} \setminus \mathfrak{e} \). Notice that

\[
\Lambda_\nu = \{ f \in \Lambda : \nu(f) = \{1, \ldots, n\} \setminus \mathfrak{e} \}
\]

is dense and Zariski open in \( \Lambda \). Additionally, we observe that for \( f \in \Lambda_\nu \), and \( j \in \{1, \ldots, n\} \setminus \mathfrak{e} \), \( f(Z_j) \neq 0 \). Next, \( \Omega_\nu \) is Zariski open in \( g^* \) since the projection map is continuous, and the cross-section mapping is rational and smooth (see [2]).

**Lemma 16.** If \( l \in \Omega_\nu \), \( \mathfrak{e}(l) \) is the set of jump indices for \( \Omega_\nu \) such that

\[
\begin{align*}
\mathfrak{e}(l) &= \{i_1, \ldots, i_d\} \cup \{j_1, \ldots, j_d\}, \\
\mathfrak{i}(l) &= \{i_1, \ldots, i_d\}, \\
\mathfrak{j}(l) &= \{j_1, \ldots, j_d\}
\end{align*}
\]

then \( \max \mathfrak{i}(l) \leq \dim n \).

**Proof.** Let us assume by contradiction that there exists some jump index \( i_t \in \mathfrak{i}(l) \) such that \( Z_{i_t} \in \mathfrak{h} \). Because, jump indices always come in pairs, and because \( j_t > i_t \), then \( Z_{j_t} \in \mathfrak{h} \). However, since \( \mathfrak{h} \) is commutative, then \( l[Z_{i_t}, Z_{j_t}] = 0 \). This is a contradiction.

**Lemma 17.** For any \( l \in \Omega_\nu \), and for all \( j \in (\mathfrak{e}(l) \setminus \mathfrak{e}^o) \setminus \mathfrak{i}(l) \), \( Z_j \in \mathfrak{h} \).

**Proof.** We have \( \mathfrak{e}(l) = \mathfrak{e}^o \cup \{i_{s_1}, \ldots, i_{s_r}\} \cup \{j_{s_1}, \ldots, j_{s_r}\} \). If \( j \in (\mathfrak{e}(l) \setminus \mathfrak{e}^o) \setminus \mathfrak{i}(l) \), then \( j \in \mathfrak{j}(l) \setminus \mathfrak{e}^o \), and there exists some \( k \) such that \( Z_j = Z_{j_{s_k}} \). Assume that \( Z_{j_{s_k}} \in n \). Since \( j_{s_k} \notin \mathfrak{e}^o \), there must exist some jump index \( i_{s_k} \) such that
Let $i_{sk} < j_{sk}$ and $l[Z_{i_{sk}}, Z_{j_{sk}}] \neq 0$. Since $Z_{i_{sk}}$ also belongs to $\mathfrak{n}$, then letting $\pi(l) = f, f[Z_{i_{sk}}, Z_{j_{sk}}] \neq 0$. Thus, both $i_{sk}, j_{sk} \in \mathfrak{e}^o$ which is a contradiction according to our assumption.

We observe that the choice of an adaptable basis mainly relies on the choice for an adaptable basis for the nilpotent Lie algebra. Any permutation of the basis elements of $\mathfrak{h}$ will not affect the ‘adaptability’ of the basis. Without loss of generality, we will assume that we have the following adaptable basis for $\mathfrak{g}$:

$$\{Z_1, \cdots, Z_n, A_{r+1}, A_r, \cdots, A_2, A_1\}$$

such that $A_r = Z_{i_{sr}}, \cdots, A_1 = Z_{i_{s1}}$. Additionally, we assume that the basis elements $A_r \cdots A_2, A_1$ with weight $\{\gamma_r, \cdots, \gamma_1\}$ are chosen such that $\text{Re} (\gamma_t (A_t)) = 1, \gamma_t (A_{t'}) = 0, t \neq t'$.

**Lemma 18.** For any $l \in \Omega_\nu, \varphi (l) = \{i_{s1}, \cdots, i_{sr}\}$.

**Proof.** We already have that $\varphi (l) \subseteq \{i_{s1}, \cdots, i_{sr}\}$. We only need to show that for any $j = i_{s1}, j \in \varphi (l)$. By definition, $\varphi (l) = \{j \in \mathfrak{e} : a_j (l) = 0\}$ and according to Proposition 4.1 in [2], $a_j (l) = 0$ if and only if $\gamma_j (U_k (l)) \neq 0$ for $j = i_k$. In order to prove the proposition, it suffices to show that $\gamma_{i_{sk}} (U_k (l)) = 0$.

$$U_k (l) = \rho_{k-1} (Z_{j_{sk}} (l), l) = \rho_{k-1} (A_{sk}) = \rho_{k-1} (A_k) = \rho_{k-2} (A_k, l) - \frac{l \rho_{k-2} (A_k, l)}{l \rho_{k-1} (l)} U_{k-1} (l) - \frac{l \rho_{k-2} (A_k, l)}{l \rho_{k-1} (l)} U_{k-1} (l).$$

A straightforward computation shows that for some coefficients $c_t$

$$\gamma_{i_{sk}} (U_k (l)) = \gamma_k (A_k) - c_{k-1} \gamma_k (A_{k-1}) - \cdots - c_1 \gamma_1 (A_1) = \gamma_k (A_k) \neq 0.$$

This completes the proof. \qed
Proposition 19. Let $g = n \times \mathfrak{k} \times \mathfrak{a}$ where, $\mathfrak{h} = \mathfrak{k} \times \mathfrak{a}$. The cross-section for the $G$-orbits in $\Omega_\nu$ is

$$\Sigma = \{ l \in \Omega_\nu : l = (f,k,0) , f \in \Sigma^o , k \in \mathfrak{t}^* \}.$$

Letting $\pi : g^* \to n^*$ be the projection map,

$$\pi (\Sigma) = \Sigma^o = \{ l \in \Lambda_\nu : |l(Z_j)| = 1 \ \forall j \in \{ i_{s_1}, \cdots , i_{s_r} \} \}.$$

