


title

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Abstract. Let $N$ be a simply connected, connected, two-step nilpotent Lie group with Lie algebra $n$ such that $n = a \oplus b \oplus z$, $[a, b] \subset z$, the algebras $a, b$ are commutative and have the same dimension. Also, we assume that $\det ([X_i, Y_j])_{1 \leq i, j \leq d}$ is a non-vanishing homogeneous polynomial. Using well-known facts from time-frequency analysis, we provide some precise sufficient conditions for the existence of sampling spaces with respect to some discrete subset with the interpolation property. Thus, the result obtained in this work can be seen as a direct application of time-frequency analysis to the theory of nilpotent Lie groups. Several explicit examples are computed to illustrate the results. This work is a generalization of recent results obtained for the Heisenberg group by Currey, and Mayeli in [2].

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1. Introduction

Let $N$ be a locally compact group, and let $\Gamma$ be a discrete subset of $N$. Let $H$ be a left-invariant closed subspace of $L^2(N)$ consisting of continuous functions. We call $H$ a sampling space (Section 2.6 [6]) with respect to $\Gamma$ (or $\Gamma$-sampling space) if the following properties hold. First, the restriction

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mapping \( R_\Gamma : H \to L^2 (\Gamma) \), \( R_\Gamma f = (f(\gamma))_{\gamma \in \Gamma} \) is an isometry. Secondly, there exists a vector \( S \in H \) such that for any vector \( f \in H \), we have the following expansion \( f(x) = \sum_{\gamma \in \Gamma} f(\gamma) S(\gamma^{-1}x) \) with convergence in the \( L^2 \)-norm. The vector \( S \) is called a sinc-type function, and if \( R_\Gamma \) is surjective, we say that the sampling space \( H \) has the interpolation property.

The simplest example of a sampling space with interpolation property over a nilpotent Lie group is provided by the well-known Whittaker, Shannon, Kotel’nikov Theorem (see Example 2.52 [6]) which we recall here. Let
\[
H = \left\{ f \in L^2 (\mathbb{R}) : \text{supp} \widehat{f} \subset [-0.5, 0.5] \right\}
\]
where \( f \mapsto \widehat{f} \) is the Fourier transform of \( f \) and is defined as \( \widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx \). Then \( H \) is a sampling space which has the interpolation property with associated sinc-type function
\[
S(x) = \frac{\sin \pi x}{\pi x}.
\]

To the best of our knowledge, the first example of a sampling space with interpolation property on a non-commutative nilpotent Lie group, using the Plancherel transform is the three-dimensional Heisenberg Lie group. This example is due to a remarkable result of Currey and Mayeli [2]. The specific definition of bandlimited spaces by the Plancherel transform used in [2], was taken from [6], Chapter 6, where a very precise characterization of sampling spaces of the Heisenberg group was provided. Moreover, sampling spaces were studied by the author in [8] and [7] for a class of nilpotent Lie groups which contains the Heisenberg Lie groups. However, nothing was said about the interpolation property of these spaces. This question of existence of sampling spaces with interpolation property on non-commutative nilpotent Lie groups is a much harder problem which we shall address in this paper.

Let \( N \) be a simply connected, connected, two-step nilpotent Lie group with Lie algebra \( n \) of dimension \( n \) satisfying the following conditions. Let \( a = \mathbb{R}\text{-span} \{X_1, \ldots, X_d\} \), \( b = \mathbb{R}\text{-span} \{Y_1, \ldots, Y_d\} \), and \( z = \mathbb{R}\text{-span} \{Z_1, \ldots, Z_{n-2d}\} \) such that \( n = a \oplus b \oplus z \), \([a, b] \subseteq z \), \( a, b \) are commutative algebras, \( a, b \) are \( d \)-dimensional real vector space and
\[
\det \begin{bmatrix}
[X_1, Y_1] & [X_1, Y_2] & \cdots & [X_1, Y_d] \\
[X_2, Y_1] & [X_2, Y_2] & \cdots & [X_2, Y_d] \\
\vdots & \vdots & \cdots & \vdots \\
[X_d, Y_1] & [X_d, Y_2] & \cdots & [X_d, Y_d]
\end{bmatrix}
\]
is a non-vanishing polynomial defined over the commutator ideal \([n, n]\). This class of groups has also been studied by the author in [8] and [7].
One very appealing fact about these groups is the following. Elements in the unitary dual of any group satisfying the conditions given above are closely related to the well-known Schrödinger representations [8, 7]. The advantage of this fact is that, we are able to exploit well-known theorems from time-frequency analysis to derive our results. Let $N$ be a nilpotent Lie group satisfying the conditions given above. We deal with the existence of left-invariant subspaces of $L^2(N)$ which are sampling spaces which have the interpolation property. More precisely, we investigate conditions under which sampling provides an orthonormal basis which is generated by shifting a single function. The work presented here provides a natural generalization of recent results obtained for the Heisenberg group in [2]. We offer precise and explicit sufficient conditions for sampling spaces which have the interpolation property with respect to some discrete set $\Gamma \subset N$. We also construct examples of nilpotent Lie groups of step-two which have these properties.

We organize this paper as follows. The second section deals with some preliminary results which can be found in [8, 7, 3]. In the third section, we introduce a natural notion of bandlimitation for the class of groups considered, and we state the main theorem (Theorem 9) of the paper. In the fourth section, we prove results related to sampling and frames for the class of groups considered here. The results obtained in the fourth section allow us to provide a proof for the main theorem in the last section, and we compute some examples to support our work.

2. Preliminaries

Let us start by setting up some notation. In this paper, all representations are strongly continuous and unitary, unless we state otherwise. All sets are measurable, and given two equivalent unitary representations $\tau$ and $\pi$, we write $\tau \cong \pi$. We also use the same notation for isomorphic Hilbert spaces. The characteristic function of a set $E$ is written as $\chi_E$, and the cardinal number of a set $I$ is denoted by $\text{card}(I)$. $V^*$ stands for the dual vector space of a vector space of $V$. Let $v$ be a vector. $v^*$ stands for the transpose of the vector $v$. The Fourier transform of a suitable function $f$ defined over a commutative domain is written as $\hat{f}$, and the conjugate of a complex number $z$ is denoted $\bar{z}$. The general linear group of $R^n$ is denoted $GL_n(R)$. Let $v, w$ be two vectors in some Hilbert space. We write $v \perp w$ to denote that the vectors are orthogonal to each other.