Proof. Let $\pi(l) = f$. So far, we have shown that $e(l) = e^o \cup \varphi (l) \cup \{ j_{s_1}, \cdots , j_{s_r} \}$. Using the description of the cross-section described in [2],

$$\Sigma = \{ l \in \Omega_\nu : l(Z_j (l)) = 0 \ \text{for} \ j \in \mathfrak{e} \setminus \varphi , \text{and} \ |b_j (l)| = 1 \ \text{for} \ j \in \varphi \}.$$

For $l \in g^*$, if $j \in \mathfrak{e} \setminus \varphi$ then $j \in e^o \cup \{ j_{s_1}, \cdots , j_{s_r} \}$. For $j \in e^o, l(Z_j (l)) = f(Z_j (f)) = 0$ and for $j \in \{ j_{s_1}, \cdots , j_{s_r} \}, l(Z_j (l)) = 0$. Thus, $A_j = 0$ for $j \in \{ j_{s_1}, \cdots , j_{s_r} \}$. Next, for $j \in \varphi (l) = \{ i_{s_1}, \cdots , i_{s_r} \}$,

$$|b_j (l)| = \left| \frac{\gamma_j(U_k(l))}{l(Z_j,U_k(l))} \right| = \left| \frac{\gamma_j(A_k)}{l(Z_j,A_K)} \right| = \left| \frac{1}{l(Z_j)} \right| = 1 \Rightarrow |l(Z_j)| = 1.$$

Thus, we conclude that $\Sigma = \{ l \in \Omega_\nu : l = (f,k,0) , f \in \Sigma^o , k \in \mathfrak{t}^* \} \text{ where}$

$$\Sigma^o = \{ l \in \Lambda_\nu : |l(Z_j)| = 1 , j \in \{ i_{s_1}, \cdots , i_{s_r} \} \}.$$

Proposition 20. $\Sigma^o$ is a cross-section for the $H$-orbits in $\Lambda_\nu$. In other words,

$$\Sigma^o = \pi (\Sigma) \simeq \Lambda_\nu / H.$$

Proof. The set $\Lambda_\nu$ is an $H$ invariant cross-section for the $N$ coadjoint orbits of a fixed layer $\Omega_{\nu_p}$, while the set $\Sigma$ is a cross-section for the $G$ coadjoint orbits of for $\Omega_\nu$. In order to prove the proposition, we must show that

Throughout the remainder of this paper, we will also use the symbol $\simeq$ to denote a homeomorphism between two topological spaces.
each $H$-orbit of any arbitrary element inside $\Lambda_\nu$ meets the set $\Sigma^\circ$ at exactly one unique point, and also any arbitrary point in $\Sigma^\circ$ belongs to an $H$ orbit of some linear functional belonging to $\Lambda_\nu$. We start by showing that $H \cdot f \cap \Sigma^\circ$ is a non empty set for $f \in \Lambda_\nu$. Given $f \in \Lambda_\nu$, we consider the element $(f, k, 0) \in \Omega_\nu$ such that $f = \pi((f, k, 0))$. We know there exists an element $x \in \Sigma$ such that $g \cdot (f, k, 0) = x$, for some $g \in G$. In fact, let $g = (n, 1)(1, h)$. If $(n, 1)(1, h) \cdot (f, k, 0) = x$, then $\pi((n, 1)(1, h) \cdot (f, k, 0)) = \pi(x)$, and $(n, 1)\pi((1, h) \cdot (f, k, 0)) = \pi(x) \in \Lambda_\nu$. Thus, $(n, 1)$ stabilizes $\pi((1, h) \cdot (f, k, 0))$ implying that $\pi((1, h) \cdot (f, k, 0)) = \pi(x) \in \Lambda_\nu$. Since $\pi((1, h) \cdot (f, k, 0)) = \pi((h \cdot f, k, 0)) = h \cdot f$

$h \cdot f \in \pi(\Sigma) = \Sigma^\circ$. Next, let us assume that there exist $h$, and $h' \in H$ such that $f \in \Lambda_\nu$ and $h \cdot f, h' \cdot f \in \Sigma^\circ$ with $h \cdot f \neq h' \cdot f$. Now consider $(h' \cdot f, k, 0), (h \cdot f, k, 0) \in \Sigma$. We have,

$$(h \cdot f, k, 0) = (1, h) \cdot (f, k, 0)$$

$$(h' \cdot f, k, 0) = (1, h') \cdot (f, k, 0).$$

Both $(h \cdot f, k, 0)$, $(h' \cdot f, k, 0)$ are elements of the $G$-orbit of $(f, k, 0)$, and since the elements $(h \cdot f, k, 0)$, and $(h' \cdot f, k, 0)$ also belong to the cross-section $\Sigma$ then $(h \cdot f, k, 0) = (h' \cdot f, k, 0)$. The latter implies that $h \cdot f = h' \cdot f$. We reach a contradiction. We conclude that $\pi(\Sigma^\circ) = \pi(\Sigma) \simeq \Lambda_\nu/H$.

**Example 21.** Let $N$ be the Heisenberg Lie group with Lie algebra $\mathfrak{n}$ spanned by the adaptable basis $\{Z, Y, X\}$ with non-trivial Lie brackets $[X, Y] = Z$. Let $H$ be a 2 dimensional commutative Lie group with Lie algebra $\mathfrak{h} = \mathbb{R}A \oplus \mathbb{R}B$ acting on $\mathfrak{n}$ as follows. $\mathbb{R}B = \mathfrak{z}(\mathfrak{g})$ and, $[A, X] = 1/2X, [A, Y] = 1/2Y, [A, Z] = Z$. Applying the procedure above, we obtain

1. $\nu = \{1\}$
2. $\Lambda_\nu = \{(z, 0, 0) \in \mathfrak{n}^*: z \neq 0\}$
3. $\Omega_\nu = \{(z, y, x, a, b) \in \mathfrak{g}^* : z \neq 0, y, x, a, b \in \mathbb{R}\}$

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4. $\Sigma = \{(\pm 1, 0, 0, 0, b) : b \in \mathbb{R}\}$