Next, we will provide a short introduction to the theory of direct integrals which is also nicely exposed in Section 3.3 of [6]. Let $\{H_a\}_{a \in A}$ be a family of separable Hilbert spaces with respect to a measure $\mu$ on the parameter space $A$. We define the direct integral of this family of Hilbert
spaces with respect to $\mu$ as the space which consists of functions $f$ defined on the parameter space $A$ such that $f(\alpha)$ is an element of $H_\alpha$ for each $\alpha \in A$, such that
\[
\int_A \|f(\alpha)\|_{H_\alpha}^2 \, d\mu(\alpha) < \infty
\]
with some additional measurability conditions which we will clarify. A family of separable Hilbert spaces $\{H_\alpha\}_{\alpha \in A}$ indexed by a Borel set $A$ is called a field of Hilbert spaces over $A$. Next, a map
\[
f : A \to \prod_{\alpha \in A} H_\alpha
\]
such that $f(\alpha) \in H_\alpha$ is called a vector field on $A$. A measurable field of Hilbert spaces over the indexing set $A$ is a field of Hilbert spaces $\{H_\alpha\}_{\alpha \in A}$ together with a countable set $\{e_j\}_j$ of vector fields such that
\begin{enumerate}
\item the functions $\alpha \to \langle e_j(\alpha), e_k(\alpha) \rangle_{H_\alpha}$ are measurable for all $j, k$,
\item the linear span of $\{e_k(\alpha)\}_k$ is dense in $H_\alpha$ for each $\alpha \in A$.
\end{enumerate}
The direct integral of the spaces $H_\alpha$ with respect to the measure $\mu$ is denoted by
\[
\int_A \oplus_{\alpha \in A} H_\alpha \, d\mu(\alpha)
\]
and is the space of measurable vector fields $f$ on $A$ such that
\[
\int_A \|f(\alpha)\|_{H_\alpha}^2 \, d\mu(\alpha) < \infty.
\]
The inner product for this Hilbert space is naturally obtained as follows. For $f, g \in \int_A \oplus_{\alpha \in A} H_\alpha \, d\mu(\alpha)$,
\[
\langle f, g \rangle = \int_A \langle f(\alpha), g(\alpha) \rangle_{H_\alpha} \, d\mu(\alpha).
\]
This theory of direct integrals will play an important role in the definition of bandlimited spaces in our work.

Let $N$ be a non-commutative connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ over the reals. Recall $\mathfrak{a} = \mathbb{R}\text{-span} \{X_1, \cdots, X_d\}$, $\mathfrak{b} = \mathbb{R}\text{-span} \{Y_1, \cdots, Y_d\}$, and $\mathfrak{z} = \mathbb{R}\text{-span} \{Z_1, \cdots, Z_{n-2d}\}$ such that the following assumptions hold.

**Condition 1.** $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{z}$, $[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{z}$, $\mathfrak{a}, \mathfrak{b}$ are commutative algebras,
\[
\dim_{\mathbb{R}}(\mathfrak{a}) = \dim_{\mathbb{R}}(\mathfrak{b}) = d,
\]
and
\[
\det \begin{bmatrix}
[X_1, Y_1] & [X_1, Y_2] & \cdots & [X_1, Y_d] \\
[X_2, Y_1] & [X_2, Y_2] & \cdots & [X_2, Y_d] \\
\vdots & \vdots & \cdots & \vdots \\
[X_d, Y_1] & [X_d, Y_2] & \cdots & [X_d, Y_d]
\end{bmatrix}
\]
is a non-vanishing polynomial defined over the commutator ideal \([n, n]\).

Let \(\mathfrak{B} = \{T_1, T_2, \cdots, T_n\}\) be a basis for the Lie algebra \(n\). We say that \(\mathfrak{B}\) is a strong Malcev basis (see Page 10 [3]) if and only if for each \(j\) the real span of \(\{T_1, T_2, \cdots, T_j\}\) is an ideal of \(n\). Fixing a strong Malcev basis of the Lie algebra \(n\), a typical element of the Lie group \(N\) is written as follows:
\[
\exp \left( \sum_{k=1}^{n-2d} z_k Z_k \right) \exp \left( \sum_{k=1}^{d} y_k Y_k \right) \exp \left( \sum_{k=1}^{d} x_k X_k \right).
\]
The subgroup
\[
\exp \left( \sum_{k=1}^{n-2d} RZ_k \right)
\]
is the center of the Lie group \(N\) and the subgroup
\[
\exp \left( \sum_{k=1}^{n-2d} RZ_k \right) \exp \left( \sum_{k=1}^{d} RY_k \right)
\]
is a maximal normal abelian subgroup of \(N\). Moreover, \(N\) is a step-two nilpotent Lie group since the commutator ideal \([n, n]\) is central. Let us now collect some additional basic facts about groups satisfying Condition 1.

**Proposition 2.** Let \(N\) be a nilpotent Lie group satisfying the conditions given above. There is a finite dimensional faithful representation of \(N\) in \(GL(n + 1, \mathbb{R})\) for \(n \geq 3\).

**Proof.** Clearly if \(n < 3\), then \(n\) must be commutative. Thus, we must assume that \(n \geq 3\). First, let \(n_1 = a \oplus b \oplus (a \ominus [n, n])\) and \(n_2 = [n, n]\) such that \(n = n_1 \oplus n_2\). Let \(a\) be a positive real number. Next, we define an element \(A_a\) in the outer derivation of \(n\) acting by a diagonalizable action such that \([A_a, U] = \ln(a) U\) for all \(U \in n_1\) and \([A_a, Z] = 2 \ln(a) Z\) for all \(Z \in n_2\). Using the Jacobi identity, it is fairly easy to see that indeed \(A_a\) defines a derivation. Next, we consider the linear adjoint representation of \(g = n \oplus RA_a : ad : g \rightarrow gl(g)\) and we define \(G = \exp (ad(g)) \leq GL(g)\).
Fixing a strong Malcev basis for the Lie algebra \( \mathfrak{n} \), the adjoint representation of \( G \) acting on the vector space \( \mathfrak{g} \) is a faithful representation. Thus, \( G = \exp(\text{ad}(\mathfrak{g})) \) is a Lie subgroup of \( GL(\mathfrak{g}) \cong GL(n + 1, \mathbb{R}) \). Since \( N \) is isomorphic to \( \exp(\text{ad}(\mathfrak{n} \oplus \{0\})) \) then \( \exp(\text{ad}(\mathfrak{n} \oplus \{0\})) \) is an isomorphic copy of the Lie group \( N \) inside \( GL(n + 1, \mathbb{R}) \).

Next, in order to make this paper self-contained, we will revisit the Plancherel theory for the class of groups considered in this paper. We start by fixing a strong Malcev basis for the Lie algebra of \( \mathfrak{n} \). Assume that \( N \) is endowed with its canonical Haar measure. Let \( \Sigma \) be a set parametrizing the unitary dual of \( N \). Since \( N \) is a nilpotent Lie group, according to the orbit method (see [3]) all irreducible representations of \( N \) are parametrized by the coadjoint orbits of \( N \) in \( \mathfrak{n}^* \). Let \( \mathcal{P} \) be the Plancherel transform on \( L^2(N) \) and \( \mathcal{F} \) the Fourier transform defined on \( L^2(N) \cap L^1(N) \) by

\[
\mathcal{F}(f)(\lambda) = \int_N \pi_\lambda(n) f(n) \, dn
\]

where \( \{\pi_\lambda : \lambda \in \Sigma\} = \widehat{N} \). It is well-known that

\[
\mathcal{P} : L^2(N) \rightarrow \int_\Sigma \mathbb{R}^d \otimes L^2(\mathbb{R}^d) \, d\mu(\lambda),
\]

such that the Plancherel transform is the extension of the Fourier transform to \( L^2(N) \) inducing the equality

\[
\|f\|^2_{L^2(N)} = \int_\Sigma \|\mathcal{P}(f)(\lambda)\|_{HS}^2 \, d\mu(\lambda).
\]

We recall that \( \|\cdot\|_{HS} \) denotes the Hilbert-Schmidt norm on \( L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \). The Hilbert space tensor product \( L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \) is defined as the space of bounded linear operators \( T : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d) \) such that

\[
\|T\|_{HS} = \sum_{k \in I} \|Te_k\|_{L^2(\mathbb{R}^d)}^2
\]

where \( (e_k)_{k \in I} \) is an orthonormal basis of \( L^2(\mathbb{R}^d) \). Given arbitrary \( S, T \in L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \), the inner product of the operators \( S, T \) is:

\[
\langle S, T \rangle_{HS} = \sum_{k \in I} \langle Se_k, Te_k \rangle_{L^2(\mathbb{R}^d)}.
\]

Also, it is useful to observe that the inner product of arbitrary rank-one operators in \( L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \) is given by

\[
\langle u \otimes v, w \otimes y \rangle_{HS} = \langle u, w \rangle_{L^2(\mathbb{R}^d)} \langle v, y \rangle_{L^2(\mathbb{R}^d)}.
\]

**Proposition 3.** Let \( \mathfrak{n} \) be a Lie algebra over the reals satisfying Condition 1 and let \( L \) be the left regular representation of the group \( N \).
For $\lambda \in \mathfrak{n}^*$, defining the $d \times d$ matrix
\[ B(\lambda) = \begin{bmatrix}
\lambda [X_1, Y_1] & \cdots & \lambda [X_1, Y_d] \\
\vdots & \ddots & \vdots \\
\lambda [X_d, Y_1] & \cdots & \lambda [X_d, Y_d]
\end{bmatrix}, \]
the unitary dual of $N$ is parametrized by the smooth manifold
\[ \Sigma = \left\{ \lambda \in \mathfrak{n}^* : \det(B(\lambda)) \neq 0, \lambda(X_1) = \cdots = \lambda(Y_1) = \cdots = \lambda(Y_d) = 0 \right\} \]
which is naturally identified with a Zariski open subset of $\mathfrak{s}^*$. 

Let $\lambda$ be the Lebesgue measure on $\Sigma$. The Plancherel measure for the group $N$ is supported on $\Sigma$ and is equal to
\[ d\mu(\lambda) = |\det(B(\lambda))|d\lambda. \]

The unitary dual of $N$ which we denote by $\hat{N}$ is up to a null set equal to $\{ \pi_{\lambda} : \lambda \in \Sigma \}$ where each representation $\pi_{\lambda}$ is realized as acting in $L^2(\mathbb{R}^d)$ such that
\[ \pi_{\lambda}\left( \exp \sum_{i=1}^{n-2d} z_iZ_i \right) f(t) = e^{2\pi i \sum_{i=1}^{n-2d} z_iZ_i} f(t), \]
\[ \pi_{\lambda}\left( \exp \sum_{i=1}^d y_iY_i \right) f(t) = e^{2\pi i (B(\lambda)y, t)} f(t), \]
\[ \pi_{\lambda}\left( \exp \sum_{i=1}^d x_iX_i \right) f(t) = f(t-x) \]
where $y = (y_1, \cdots, y_d)^{tr}$ and $x = (x_1, \cdots, x_d)$.

$L \cong \mathcal{P} \circ L \circ \mathcal{P}^{-1} = \int_{\Sigma}^{\oplus} \pi_{\lambda} \otimes 1_{L^2(\mathbb{R}^d)} d\mu(\lambda)$ and $1_{L^2(\mathbb{R}^d)}$ is the identity operator on $L^2(\mathbb{R}^d)$. Moreover for $\lambda \in \Sigma$, we have
\[ \mathcal{P}(L(x)\phi)(\lambda) = \pi_{\lambda}(x) \circ (\mathcal{P}\phi)(\lambda). \]

The results in the proposition above are some elementary facts, which are well-known in the theory of harmonic analysis of nilpotent Lie groups. See [8], where we specialized to the class of groups considered here. For general nilpotent Lie groups, we refer the interested reader to Section 4.3 in [3] which contains a complete presentation of the Plancherel theory for nilpotent Lie groups.

We will now provide a few examples of Lie groups satisfying Condition 1.
Example 4. Let $N$ be a nilpotent Lie group with Lie algebra $\mathfrak{n}$ spanned by $Z, Y, X$ with non-trivial Lie brackets $[X, Y] = Z$. Then $N$ is the 3-dimensional Heisenberg Lie group and satisfies all properties given in Condition 1.

Example 5. Let $N$ be a nilpotent Lie group with Lie algebra $\mathfrak{n}$ spanned by the strong Malcev basis $Z_1, Z_2, Y_1, Y_2, X_1, X_2$ with non-trivial Lie brackets

\[
[X_1, Y_1] = Z_1, [X_2, Y_1] = -Z_2, \\
[X_1, Y_2] = Z_2, [X_2, Y_2] = Z_1.
\]

Clearly, $N$ satisfies all properties described in Condition 1 and

\[
\det \left( \left( [X_i, Y_j] \right)_{1 \leq i, j \leq 2} \right) = \det \left[ \begin{array}{cc} Z_1 & Z_2 \\ -Z_2 & Z_1 \end{array} \right] = Z_1^2 + Z_2^2.
\]

Applying Proposition 2, we define the monomorphism $\pi : N \rightarrow GL_7(\mathbb{R})$ such that for

\[
p = \exp (z_1 Z_1) \exp (z_2 Z_2) \exp (y_1 Y_1) \exp (y_2 Y_2) \exp (x_1 X_1) \exp (x_2 X_2),
\]

the image of $p$ under the representation $\pi$ is the following matrix:

\[
\begin{pmatrix}
1 & 0 & x_1 & x_2 & -y_1 & -y_2 & 2z_1 \\
0 & 1 & -x_2 & x_1 & y_1 & y_2 & 2z_2 \\
0 & 0 & 1 & 0 & 0 & 0 & y_1 \\
0 & 0 & 0 & 1 & 0 & 0 & y_2 \\
0 & 0 & 0 & 0 & 1 & 0 & x_1 \\
0 & 0 & 0 & 0 & 0 & 1 & x_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Next, referring to Proposition 3, the Plancherel measure is supported on the manifold

\[
\Sigma = \left\{ \lambda \in \mathfrak{n}^* : \lambda (Z_1)^2 + \lambda (Z_2)^2 \neq 0, \lambda (Y_j) = 0, \lambda (X_i) = 0 \text{ for } 1 \leq i \leq 3 \right\}
\]

and the Plancherel measure is $|\lambda_1^2 + \lambda_2^2| d\lambda_1 d\lambda_2$ where $\lambda_k = \lambda (Z_k)$.

Example 6. Let $N$ be a nilpotent Lie group with Lie algebra $\mathfrak{n}$ spanned by the following fixed basis:

\[
\{Z_1, Z_2, Y_1, Y_2, Y_3, X_1, X_2, X_3\}
\]

with non trivial Lie brackets.

\[
[X_1, Y_1] = [X_3, Y_1] = Z_1 \\
[X_3, Y_2] = [X_3, Y_3] = Z_1, \\
[X_2, Y_1] = [X_2, Y_2] = Z_2.
\]
Since
\[
\begin{vmatrix}
Z_1 & 0 & 0 \\
Z_2 & Z_2 & 0 \\
Z_1 & Z_1 & Z_1
\end{vmatrix} = Z_1^2 Z_2,
\]
referring to the non-trivial brackets given above, it is not hard to see that \( N \) belongs to the class of groups considered here. Applying Proposition 2, there exists a monomorphism \( \pi : N \rightarrow GL_9(\mathbb{R}) \) such that for
\[
p = \prod_{k=1}^{2} \exp (z_k Z_k) \prod_{k=1}^{3} \exp (y_k Y_k) \prod_{k=1}^{3} \exp (x_k X_k)
\]
\( \pi (p) \) is equal to
\[
\begin{bmatrix}
1 & 0 & x_1 + x_3 & x_3 & x_3 & -y_1 & 0 & -(y_1 + y_2) & 2z_1 \\
0 & 1 & x_2 & x_2 & 0 & 0 & 0 & -(y_1 + y_2) & 0 & 2z_2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & y_1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & y_2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & y_3 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & x_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & x_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Finally, the Plancherel measure is supported on the manifold
\[
\Sigma = \left\{ \lambda \in \mathfrak{n}^* : \lambda (Z_1)^2 \lambda (Z_2) \neq 0, \lambda (Y_j) = 0, \lambda (X_j) = 0 \text{ for } 1 \leq j \leq 3 \right\}
\]
and is equal to \( \lambda_1^2 |\lambda_2| d \lambda_1 d \lambda_2 \).