5. $\Sigma^o = \{(\pm 1, 0, 0) \in n^*\}$

**Example 22.** Let $g = (\mathbb{R}Z_1 \oplus \mathbb{R}Z_2 \oplus \mathbb{R}Y_1 \oplus \mathbb{R}Y_2 \oplus \mathbb{R}X_1 \oplus \mathbb{R}X_2) \oplus \mathbb{R}A$ with $n = \mathbb{R}Z_1 \oplus \mathbb{R}Z_2 \oplus \mathbb{R}Y_1 \oplus \mathbb{R}Y_2 \oplus \mathbb{R}X_1 \oplus \mathbb{R}X_2$

and non-trivial Lie brackets

\[
\begin{align*}
[X_1 + iX_2, Y_1 + iY_2] &= Z_1 + iZ_2, \\
[X_1 - iX_2, Y_1 - iY_2] &= Z_1 - iZ_2 \\
[A, X_1 + iX_2] &= (1 + i)/2 (X_1 + iX_2), \\
[A, Y_1 + iY_2] &= (1 + i)/2 (Y_1 + iY_2) \\
[A, Z_1 + iZ_2] &= (1 + i)(Z_1 + iZ_2).
\end{align*}
\]

Then

1. $\nu = \{1, 2\}$

2. $\Lambda_\nu = \{(z, \overline{z}, 0, 0, 0, 0) : z \neq 0\}$

3. $\Omega_\nu = \{(z, \overline{z}, y, \overline{y}, x, \overline{x}, a) : z \neq 0, y, x \in \mathbb{C}, a \in \mathbb{R}\}$

4. $\Sigma = \{(z, \overline{z}, 0, 0, 0, 0) : z \neq 0\}$

5. $\Sigma^o = \{(z, \overline{z}, 0, 0, 0, 0) : z \neq 0\}$

Now, that we have a precise description of the orbital parametrization of the unitary dual of the group, we will take a closer look at the quasiregular representation $\tau$ of $G$ in the next section.

### 4 Decomposition of the Quasiregular Representation

In this section, we will provide a precise decomposition of $\tau$ as a direct integral of irreducible representations of $G$. As a result, we will be able to compare
the quasiregular representation with the left regular representation of $G$, and to completely settle the question of admissibility for $\tau$.

There is a well-known algorithm available for the computation of the Plancherel measure of $N$. It is simply obtained by computing the Pfaffian of a certain skew-symmetric matrix. More precisely, the Plancherel measure on $\Lambda_\nu$ is

$$d\mu(\lambda) = |\det(M_{e^\phi}(\lambda))|^{1/2} \, d\lambda = |\text{Pf}(\lambda)| \, d\lambda,$$

where $M_{e^\phi}(\lambda) = (\lambda[Z_i, Z_j])_{1 \leq i,j \leq e^\phi}$. In this section, we will focus on the decomposition of the quasiregular representation $\tau = \text{Ind}_H^G 1$, which is a unitary representation of $G$ realized as acting in $L^2(N)$ in the following ways,

$$\tau(n, 1) \phi(m) = \phi(n^{-1}m),$$
$$\tau(1, h) \phi(m) = |\delta(h)|^{-1/2} \phi(h^{-1} \cdot m), \text{ with } \delta(h) = \det(Ad(h)).$$

Let $F$ be the Fourier transform defined on $L^2(N) \cap L^1(N)$, which we extend to $L^2(N)$. Define

$$\hat{\tau}(\cdot) = F \circ \tau(\cdot) \circ F^{-1}.$$

**Definition 23.** Let $\lambda \in \Lambda_\nu$ a linear functional. A polarization algebra subordinated to $\lambda$ is a maximal subalgebra of $n_C$ satisfying the following conditions. Firstly, it is isotropic for the bilinear form $B_\lambda$ defined as $B_\lambda(X,Y) = \lambda[X,Y]$. In other words, it is a maximal subalgebra $p$ such that $\lambda([p,p]) = 0$. Secondly, $p + \overline{p}$ is a subalgebra of $n_C$. We will denote a polarization subalgebra subordinated to $\lambda$ by $p(\lambda)$. A polarization is said to be real if $p(\lambda) = \overline{p(\lambda)}$. Also, we say that the polarization $p(\lambda)$ is positive at $\lambda$ if $i\lambda[X,\overline{X}] \geq 0$ for all $X \in p(\lambda)$.

Let $e^o$ be the set of jump indices corresponding to the linear functionals in $\Lambda_\nu$, and let $e^o = \frac{d^o}{2}$. Referring to Lemma 3.5 in [2], for any given linear functional $\lambda$, a polarization subalgebra subordinated to $\lambda$ is given by $p(\lambda) = h_{d^o}(\lambda)$. See formula below Equation(3.1). Unfortunately, in general the polarization obtained as $h_{d^o}(\lambda)$ is not real and we must in that case proceed by holomorphic induction in order to construct irreducible representations of $N$. For the interested reader, a very short introduction to holomorphic induction is available on page 78 in the book [4].
The following discussion can also be found in [15] Page 124. Given \( \lambda \in \Lambda \nu \), let \( \pi_\lambda \) be an irreducible representation of \( N \) acting in the Hilbert space \( \mathcal{H}_\lambda \) and realized via holomorphic induction. Let \( \mathcal{X} \) be the domain of \( \mathcal{H}_\lambda \) on which the irreducible representation \( \pi_\lambda \) is acting on. It is well-known that \( \mathcal{X} \) can be identified with \( n/e \times e/d \), where \( d = n \cap p(\lambda) \), \( e = (p(\lambda) + p(\lambda)) \cap n \), and \( p(\lambda) \) is an \( H \)-invariant positive polarization inside \( n \). Finally, \( \mathcal{H}_\lambda = L^2(n/e) \otimes \text{Hol}(e/d) \) with \( \text{Hol}(e/d) \) denoting the holomorphic functions which are square integrable with respect to some Gaussian function. It is worth mentioning here that, if the polarization \( p(\lambda) \) is real, then \( \mathcal{H}_\lambda = L^2(n/e) \), \( \mathcal{X} = n/e \), and holomorphic induction here is a just a regular induction.