The following example exhausts all elements in the class of groups considered in this paper.

**Example 7.** Fix two natural numbers \( n \) and \( d \), such that \( n - 2d > 0 \). Let \( M \) be a matrix of order \( d \) with entries in \( \mathbb{R}Z_1 \oplus \cdots \oplus \mathbb{R}Z_{n-2d} \) such that \( \det (M) \) is a non-vanishing homogeneous polynomial on \( \mathbb{R}Z_1 \oplus \cdots \oplus \mathbb{R}Z_{n-2d} \). Now let \( a = \mathbb{R}\text{-span} \{X_1, \cdots, X_d\} \) and \( b = \mathbb{R}\text{-span} \{Y_1, \cdots, Y_d\} \) such that \( [X_i, Y_j] = M_{ij} \) and \( M_{ij} \) is the entry of \( M \) located at the intersection of the \( i \)-th row and \( j \)-th column. The Lie algebra
\[
n = a \oplus b \oplus (\mathbb{R}Z_1 \oplus \cdots \oplus \mathbb{R}Z_{n-2d})
\]
satisfies all properties given in Condition 1.

Now, we define \( \Gamma_b = \exp (Z Y_1 + \cdots + Z Y_d), \Gamma_a = \exp (Z X_1 + \cdots + Z X_d), \Gamma_3 = \exp (Z Z_1 + \cdots + Z Z_{n-2d}) \)
\( \Gamma = \Gamma^z \Gamma^b \Gamma^a \subset N. \)

Then \( \Gamma \) is a discrete subset of \( N \). In fact, \( \Gamma \) is generally not a subgroup of \( N \). Here is a simple example. Let \( N \) be a nilpotent Lie group with Lie algebra \( \mathfrak{n} \) spanned by the strong Malcev basis \( Z_1, Z_2, Y_1, Y_2, X_1, X_2 \) such that we have for non-trivial Lie brackets

\[
[X_1, Y_1] = Z_1, [X_2, Y_1] = -Z_2, [X_1, Y_2] = \frac{1}{3}Z_2, [X_2, Y_2] = Z_1.
\]

Then

\[
\exp X_1 \exp Y_2 \exp (-X_1) = \exp \left( Y_2 \exp \left( \frac{1}{3}Z_2 \right) \right) \notin \Gamma.
\]

Thus, \( \Gamma \) does not have a group structure.

3. Overview of the Main Results

In this section, we will present a short overview of our main results. In order to do so, we will need a few important definitions.

**Definition 8.** We say a function \( f \in L^2(N) \) is bandlimited if its Plancherel transform is supported on a bounded measurable subset of \( \Sigma \). Fix a measurable field of unit vectors \( e = \{ e_\lambda \}_{\lambda \in E} \) where \( e_\lambda \in L^2(\mathbb{R}^d) \). We say a Hilbert space is a multiplicity-free left-invariant subspace of \( L^2(N) \) if

\[
\mathcal{H}(e) = \mathcal{P}^{-1} \left( \int_{\Sigma} \mathbb{R}^d \otimes e_\lambda \, d\mu(\lambda) \right).
\]

We observe here that the Hilbert space \( \mathcal{P}(\mathcal{H}(e)) \) is naturally identified with \( L^2(\Sigma \times \mathbb{R}^d) \). Let us now define the set

\[
E = \{ \lambda \in z^*: |\det B(\lambda)| \neq 0, \text{ and } |\det B(\lambda)| \leq 1 \}.
\]

It is easy to see that \( E \) is the intersection of a Zariski open subset of \( z^* \) and a closed subset of \( z^* \). Also, \( E \) is not bounded in general and \( E \) is necessarily a set of positive Lebesgue measure on \( z^* \). In order to develop a theory of bandlimitation, we will need to consider some bounded subset of \( E \).

For any given bounded set \( A \subset \Sigma \), we define the corresponding **multiplicity-free, bandlimited**, left-invariant Hilbert subspace \( \mathcal{H}(e, A) \) as follows

\[
\mathcal{H}(e, A) = \mathcal{P}^{-1} \left( \int_{A} \mathbb{R}^d \otimes e_\lambda \, |\det B(\lambda)| \, d\lambda \right).
\]

Let \( \phi \in \mathcal{H}(e, A) \) and define the linear map \( W_\phi : \mathcal{H}(e, A) \rightarrow L^2(N) \), such that \( W_\phi \psi(x) = \langle \psi, L(x) \phi \rangle \). It is easy to see that the space \( W_\phi(\mathcal{H}(e, A)) \) is a subspace of \( L^2(N) \) which consists of continuous functions.
Let \( C \subset \mathbb{Z}^n \) be a bounded set such that
\[
\left\{ e^{2\pi i (k, \lambda)} \chi_C(\lambda) : k \in \mathbb{Z}^n \right\}
\] is a Parseval frame for \( L^2(C, d\lambda) \). For example, it suffices to pick \( C \subset I \) such that the collection \( \{I + k : k \in \mathbb{Z}^n\} \) forms a measurable partition of \( \mathbb{R}^n = \mathbb{Z}^n \) and \( I \) is bounded. Recalling that \( d\mu(\lambda) = |\det B(\lambda)| d\lambda \), our main result is summarized as follows.

**Theorem 9.** Let \( \Lambda \) be a connected, simply connected nilpotent Lie group satisfying Condition 1.

1. There exists \( \phi \in H(e, E \cap C) \) such that \( W_\phi(H(e, E \cap C)) \) is a \( \Gamma \)-sampling subspace of \( L^2(\Lambda) \).
2. In general, \( W_\phi(H(e, E \cap C)) \) does not have the interpolation property. However if \( W_\phi(H(e, A)) \) is a \( \Gamma \)-sampling space and \( \mu(A) = 1 \) then \( W_\phi(H(e, A)) \) has the interpolation property.
3. There exist precise sufficient conditions related to the structure constants of the Lie algebra \( \mathfrak{n} \) for which \( W_\phi(H(e, A)) \) is a \( \Gamma \)-sampling space which has the interpolation property.

The proof of Theorem 9 will be given in the last section of this paper.

### 4. Results on Frames and Orthonormal Bases

We will need to be familiar with the theory of frames (see [1], [9] and [4]). Given a countable sequence \( \{f_i\}_{i \in I} \) of vectors in a Hilbert space \( H \), we say \( \{f_i\}_{i \in I} \) forms a **frame** if and only if there exist strictly positive real numbers \( A, B \) such that for any vector \( f \in H \),
\[
A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B \|f\|^2.
\]
In the case where \( A = B \), the sequence of vectors \( \{f_i\}_{i \in I} \) forms what is called a **tight frame**, and if \( A = B = 1 \), \( \{f_i\}_{i \in I} \) is called a **Parseval frame** because it satisfies the Parseval equality:
\[
\sum_{i \in I} |\langle f, f_i \rangle|^2 = \|f\|^2 \text{ for all } f \in H.
\]
Also, if \( \{f_i\}_{i \in I} \) is a Parseval frame such that for all \( i \in I, \|f_i\| = 1 \) then \( \{f_i\}_{i \in I} \) is an orthonormal basis for \( H \).

Let \( \Lambda = A\mathbb{Z}^d \) for some matrix \( A \). We say \( \Lambda \) is a full-rank lattice if \( A \) is non-singular. We say a lattice is separable if \( \Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d \). A **fundamental domain** \( D \) for a lattice in \( \mathbb{R}^d \) is a measurable set which satisfies the following: \( (D + l) \cap (D + l') \neq \emptyset \) for distinct \( l, l' \) in \( \Lambda \), and
\[ \mathbb{R}^d = \bigcup_{l \in \Lambda} (D + l). \] We say \( D \) is a packing set for \( \Lambda \), if \((D + l) \cap (D + l')\) has Lebesgue measure zero for any \( l \neq l' \). Let \( \Lambda = A \mathbb{Z}^d \times B \mathbb{Z}^d \) be a full-rank lattice in \( \mathbb{R}^{2d} \) and \( f \in L^2(\mathbb{R}^d) \). The family of functions in \( L^2(\mathbb{R}^d) \),

\[ G(f, A \mathbb{Z}^d \times B \mathbb{Z}^d) = \left\{ e^{2\pi i (k,x)} f(x-n) : k \in B \mathbb{Z}^d, n \in A \mathbb{Z}^d \right\} \]

is called a Gabor system. Gabor frames are a particular type of frame whose elements are generated by time-frequency shifts of a single vector. A Gabor system which is a Parseval frame is called a Gabor Parseval frame.

Let \( r \) be a natural number. Let \( \Lambda = A \mathbb{Z}^r \) be a full-rank lattice in \( \mathbb{R}^r \). The volume of \( \Lambda \) is defined as

\[ \text{vol}(\Lambda) = |\det A| \]

and the density of the lattice \( \Lambda \) is defined as \( d(\Lambda) = |\det A|^{-1} \).

**Lemma 10. (Density Condition)** Let \( \Lambda = A \mathbb{Z}^d \times B \mathbb{Z}^d \) be a full-rank lattice in \( \mathbb{R}^{2d} \). There exists \( f \in L^2(\mathbb{R}^d) \) such that \( G(f, A \mathbb{Z}^d \times B \mathbb{Z}^d) \) is a Parseval frame in \( L^2(\mathbb{R}^d) \) if and only if \( \text{vol}(\Lambda) = |\det A \det B| \leq 1 \).

A proof of Lemma 10 is given in [4] Theorem 3.3.

**Lemma 11.** Let \( \Lambda \) be a full rank lattice in \( \mathbb{R}^{2d} \). There exists \( f \in L^2(\mathbb{R}^d) \) such that \( G(f, \Lambda) \) is an orthonormal basis if and only if \( \text{vol}(\Lambda) = 1 \). Also, if \( G(f, \Lambda) \) is a Parseval frame for \( L^2(\mathbb{R}^d) \), then \( \|f\|^2 = \text{vol}(\Lambda) \).

Lemma 11 is due to Theorem 1.3 and the proof of Lemma 3.2 in [4].

Next, from the definition of the irreducible representations of \( N \), provided in Proposition 3, it is easy to see that for \( f \in L^2(\mathbb{R}^d) \), \( \pi_\lambda (\Gamma_b \Gamma_a) f \) is a Gabor system for each fixed \( \lambda \in \Sigma \). Moreover, following the notation given in (6), we write:

\[ \pi_\lambda (\Gamma_b \Gamma_a) f = G(f, \mathbb{Z}^d \times B(\lambda) \mathbb{Z}^d). \]

We recall that the set \( C \) satisfies the following conditions: \( C \subset \mathfrak{z}^* \) is a bounded set such that the system

\[ \left\{ e^{2\pi i (k,\lambda)} \chi_C(\lambda) : k \in \mathbb{Z}^{n-2d} \right\} \]

is a Parseval frame for \( L^2(C, d\lambda) \). Also, we defined the set \( E \) such that

\[ E = \{ \lambda \in \mathfrak{z}^* : |\det B(\lambda)| \neq 0, \text{ and } |\det B(\lambda)| \leq 1 \}. \]

Therefore,

\[ \left\{ e^{2\pi i (k,\lambda)} \chi_{E \cap C}(\lambda) : k \in \mathbb{Z}^{n-2d} \right\} \]

is a Parseval frame for the Hilbert space \( L^2(E \cap C, d\lambda) \).

Lemma 12. There exists a function \( \phi \in H(e, E \cap C) \) such that \( L(\Gamma)\phi \) is a Parseval frame in \( H(e, E \cap C) \).

Proof. We know that by the Density Condition (see Lemma 10), for \( \lambda \in E \cap C \), there exists a rank-one operator \( T_{\lambda} = u_{\lambda} \otimes e_{\lambda} \), such that

\[
\mathcal{P}\phi(\lambda) = \frac{T_{\lambda}}{\sqrt{|\det B(\lambda)|}},
\]

and the system \( G(u_{\lambda}, Z^d \times B(\lambda) \mathbb{Z}^d) \) is a Gabor Parseval frame in \( L^2(\mathbb{R}^d) \).

Next, given any vector \( \psi \in H(e, E \cap C) \), we obtain that

\[
\sum_{\gamma \in \Gamma} \left| \langle \psi, L(\gamma)\phi \rangle_{H(e, E \cap C)} \right|^2 = \sum_{\gamma \in \Gamma} \left| \int_{E \cap C} \langle \mathcal{P}\psi(\lambda), \pi_{\lambda}(\gamma) \mathcal{P}\phi(\lambda) \rangle_{HS} d\mu(\lambda) \right|^2
\]

\[
= \sum_{\gamma \in \Gamma} \left| \int_{E \cap C} \langle \mathcal{P}\psi(\lambda), \pi_{\lambda}(\gamma) \left(|\det B(\lambda)|^{-1/2} u_{\lambda} \otimes e_{\lambda}\right) \rangle_{HS} |\det B(\lambda)| d\lambda \right|^2
\]

\[
= \sum_{\gamma \in \Gamma} \left| \int_{E \cap C} \langle \mathcal{P}\psi(\lambda), \pi_{\lambda}(\gamma) u_{\lambda} \otimes e_{\lambda} \rangle_{HS} |\det B(\lambda)|^{1/2} d\lambda \right|^2.
\]

Using the fact that

\[
\left\{ e^{2\pi i(k,\lambda)} \chi_{E \cap C}(\lambda) : k \in \mathbb{Z}^{n-2d} \right\}
\]

is a Parseval frame for \( L^2(E \cap C, d\lambda) \), and letting

\[
f(\lambda) = \langle \mathcal{P}\psi(\lambda), \pi_{\lambda}(\gamma_1) u_{\lambda} \otimes e_{\lambda} \rangle_{HS} |\det B(\lambda)|^{1/2},
\]

we obtain

\[
\sum_{\gamma \in \Gamma} \left| \langle \psi, L(\gamma)\phi \rangle_{H(e, E \cap C)} \right|^2 = \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \sum_{m \in \mathbb{Z}^{n-2d}} \left| \int_{E \cap C} e^{-2\pi i(\lambda, m)} f(\lambda) \right|^2 d\lambda
\]

\[
= \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \sum_{m \in \mathbb{Z}^{n-2d}} \left| \hat{f}(m) \right|^2 d\lambda
\]

\[
= \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \left\| \hat{f} \right\|^2_{L^2(\mathbb{Z}^{n-2d})}
\]

\[
= \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \left\| f \right\|^2_{L^2(E \cap C, d\lambda)}.
\]
The last equality above is due to the Plancherel Theorem on $L^2(\mathbb{Z}^{n-2d})$.