The choice of how we realize the irreducible representations of \( N \) really depends on the action of the dilation group \( H \) on \( N \). For example, if the group \( N \rtimes H \) is completely solvable, there is no need to consider the complexification of \( n \) since the existence of a positive polarization always exists for exponential solvable Lie groups. From now on, we will assume that a convenient choice for a positive polarization subalgebra has been made for each \( \lambda \in \Lambda \nu \), and we denote \( \mathcal{H}_\lambda \) the Hilbert space on which we realize the corresponding irreducible representation \( \pi_\lambda \), and \( \mathcal{X} \) is a domain on each we realize the action of \( \pi_\lambda \).

We fix an \( H \)-quasi-invariant measure on \( \mathcal{X} \), which we denote by \( dn \), and we define
\[
\delta_X(h) = \frac{d(h^{-1} \cdot n)}{dn}.
\]
Furthermore, put \( C(h, \lambda) : \mathcal{H}_\lambda \to \mathcal{H}_{h \cdot \lambda} \) defined by
\[
C(h, \lambda) f(x) = |\delta_X(h)|^{-1/2} f(h^{-1} \cdot x)
\]
such that \( \pi_\lambda(h^{-1} \cdot n) C(h, \lambda) = C(h, \lambda) \pi_{h \cdot \lambda}(n) \) for all \( n \in N \). We set the following notations. \( \Delta \) denotes the modular function of \( G \) where \( \Delta(h) = \det(\text{Ad}(h^{-1})) \), and \( \delta(h) = \Delta(h)^{-1} \).

**Proposition 24.** Let \( \phi \in \mathbf{F}(L^2(N)) \), we have
\[
\hat{\pi}_\lambda(n) (\mathbf{F}\phi)(\lambda) = \pi_\lambda(n) (\mathbf{F}\phi)(\lambda)
\]
\[
\hat{\pi}_\lambda(h) (\mathbf{F}\phi)(\lambda) = |\delta(h)|^{1/2} C(h, h^{-1} \cdot \lambda) (\mathbf{F}\phi)(h^{-1} \cdot \lambda) C(h, h^{-1} \cdot \lambda)^{-1}.
\]

The proof is elementary. Thus we will omit it. Now, we will describe how to obtain almost all of the irreducible representations of $G$ via an application of the Mackey Machine.

**Lemma 25.** If there exists some non zero linear $\lambda \in \Lambda_\nu$, and a non trivial subgroup $K \leq H$ fixing $\lambda$, then $K$ must fix all elements in $\Lambda_\nu$.

**Proof.** Recall the definition of $\Lambda_\nu$:

$$\Lambda_\nu = \{ f \in \Lambda : f(Z_j) \neq 0, j \in \{1, 2, \ldots , n\} \} \setminus \mathfrak{e}^\circ \}.$$  

Suppose there exists a linear functional $f \in \Lambda_\nu$ and $h \neq 1$, such that $h \cdot f = f$. Since the action of $h$ is a diagonal action, then it must be the case that $ad \log h(Z_j) = 0$ for all $j \in \{1, 2, \ldots , n\} \setminus \mathfrak{e}^\circ$. Thus for any $f \in \Lambda_\nu$, we have that

$$K = \{ h \in H : ad \log h(Z_j) = 0 \text{ for } j \in \{1, 2, \ldots , n\} \setminus \mathfrak{e}^\circ \}.$$  

This completes the proof.

**Lemma 26.** Let $\pi_\lambda$ be an irreducible representation of $N$ corresponding to a linear functional $\lambda \in \Lambda_\nu$ via Kirillov’s map, and let $K$ the stabilizer subgroup of the coadjoint action of $H$ on $\Lambda_\nu$. We define the extension of $\pi_\lambda$ as $\tilde{\pi}_\lambda$, which is an irreducible representation of $N \rtimes K$ acting in $H_\lambda = L^2(n/e) \otimes \text{Hol}(e/d)$ such that if $\gamma_\lambda(\cdot)$ is the restriction of $C(\lambda, \cdot)$ to $K$. More precisely, the definition of such extension is given by $\tilde{\pi}_\lambda(n,k) \phi(x) = \pi_\lambda(n) \gamma_\lambda(h) \phi(x)$. Furthermore, let $\{ \chi_\sigma : \sigma \in \mathfrak{t}^* \} = \hat{K}$, and recall that $\Sigma^o$ is the cross-section for the coadjoint orbits of $H$ in $\Lambda_\nu$. The following set

$$\{ \text{Ind}_{NK}^{NH}(\tilde{\pi}_\lambda \otimes \chi_\sigma) : (\lambda, \sigma) \in \Sigma^o \times \mathfrak{t}^* \}$$

exhausts almost all of the irreducible representations of $G$ which will appear in the Plancherel transform of $G$, and if $L$ denotes the left regular representation of $G$, we have

$$L \simeq \int_{\Sigma^o \times \mathfrak{t}^*} \text{Ind}_{NK}^{NH}(\tilde{\pi}_\lambda \otimes \chi_\sigma) \otimes 1_{L^2(H/K,H_\lambda)} d\mu(\lambda, \sigma)$$

and $d\mu(\lambda, \sigma)$ is absolutely continuous with respect to the natural Lebesgue measure on $\Sigma^o \times \mathfrak{t}^*$.  

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The claims in Lemma 26 summarize some standard facts in the analysis of exponential Lie groups. We refer the reader to Theorem 10.2 in [12] where the general case of group extensions is presented, and to [6] which specializes to the class of groups considered in this paper.

**Lemma 27.** For any \( \lambda \in \Lambda_\nu \), let \( K = \text{Stab}_G(\lambda) \), such that \( K \neq \{1\} \).

There exists a non trivial representation of \( K \) inside the symplectic group \( \text{Sp}(n/\mathfrak{n}(\lambda)) \), and \( \mathfrak{n}(\lambda) \) is the null-space of the matrix \( (\lambda[Z_i,Z_j])_{1 \leq i,j \leq n} \).