Using Equation (8), letting $P\psi(\lambda) = w_\lambda^{\psi} \otimes e_\lambda$, where $w_\lambda^{\psi} = w_\lambda \in L^2(\mathbb{R}^d)$, and coming back to (7), it follows that:

$$
\sum_{\gamma \in \Gamma} \left| \langle \psi, L(\gamma) \phi \rangle_{H(e,E \cap C)} \right|^2
= \sum_{\gamma \in \Gamma} \left| \int_{E \cap C} \langle P\psi(\lambda), \pi_{\Lambda}(\gamma) u_\lambda \otimes e_\lambda \rangle_{H^S} \det B(\lambda) \right|^2 \frac{1}{d\lambda}
= \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \int_{E \cap C} \left| \langle w_\lambda \otimes e_\lambda, \pi_{\Lambda}(\gamma_1) u_\lambda \otimes e_\lambda \rangle_{H^S} \right|^2 \det B(\lambda) \frac{1}{d\lambda}
= \int_{E \cap C} \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \left| \langle w_\lambda \otimes e_\lambda, \pi_{\Lambda}(\gamma_1) u_\lambda \otimes e_\lambda \rangle_{H^S} \right|^2 \det B(\lambda) \frac{1}{d\lambda}
= \int_{E \cap C} \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \left| \langle w_\lambda, \pi_{\Lambda}(\gamma_1) u_\lambda \rangle_{L^2(\mathbb{R}^d)} \right|^2 \det B(\lambda) \frac{1}{d\lambda}.
$$

Since $\pi_{\Lambda}(\Gamma_b \Gamma_a) u_\lambda$ is a Parseval Gabor frame for each fixed $\lambda \in E \cap C$,

$$
\sum_{\gamma_1 \in \Gamma_b \Gamma_a} \left| \langle w_\lambda, \pi_{\Lambda}(\gamma_1) u_\lambda \rangle_{L^2(\mathbb{R}^d)} \right|^2 = \|w_\lambda\|^2_{L^2(\mathbb{R}^d)}
$$

then

$$
\sum_{\gamma \in \Gamma} \left| \langle \psi, L(\gamma) \phi \rangle_{H(e,E \cap C)} \right|^2 = \int_{E \cap C} \|w_\lambda\|^2_{L^2(\mathbb{R}^d)} \det B(\lambda) \frac{1}{d\lambda}
= \int_{E \cap C} \|P\psi(\lambda)\|^2_{H^S} \det B(\lambda) \frac{1}{d\lambda}
= \|\psi\|^2_{H(e,E \cap C)}.
$$

Finally, we obtain that $L(\Gamma) \phi$ is a Parseval frame in $H(e,E \cap C)$.

**Lemma 13.** If $L(\Gamma) \phi$ is a Parseval frame in $H(e,E \cap C)$ as described in Lemma 12 and if

$$
\int_{E \cap C} \det B(\lambda) \frac{1}{d\lambda} = 1
$$

then $L(\Gamma) \phi$ is an orthonormal basis.

**Proof.** Recall from Lemma 12 that for $\lambda \in E \cap C$,

$$
P\phi(\lambda) = |\det B(\lambda)|^{-1/2} u_\lambda \otimes e_\lambda,
$$

such that $G(u_\lambda, \mathbb{Z}^d \times B(\lambda) \mathbb{Z}^d)$ is a Gabor Parseval frame in $L^2(\mathbb{R}^d)$. Referring to the proof of Theorem 1.3 in [4], $\|u_\lambda\|^2_{L^2(\mathbb{R}^d)} = |\det B(\lambda)|$ for
\(\lambda \in E \cap C\). Now,
\[
\|\phi\|^2_{H(e, E \cap C)} = \int_{E \cap C} \|P\phi(\lambda)\|^2_{H^S} |\text{det} B(\lambda)| \, d\lambda \\
= \int_{E \cap C} \left\| |\text{det} B(\lambda)|^{-1/2} u\lambda \otimes e\lambda\right\|^2_{H^S} |\text{det} B(\lambda)| \, d\lambda \\
= \int_{E \cap C} \left\| u\lambda \otimes e\lambda\right\|^2_{H^S} \, d\lambda \\
= \int_{E \cap C} \left\| u\lambda\right\|^2_{L^2(\mathbb{R}^d)} \, d\lambda \\
= \int_{E \cap C} |\text{det} B(\lambda)| \, d\lambda \\
= \mu(E \cap C) = 1.
\]

Since any unit-norm Parseval frame is an orthonormal basis, the proof is completed. \(\square\)

**Remark 14.** Theorem 3.3 in [9] guarantees that for each \(\lambda \in E \cap C\), it is possible to pick
\[
u\lambda = |\text{det} B(\lambda)|^{1/2} \chi_{E(\lambda)}
\]
such that \(E(\lambda)\) tiles \(\mathbb{R}^d\) by \(\mathbb{Z}^d\) and packs \(\mathbb{R}^d\) by \(B(\lambda)^{-1} \mathbb{Z}^d\) and \(G(u\lambda, \mathbb{Z}^d \times B(\lambda) \mathbb{Z}^d)\) is a Gabor Parseval frame in \(L^2(\mathbb{R}^d)\).

In light of Lemma 13, the conditions given below provide sufficient conditions for the existence of an orthonormal basis of the type \(L(\Gamma)\).\(\phi\).

**Condition 15.** There exists \(\phi \in H(e, E \cap C)\) such that \(L(\Gamma)\phi\) is a Parseval frame in \(H(e, E \cap C)\).

**Condition 16.** \(\int_{E \cap C} |\text{det} B(\lambda)| \, d\lambda = 1\).

5. **Proof of Main Results and Examples**

In this section, we will provide a proof for Theorem 9 which is the main theorem of the paper. We will also present some clear sufficient conditions for the existence of sampling subspaces which possess the interpolation property. Recall that
\[n = a \oplus b \oplus \mathcal{g},\]
\([a, b] \subseteq \mathcal{g},\ a, b\) are commutative algebras, \(\dim_{\mathbb{R}}(a) = \dim_{\mathbb{R}}(b) = d\), and
\[\det \left( \left[ [X_{ij}, Y_{ij}] \right]_{1 \leq i, j \leq d} \right)\]
is a non-vanishing polynomial defined over the commutator ideal \([n,n]\).