**Proof.** It is well-known that \( n/\mathfrak{n}(\lambda) \) has a smooth symplectic structure since the bilinear form \( B_\lambda(X,Y) = \lambda[X,Y] \) is a non degenerate, skew-symmetric 2-form on \( n/\mathfrak{n}(\lambda) \). Let \( h \in K \), since \( h \cdot \lambda = \lambda \), then the bilinear form \( B_\lambda(X,Y) \) is \( K \)-invariant. In other words, for any \( h \in K \), \( B_\lambda(h \cdot X, h \cdot Y) = B_\lambda(X,Y) \). Thus, there is a natural matrix representation \( \beta \) of \( K \) such that \( \beta(K) \) is a closed subgroup of the symplectic group \( \text{Sp}(n/\mathfrak{n}(\lambda)) \). Identifying \( n/\mathfrak{n}(\lambda) \) with a supplementary basis of \( \mathfrak{n}(\lambda) \) in \( n \), which we denote \( \mathcal{B} \), this representation is nothing but the adjoint representation of \( K \) acting on \( \mathcal{B} \). \( \square \)

In this paper, \( Z(G) \) stands for the center of the Lie group \( G \), and \( \mathfrak{z}(\mathfrak{g}) \) stands for its Lie algebra. Also, we remind the reader that \( \gamma_\lambda(\cdot) \) is the restriction of the representation \( C(\lambda, \cdot) \) to the group \( K \).

**Lemma 28.** Assume that \( K_1 \) is a subgroup of \( K \). \( \gamma_\lambda(K_1) = \{1\} \) if and only \( K_1 \leq Z(G) \).

**Proof.** Clearly if there exists a non trivial subgroup such that \( K_1 \leq Z(G) \) then \( \gamma_\lambda(K_1) = \{1\} \). For the other way around, let \( k \in K_1 \). Notice that

\[
\gamma_\lambda(k) \phi(x) = |\delta_X(h)|^{-1/2} \phi(\beta(k)^{-1}x).
\]

We have already seen that \( \beta(k) \) is a symplectic matrix, and at least half of its eigenvalues are 1. Since for any symplectic matrix, the multiplicity of eigenvalues 1 if they occur is even, then it follows that \( \beta(k) \) is the identity. Thus, \( k \) is a central element. \( \square \)
Remark 29. Let $\beta$ be the finite dimensional representation of $K$ in $\text{Sp}(n/n_\lambda)$. By the first isomorphism theorem, $\beta(K) \simeq K/(Z(G) \cap H)$.

Lemma 30. If there exists some $x \in \mathcal{X}$ with $\phi_x : K \to \mathcal{X}$ and $\phi_x(k) = k \cdot x$ such that $\text{rank} (\phi_x) = \max_{y \in \mathcal{X}} (\text{rank} (\phi_y))$ then the number of elements in the cross-section for the $K$ orbit in $\mathcal{X}$ is equal to $2^{\text{dim } \mathcal{X}}$ if $\text{rank} (\phi_x) = \text{dim } \mathcal{X}$, and is infinite otherwise.

Proof. Fix a cross-section $\mathcal{C} \simeq \mathcal{X}/K$, for $\mathcal{C} \subseteq \mathcal{X}$. For each $x \in \mathcal{C}$, let $r = \max_{x \in \mathcal{C}} (\text{rank} (\phi_x))$ and $\mathcal{X}_1 = \{ x \in \mathcal{X} : \text{rank} (\phi_x) = r \}$. Then, $\mathcal{X}_1$ is open and dense in $\mathcal{X}$. Assume that there exists some $y$ in $\mathcal{C}$ such that $\text{rank} (\phi_y) = \text{dim } (\mathcal{X})$. If $r = \text{dim } (\mathcal{X})$, then $\phi_y$ defines a submersion, which means that $\phi_y$ is an open map. Furthermore, $\phi_y(K)$ which is the orbit of $y$ is open in $\mathcal{X}_1$. From the definition of the action of $K$ this is only possible if and only if $K$ acts with real eigenvalues, and in that case, the number of orbits is simply equal to $2^{\text{dim } \mathcal{X}}$. Now, assume that there exists no $y$ in $\mathcal{C}$ such that $\text{rank} (\phi_y) = \text{dim } \mathcal{X}$ then the orbits in $\mathcal{X}_1$ are always meagre in $\mathcal{X}_1$. So a cross-section will contain an infinite amount of points. \qed

Lemma 31. Let $\gamma_\lambda (\cdot)$ be the restriction of $C (\lambda, \cdot)$ to $K$. We obtain the direct integral decomposition

$$\gamma_\lambda \simeq \int^{\oplus} (t/h \cap \mathcal{Y}(q))^* \chi_\sigma \otimes 1_{\mathbb{C}^m} d\sigma,$$

where the multiplicity function is uniformly constant, and we have $m : t^* \to \mathbb{N} \cup \{\infty\}$ with $m(\sigma)$ being equal to the number of elements in the cross-section $\mathcal{X}/K$.

Proof. Recall that $\gamma_\lambda (h) f (x) = |\delta_\mathcal{X} (h)|^{-1/2} f (h^{-1} \cdot x)$ and let $m$ be the number of elements in the cross-section for the $K$-orbits in $\mathcal{X}$. If $K = \{1\}$ then clearly, each point in $\mathcal{X}$ is its own orbit and $m = \infty$. If $K$ acts on some invariant open subset of $\mathcal{X}$ by spirals, then the cross-section will contain an infinite number of elements. Let $\mathcal{X}_1$ as defined in Lemma 30. We have the following natural diffeomorphism $\alpha : \mathcal{X}_1/K \times K/(H \cap Z(G)) \to \mathcal{X}_1$ such
that \( \alpha(x, k) = \bar{k} \cdot x \). Thus, \( \mathcal{X}_1 \) becomes a total space with base space \( \mathcal{X}_1/K \), and fibers \( K/(H \cap Z(G)) \cdot x \) such that

\[
\mathcal{X}_1 = \bigcup_{x \in \mathcal{X}_1/K} (K/(H \cap Z(G)) \cdot x).
\]

First, for each \( x \) in the cross-section \( \mathcal{X}_1/K \), identify \( K/(H \cap Z(G)) \cdot x \) with \( K/(H \cap Z(G)) \), and the Hilbert space

\[
\mathcal{H}_\lambda \cong \left( L^2(K/(H \cap Z(G))) \right)^m \cong L^2(K/(H \cap Z(G))) \otimes \mathbb{C}^m.
\]

In fact for each linear functional \( \lambda \), the representation \( \gamma_\lambda \) can be modelled as being quasi-equivalent to the left regular representation on \( K/(H \cap Z(G)) \).

Let \( \phi \) be a function in \( \mathcal{H}_\lambda \) and for each \( x \in \mathcal{X}_1/K \), we define \( \phi_x \) as the restriction of the function \( \phi \) to the orbit of \( x \). It is easy to see that the action of \( \gamma_\lambda (\cdot) \) becomes just a left translation acting on \( \phi_x \) for each \( x \in \mathcal{X}_1/K \). \( K/(H \cap Z(G)) \) being a commutative Lie group, we can decompose its left regular representation by using its group Fourier transform. Letting \((\mathfrak{k}/\mathfrak{h} \cap \mathfrak{z}(\mathfrak{g}))^*, \) the unitary dual of the group \( K/(H \cap Z(G)) \), we obtain a decomposition of the representation \( \gamma_\lambda \) into its irreducible components as follows.

\[
\gamma_\lambda \cong \int_{(\mathfrak{k}/\mathfrak{h} \cap \mathfrak{z}(\mathfrak{g}))^*} \chi_\sigma \otimes 1_{\mathbb{C}^m} d\sigma,
\]

where \( \chi_\sigma \) are characters defined on \( Z(G) \cap H \) and \( \int_{(\mathfrak{k}/\mathfrak{h} \cap \mathfrak{z}(\mathfrak{g}))^*} \chi_\sigma \otimes 1_{\mathbb{C}^m} d\sigma \) is modelled as acting in the Hilbert space \( \int_{(\mathfrak{k}/\mathfrak{h} \cap \mathfrak{z}(\mathfrak{g}))^*} \mathbb{C} \otimes 1_{\mathbb{C}^m} d\sigma \). This completes the proof.

**Lemma 32.** Let \( \Lambda_\nu \to \Sigma^0 \simeq \Lambda_\nu/H \) be the quotient map induced by the action of \( H \). The push-forward of the Lebesgue measure on \( \Lambda_\nu \) via the quotient map is a measure equivalent to the Lebesgue measure on \( \Sigma^0 \simeq \Lambda_\nu/H \).

**Proof.** This Lemma follows from the following facts. The quotient map is a submersion everywhere, and the push-forward of a Lebesgue measure via a submersion is equivalent to a Lebesgue measure on the image set.
Now, we will compute an explicit decomposition of the Plancherel measure on \( \Lambda_\nu \) under the action of the dilation group \( H \). We first recall the more general theorem for disintegration of Borel measures.

**Lemma 33.** Let \( G \) be a locally compact group. Let \( X \) be a left Borel \( G \)-space and \( \mu \) a quasi-invariant \( \sigma \)-finite positive Borel measure on \( X \). Assume that there is a \( \mu \)-null set \( X_0 \) such that \( X_0 \) is \( G \)-invariant and \( X - X_0 \) is standard. Then for all \( x \in X - X_0 \), the orbit \( G \cdot x \) is Borel isomorphic to \( G/G_x \) under the natural mapping, and there is a quasi-invariant measure \( \mu_x \) concentrated on the orbit \( G \cdot x \) such that for all \( f \in L^1(X,\mu) \),

\[
\int_X f(x) \, d\mu(x) = \int_{(X-X_0)/G} \int_{G/G_x} f(g \cdot x) \, d\mu_x(gG_x) \, d\overline{\mu}(x),
\]

where \( G_x \) is the stability group at \( x \).

We refer the interested reader to [12] for a proof of the above lemma.

**Proposition 34.** (Disintegration of the Plancherel measure) Under the action of \( H \) the Plancherel measure on \( \Lambda_\nu \) is decomposed into a measure on the cross-section \( \Sigma^\circ \) and a family of measures on each orbit such that for any non negative measurable function \( F \in L^1(\Lambda_\nu) \), we have

\[
\int_{\Lambda_\nu} F(f) |Pf(f)| \, df = \int_{\Sigma^\circ} \int_{H/K} F(\overline{h} \cdot \sigma) \, d\omega_\sigma(\overline{h}) \, |Pf(\sigma)| \, d\sigma
\]

where for each \( \sigma \in \Sigma^\circ \), \( d\omega_\sigma(\overline{h}) = \Delta(\overline{h}) \, d\overline{h} \) is the natural Haar measure on \( H/K \), and \( d\sigma \) is a Lebesgue measure on \( \Sigma^\circ \) with \( \overline{h} = hK \), and \( \Delta \) is the modular function defined on the group \( H/K \).

The proof is obtained via some elementary computations involving changing variables. It is quite trivial. Thus, we shall omit it.

**Theorem 35.** The quasiregular representation is unitarily equivalent to the following direct integral decomposition:

\[
\int_{\Sigma^\circ} \left( \int_{(t/\Delta(g) \cap \overline{h})} \text{Ind}_{NK}^{NH} (\hat{\pi}_\lambda \otimes \chi_\sigma) \otimes 1_{C^0} \, d\overline{\sigma} \right) |Pf(\lambda)| \, d\lambda,
\]

with multiplicity function \( m \) equal to \( 2^{\dim X} \) if \( \text{rank} (\phi_x) = \dim X \), or infinite otherwise.

Proposition 36. The quasiregular representation \( \tau = \text{Ind}^G_H (1) \) is contained in the left regular representation if and only if \( \dim(Z (G) \cap H) = 0 \).