Also, we recall that

\[
B(\lambda) = \begin{bmatrix}
\lambda [X_1, Y_1] & \cdots & \lambda [X_1, Y_d] \\
\vdots & \ddots & \vdots \\
\lambda [X_d, Y_1] & \cdots & \lambda [X_d, Y_d]
\end{bmatrix}
\]

and

\[
d\mu(\lambda) = |\det B(\lambda)| \, d\lambda.
\]

Moreover, the unitary dual of \(N\) is parametrized by the smooth manifold

\[
\Sigma = \left\{ \lambda \in \mathfrak{n}^* : \det (B(\lambda)) \neq 0, \lambda (X_1) = \cdots = \lambda (X_d) = \lambda (Y_1) = \cdots = \lambda (Y_d) = 0 \right\}
\]

which is naturally identified with a Zariski open subset of \(z^*\).

**Lemma 17.** The basis elements of \(n\) can be rescaled so that \(\mu(E) \geq 1\).

**Proof.** Let \(E = \{ \lambda \in \Sigma : |\det B(\lambda)| \leq 1 \}\). If \(\mu(E) \geq 1\) then we are done. So let us suppose that

\[
\mu(E) = \int_E |\det B(\lambda)| \, d\lambda < 1.
\]

Pick \(\alpha > 1\). Put \(B(\lambda) = (\lambda [X_i, Y_j])_{1 \leq i,j \leq d}\) and \(B'(\lambda) = (\lambda [\alpha X_i, Y_j])_{1 \leq i,j \leq d}\). It follows that \(\det B'(\lambda) = \alpha \det B(\lambda)\). Now let

\[
E' = \{ \lambda \in \Sigma : |\det B'(\lambda)| \leq 1 \}.
\]

Notice that because \(\alpha > 1\), \(E' \subseteq E\) and \(\int_{E'} |\det B(\lambda)| \, d\lambda\) is finite since

\[
\int_{E'} |\det B(\lambda)| \, d\lambda \leq \int_E |\det B(\lambda)| \, d\lambda.
\]

Put \(d\mu'(\lambda) = |\det B'(\lambda)| \, d\lambda\) and pick \(\alpha\) large enough so that

\[
\mu'(E') = \alpha \times \int_{E'} |\det B(\lambda)| \, d\lambda \geq 1.
\]

Our new basis is then \(\{ Z_1, \cdots, Z_{n-2d}, Y_1, \cdots, Y_d, \alpha X_1, \cdots, \alpha X_d \}\). \(\square\)

From now on, we will assume that a choice of a basis has been made so that the statement \(\mu(E) \geq 1\) holds. Thus, there exists a compact subset \(C\) of \(z^*\) such that \(\mu(E \cap C) = 1\). Put

\[
H(e, E \cap C) = \mathcal{P}^{-1} \left( \int_{E \cap C} L^2 \left( \mathbb{R}^d \right) \otimes e_\lambda |\det B(\lambda)| \, d\lambda \right).
\]

Let \(\iota : \mathbb{R}^{n-2d} \to z^*\) be a map defined by

\[
\iota(\lambda_1, \cdots, \lambda_{n-2d}) = (\lambda_1, \cdots, \lambda_{n-2d}, 0, \cdots, 0).
\]
Then $\iota$ is a measurable bijection. Identifying $\mathbb{R}^{n-2d}$ with $\mathfrak{z}^*$ via the map $\iota$, we slightly abuse the notations when we say $\mathfrak{z}^* = \mathbb{R}^{n-2d}$. In order to make the presentation simpler, we will adopt this abuse of notation for the remainder of the paper.

The example below suggests that in general it is quite difficult to construct an orthonormal basis of the type $L(\Gamma) \phi$.

**Example 18.** Let $N$ be a Lie group with Lie algebra spanned by the vectors $Z_1, Z_2, Y_1, Y_2, X_1, X_2$

such that

$$[X_1, Y_1] = Z_1, [X_1, Y_2] = -Z_2, [X_2, Y_1] = Z_2, [X_2, Y_2] = Z_1.$$ 

Here $E$ is the punctured disk in $\mathfrak{z}^*$ centered around zero. Choosing $C$ to be the disk centered at the origin of $\mathfrak{z}^*$ with radius $\sqrt{2/\pi}$ then

$$\int_C \left( \lambda_1^2 + \lambda_2^2 \right) d\lambda_1 d\lambda_2 = 1.$$ 

However, the area of $C$ being equal to $\sqrt{2\pi}$, it is clear that the system

$$\left\{ e^{2\pi i \langle k, \lambda \rangle} \chi_{E \cap C}(\lambda) : k \in \mathbb{Z}^2 \right\}$$

is not Parseval frame for $L^2(E \cap C, d\lambda)$. Thus, Lemma 12 cannot be applied.

**Proposition 19.** Let $C$ be given such that Condition 16 holds. Then the set $E \cap C$ cannot be contained in a fundamental domain of the lattice $\mathbb{Z}^{n-2d}$.

**Proof.** If Condition 16 holds then

$$\mu(E \cap C) = \int_{E \cap C} |\det B(\lambda)| d\lambda = 1.$$ 

However, the function $\lambda \mapsto |\det B(\lambda)|$ is a non-constant continuous function which is bounded above by 1 on $E \cap C$. Therefore,

$$1 = \int_{E \cap C} |\det B(\lambda)| d\lambda < \int_{E \cap C} d\lambda = m(E \cap C)$$

where $m$ is the Lebesgue measure on $\Sigma$. By contradiction, let us assume that $E \cap C$ is contained in a fundamental domain of a lattice $\mathbb{Z}^{n-2d}$. Then

$$1 < m(E \cap C) \leq 1$$

and we reach a contradiction. \qed

From now on, put

$$E^\circ = E \cap C.$$  

**Lemma 20.** There exists a finite partition of $E^\circ$ denoted $P$ such that

(9)
(1) \( E^o = \bigcup_{A \in P} A_j \) and
\[
H(e, E^o) = \bigoplus_{j=1}^{\text{card}(P)} H(e, A_j).
\]

(2) For each \( j \) where \( 1 \leq j \leq \text{card}(P) \), \( A_j \) is contained in a fundamental domain for \( \mathbb{Z}^{n-2d} \).

(3) For each \( j \) where \( 1 \leq j \leq \text{card}(P) \), there exists a Parseval frame of the type \( L(\Gamma) \phi_j \) for the Hilbert space
\[
H(e, A_j) = P^{-1} \left( \int_{A_j} L^2(R^d) \otimes e_\lambda |\det B(\lambda)| d\lambda \right).
\]

Proof. Parts 1, 2 are obviously true. For the proof for Part 3, we observe that if \( A_j \) is contained in a fundamental domain of \( \mathbb{Z}^{n-2d} \) then
\[
\left\{ e^{2\pi i(k,\lambda)} \chi_{A_j}(\lambda) : k \in \mathbb{Z}^{n-2d} \right\}
\]
is a Parseval frame for the Hilbert space \( L^2(A_j, d\lambda) \). Thus Lemma 12 gives us Part 3.