Proof. Assume that \( Z (G) \cap H \) is not equal to the trivial group \( \{1\} \). We have proved that \( \gamma_\lambda \simeq \int_{(t/(j(g) \cap h))^*} \chi_\sigma \otimes 1_{CM(\sigma)} d\bar{\sigma} \). By Proposition 32 and also, Theorem 3.1 in [15], we have

\[
\tau \simeq \int_{\Sigma^o} \int_{(t/(j(g) \cap h))^*} \text{Ind}_{NK}^{NH} (\tilde{\pi}_\lambda \otimes \chi_\sigma) \otimes 1_{CM(\lambda, \sigma)} d\bar{\sigma} d\lambda.
\]

The measure \( d\bar{\sigma} \) is a measure belonging to the Lebesgue class measure on \((t/3 (g) \cap h)^*\), which we identify with \( \mathbb{R}^{\dim(t/(j(g) \cap h))} \). The Plancherel measure of the group \( G \) is supported on \( \Sigma^o \times t^* \) and belongs to the Lebesgue class measure \( d\lambda d\sigma \) such that \( d\sigma \) is the Lebesgue measure on \( t^* = \mathbb{R}^{\dim(t)} \). Clearly, if \( \dim(Z (G) \cap H) > 0 \), then \( \mathbb{R}^{\dim(t/3 (g) \cap h)} \) is meagre in \( \mathbb{R}^{\dim(t)} \). Thus, the measure occurring in the decomposition of the quasiregular representation, and the measure occurring in the decomposition of the left regular representation are mutually singular if and only if \( \dim(Z (G) \cap H) > 0 \). Finally, we have

\[
L \simeq \int_{\Sigma^o} \int_{t^*} \text{Ind}_{NK}^{NH} (\tilde{\pi}_\lambda \otimes \chi_\sigma) \otimes 1_{L^2(H/K, \mathcal{H}_\lambda)} d\sigma d\lambda
\]

\[
\simeq \int_{\Sigma^o} \int_{t^*} \text{Ind}_{NK}^{NH} (\tilde{\pi}_\lambda \otimes \chi_\sigma) \otimes 1_{C^\infty} d\sigma d\lambda.
\]

Since the irreducible representations occurring in the decomposition of \( L \) have uniform infinite multiplicities, the quasiregular representation \( \tau = \text{Ind}^G_H 1 \) is contained in the left regular representation if and only if \( \dim(Z (G) \cap H) = 0 \).

Finally we have our main result.

Theorem 37. Assume that \( G = N \rtimes H \) is unimodular. Then \( \tau \) is never admissible. Assume that \( G \) is nonunimodular. \( \tau \) is admissible if and only if \( \dim(Z (G) \cap H) = 0 \).
Proof. First, assume that \( G \) is unimodular. Clearly if

\[
\dim (Z(G) \cap H) = 0
\]

then, \( \tau \) will be contained in the left regular representation. However \( G \) being unimodular, it is known (see [10]) that any subrepresentation of the left regular representation is admissible if and only if

\[
\int \Sigma m(\lambda, \sigma) d\mu(\lambda, \sigma) < \infty. \quad (4.1)
\]

However that is not possible because, the multiplicity is constant a.e., \( m(\lambda, \sigma) = m \)

\[
\int \Sigma m(\lambda, \sigma) d\mu(\lambda, \sigma) = \int \Sigma m \cdot d\mu(\lambda, \sigma) = m \cdot \mu(\Sigma).
\]

Now for the first case. Assume that \( m \) is infinite, then clearly, the integral will diverge. For the second case, assume that \( m \) is finite. Then, there exists at least a non trivial \( k \in k \) such that \( \Sigma = \Sigma^o \times k^* \) and, using Currey’s measure([6]), up to multiplication by a constant,

\[
d\mu(\lambda, \sigma) = |Pf_e (\lambda, \sigma)| d\lambda d\sigma.
\]

where \( Pf_e (\lambda, \sigma) = \det ((\lambda, \sigma) [Z_i, Z_j])_{1 \leq r, s \leq d}. \) It is thus clear from the definition of the action of \( H \), that the function \( Pf_e (\lambda, \sigma) \) is really a function of \( \lambda \). Thus, we just write \( Pf_e (\lambda, \sigma) = Pf_e (\lambda) \) and

\[
\int \Sigma m(\lambda, \sigma) d\mu(\lambda, \sigma) = m \int \Sigma^o \int k^* |Pf_e (\lambda)| d\lambda d\sigma = \infty.
\]

If \( G \) is unimodular and \( \dim (Z(G) \cap H) > 0 \) then, \( \tau \) must be disjoint from the left regular representation. Now assume that \( G \) is nonunimodular. We have 2 different cases. If \( \dim (Z(G) \cap H) > 0 \) then the quasiregular representation is disjoint from the left regular representation which automatically prevents
τ from being admissible. Secondly, assume that \( \dim (Z(G) \cap H) = 0 \). We have
\[
\tau \simeq \int_{\Sigma^*} \int_{t^*} \text{Ind}_{NH}^{NK} (\pi_{\lambda} \otimes \chi_{\sigma}) \otimes 1_{C^m(\lambda, \sigma)} d\sigma d\lambda,
\]
and of course, as seen previously, the multiplicity function is uniformly constant and, \( m(\lambda, \sigma) \leq \infty \). Thus, \( \tau \) is quasi-equivalent with the left regular representation. \( G \) being nonunimodular, it follows that \( \tau \) is admissible. \[\square\]

**Remark 38.** We bring the attention of the reader to the fact that the theorem above supports Conjecture 3.7 in [8] which states that a monomial representation of a unimodular exponential solvable Lie group \( G \) never has admissible vectors. The general case remains an open problem.

Based on our main theorem, we can assert the following.

**Remark 39.** Let \( N \) be a nilpotent Lie group with Lie algebra \( n \). Let \( H \) be given such that at least one of the basis element of \( h \) commutes with all basis elements of \( n \). Then \( Z(N \rtimes H) \cap H \) is clearly non trivial, and \( \tau \) cannot be admissible as a representation of \( G \).

## 5 Examples

In this section, we will present several examples, and we will show how to apply our results in order to settle the admissibility of \( \tau \) in each case.

**Example 40.** Coming back to Example 21, clearly \( G \) is not unimodular. Since the center of the group has a non-trivial intersection with \( H \) then \( \tau \) is not an admissible representation.

**Example 41.** Recall Example 22. Since \( G \) is nonunimodular and since the center of the group is trivial, then \( \tau \) is an admissible representation of \( G \).

**Example 42.** Let \( G \) a Lie group with Lie algebra \( g \) spanned by \( \{Z,Y,X,A_1,A_2,A_3\} \)
such that
\[
\begin{align*}
[X, Y] &= Z, [A_1, X] = X, \\
[A_2, X] &= X, [A_3, X] = 2X, \\
[A_1, Y] &= Y, [A_2, Y] = -Y, \\
[A_3, Y] &= -Y, [A_1, Z] = Z, \\
[A_3, Z] &= Z.
\end{align*}
\]

Since the center of $G$ is equal to
\[
\exp \left( \mathbb{R} \left( -\frac{1}{2} A_1 - \frac{3}{2} A_2 + A_3 \right) \right) < H
\]
then $\tau$ is not admissible.

**Example 43.** Let $G$ a Lie group with Lie algebra $\mathfrak{g}$ spanned by
\[
\{ Z, Y, X, W, A_1, A_2, A_3, A_4 \}
\]
with non-trivial Lie brackets
\[
\begin{align*}
[A_1, W] &= \frac{1}{3} W, [A_1, X] = \frac{1}{3} X, \\
[A_1, Y] &= \frac{2}{3} Y, [A_1, Z] = Z \\
[A_3, X] &= 2/5 X, [A_2, Y] = 3/5 Y, \\
[A_4, Y] &= 1/2 Y, [A_4, Z] = Z.
\end{align*}
\]

In this example the Lie algebra $\mathfrak{h}$ is spanned by the vectors $A_1, A_2, A_3, A_4$. The center of $G$ is equal to
\[
\exp \left( \mathbb{R} \left( -\frac{9}{10} A_1 - \frac{1}{10} A_2 - A_3 \right) \right) \exp \left( \mathbb{R} \left( -\frac{3}{4} A_1 - \frac{1}{4} A_2 + A_4 \right) \right) < H
\]
then $\tau$ is not admissible.
Example 44. Let us suppose that $\mathfrak{g}$ is spanned by the vectors

$$U_1, U_2, Z_1, Z_2, Z_3, X_1, X_2, X_3, A$$

and $\mathfrak{h}$ is spanned by the vector $A$. Furthermore, assume that we have the following non-trivial Lie brackets

$$[X_3, X_2] = Z_1, [X_3, X_1] = Z_2, [X_2, X_1] = Z_3, [A, U_1 + iU_2] = (1 + i)(U_1 + iU_2).$$

We remark that in this example, the nilradical of $\mathfrak{g}$ contains a step-two freely generated nilpotent Lie algebra with 3 generators. Since $G$ is nonunimodular, and since the center of $G$ is trivial, then $\tau$ is admissible.

Example 45. Let $N$ be the Heisenberg group

$$N = \left\{ \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} z \\ y \\ x \end{pmatrix} \in \mathbb{R}^3 \right\},$$

and the dilation group $H$ is isomorphic to $\mathbb{R}^2$ such that

$$H = \left\{ \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{t-r} & 0 & 0 \\ 0 & 0 & e^r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} t \\ r \end{pmatrix} \in \mathbb{R}^2 \right\}.$$

The action of $H$ on $N$ is given as follows.

$$\begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{t-r} & 0 & 0 \\ 0 & 0 & e^r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & xe^r & ye^r e^{t-r} & ze^t \\ 0 & 1 & 0 & ye^{t-r} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$
It is easy to see that the Lie algebra of $G$ is spanned by \{Z, Y, X, A\} with non-trivial Lie brackets

\[
[A_1, Z] = Z, \quad [A_1, Y] = Y \\
[A_2, Y] = -Y, \quad [A_2, X] = X.
\]

Here

\[
K = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-r} & 0 & 0 \\ 0 & 0 & e^r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : r \in \mathbb{R} \right\}
\]

but the center of the $G$ is trivial. Thus, there is a non-trivial subgroup of the dilation group stabilizing the center of $N$ and thus stabilizing almost all of elements of the unitary dual of $N$. The spectrum of the left regular representation of $G = N \rtimes H$ is supported on the two disjoint lines, and the irreducible representations occurring in decomposition of the left regular representation occur with infinite multiplicities. Also, the spectrum of the quasiregular representation $\tau$ is parametrized by two disjoint lines, but the irreducible representations occurring in the decomposition of $\tau$ occur twice almost everywhere. Since the group $G$ is nonunimodular, and $\tau$ is contained in $L$ then $\tau$ is admissible.

**Example 46.** Let us suppose that $\mathfrak{n}$ is spanned by $T_1, T_2, Z, Y, X$ such that $[X, Y] = Z$, $\mathfrak{h}$ is spanned by $A_1, A_2, A_3, A_4, A_5$ such that

\[
[A_2, X] = 1/2X, \\
[A_2, Y] = 1/2Y, \\
[A_2, Z] = Z, \\
[A_3, X] = X, \\
[A_3, Y] = -Y, \\
[A_5, X] = X, [A_6, Y] = Y \\
[A_3, T_1 + iT_1] = (1 + i) (T_1 + iT_1) \\
[A_4, T_1 + iT_1] = (2 + 2i) (T_1 + iT_1) \\
[A_1, T_1 + iT_1] = (1 + i) (T_1 + iT_1).
\]
The center of $G$ is given by

$$\exp(R(A_1 - 2A_2 - 1/2A_4))\exp(R(A_3 - 1/2A_4 - A_5)) < H.$$ 

Thus $\tau$ is not admissible.

Acknowledgements

Thanks go to Professor Bradley Currey who introduced me to the theory of admissibility. Without his support, and guidance this work would have never been possible. I also thank my wife Lindsay and my three children Senami, Kemi and Donah for their support. Finally, the author is grateful to the anonymous referee for his careful reading, comments, corrections, and helpful suggestions.

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