Lemma 21. For each \( 1 \leq j \leq \text{card}(P) \), we can construct a Parseval frame of the type \( L(\Gamma) \phi_j \), such that
\[
\left\| \sum_{j=1}^{\text{card}(P)} \phi_j \right\|_{H(e, E^o)}^2 = 1.
\]

Proof. The construction of a Parseval frame for each \( H(e, A_j) \), \( 1 \leq j \leq \text{card}(P) \) of the type \( L(\Gamma) \phi_j \) is given in Lemma 12, and
\[
\left\| \sum_{j=1}^{\text{card}(P)} \phi_j \right\|^2_{H(e, E^o)} = \sum_{j=1}^{\text{card}(P)} \left\| \phi_j \right\|^2_{H(e, A_j)}
\]
\[
= \int_{j=1}^{\text{card}(P)} |\det B(\lambda)| d\lambda
\]
\[
= \int_{E^o} |\det B(\lambda)| d\lambda
\]
\[
= 1.
\]
Lemma 22. Let $\phi = \sum_{j=1}^{\text{card}(P)} \phi_j$ such that for each $1 \leq j \leq \text{card}(P)$, $L(\Gamma) \phi_j$ is a Parseval frame for $H(e,A_j)$ and $\|\phi\|^2_{H(e,E^o)} = 1$. If $L(\Gamma)(\phi)$ is a Parseval frame then $L(\Gamma)\phi$ is an orthonormal basis for $H(e,E^o)$.

Proof. If $L(\Gamma)\phi$ is a Parseval frame for $H(e,E^o)$ then it must be an orthonormal basis since $\|\phi\|^2_{H(e,E\cap C)} = 1$.

We would like to remark that in general the direct sum of Parseval frames is not a Parseval frame. Let us consider the following finite collection of subsets of $\mathbb{R}^{n-2d}$:

1. $\Lambda$ is a fundamental domain for $\mathbb{Z}^{n-2d}$.
2. $E^o \subset \bigcup_{\kappa_j \in S} (\Lambda - \kappa_j)$.
3. For $\kappa_j, \kappa_j' \in S$, $\kappa_j \neq \kappa_j'$, the set $(\Lambda - \kappa_j) \cap (\Lambda - \kappa_j')$ is a null set with respect to the Lebesgue measure on $\mathbb{R}^{n-2d}$.
4. $E^o \cap (\Lambda - \kappa_j)$ is a set of positive Lebesgue measure for all $\kappa_j \in S$.

Now, we pick a partition $P$ of $E^o$ as described above. That is, for each $j, 1 \leq j \leq \text{card}(P) = \text{card}(S)$, there exists a unique $\kappa_j \in S$ where $A_j = E^o \cap (\Lambda - \kappa_j)$ and

$$\bigcup_{j=1}^{\text{card}(P)} A_j = E^o.$$

Example 23. Let $N$ be a Lie group with Lie algebra spanned by the vectors $Z_1, Z_2, Y_1, Y_2, X_1, X_2$ with the following non-trivial Lie brackets

$$[X_1, Y_1] = Z_1, [X_1, Y_2] = Z_2,$$

$$[X_2, Y_1] = Z_2, [X_2, Y_2] = Z_1.$$ 

We construct $E^o$ as follows:

$$E^o = \left\{ (\lambda_1, \lambda_2)^2 \in \left[ -\frac{3^{1/4}}{\sqrt{2}}, \frac{3^{1/4}}{\sqrt{2}} \right]^2 : \lambda_1^2 - \lambda_2^2 \neq 0, \left| \lambda_1^2 - \lambda_2^2 \right| \leq 1 \right\}.$$ 

Following the description above, let $\Lambda = [0,1)^2$ and define

$$A_1 = \Lambda \cap E^o, A_2 = (\Lambda - (1,0)) \cap E^o,$$

$$A_3 = (\Lambda - (1,1)) \cap E^o, \text{ and } A_4 = (\Lambda - (0,1)) \cap E^o$$

as shown in the figure below. $E$ is the set in black, and $E^o$ is partitioned into four disjoint squares satisfying the conditions given above.
Proposition 24. For each \(1 \leq j \leq \text{card}(P)\), there exists \(\phi_j \in H(e, A_j)\) such that the following holds.

1. \(L(\Gamma) \phi_j\) is a Parseval frame for \(H(e, A_j)\)
2. \(P\phi_j(\lambda) = (u_j^i \otimes e_\lambda) |\det B(\lambda)|^{-1/2}\) with \(u^i_\lambda = |\det B(\lambda)|^{1/2} \chi_{E(\lambda)} \in L^2(\mathbb{R}^d)\).

Moreover, \(E(\lambda)\) is a \(\mathbb{Z}^d\)-tiling set, and a \(B(\lambda)^{-tr} \mathbb{Z}^d\)-packing set for \(\mathbb{R}^d\).

Proof. See Lemma 13 and Remark 14.

Let us now define \(\phi = \phi_1 + \cdots + \phi_{\text{card}(P)}\) such that each \(\phi_j\) is as described in Proposition 24. Clearly

\[(10) \quad P\phi(\lambda) = |\det B(\lambda)|^{-1/2} u_\lambda \otimes e_\lambda \quad \text{and} \quad \]
\[(11) \quad u^i_\lambda = u^i_\lambda = |\det B(\lambda)|^{1/2} \chi_{E(\lambda)} \text{ for } \lambda \in A_j.\]

Theorem 25. If for each \(1 \leq j, j' \leq \text{card}(P), j \neq j'\), and for arbitrary functions \(f, g \in L^2(\mathbb{R}^d)\), \(\kappa_j, \kappa_{j'} \in S\)

\[ \left( \left\langle f, \pi_{\lambda - \kappa_j}(\gamma_1) u_{\lambda - \kappa_j} \right\rangle \right)_{\gamma_1 \in \Gamma_b \Gamma_a} \perp \left( \left\langle g, \pi_{\lambda - \kappa_{j'}}(\gamma_1) u_{\lambda - \kappa_{j'}} \right\rangle \right)_{\gamma_1 \in \Gamma_b \Gamma_a} \]

for \(\lambda \in \Lambda\) then \(L(\Gamma)(\phi)\) is an orthonormal basis for the Hilbert space \(H(e, E^c)\).
Proof. Let $\psi$ be any arbitrary element in

$$H(e, E^\circ) = \bigoplus_{j=1}^{\text{card}(P)} H(e, A_j)$$

such that $\psi = \sum_{j=1}^{\text{card}(P)} \psi_j$, for $\psi_j \in H(e, A_j)$. Let $r(\lambda) = |\det B(\lambda)|$. Then

$$\|\psi_j\|_{H(e, A_j)}^2 = \int_{A_j} \|P\psi_j(\sigma)\|_{H^S}^2 r(\sigma) d\sigma$$

where $\sigma = \lambda - \kappa_j, \lambda \in \Lambda, \kappa_j \in S$.

Since for each $\sigma \in A_j$ there exists a unique $\lambda \in \Lambda, \kappa_j \in S$ such that $\sigma = \lambda - \kappa_j$. Moreover, if $\psi_j \in H(e, A_j)$ then

$$\|\psi_j\|_{H(e, A_j)}^2 = \int_{\Lambda} \|P\psi_j(\lambda - \kappa_j)\|_{H^S}^2 r(\lambda - \kappa_j) d\lambda.$$

Let $P\psi_j(\lambda - \kappa_j) = w_{\lambda - \kappa_j} \otimes e_{\lambda - \kappa_j} \in L^2(\mathbb{R}^d) \otimes e_{\lambda - \kappa_j}$ for $\lambda \in \Lambda$. Then

$$\sum_{j=1}^{\text{card}(P)} \|\psi_j\|_{H(e, A_j)}^2$$

$$= \sum_{j=1}^{\text{card}(P)} \int_{\Lambda} \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \left| \left\langle w_{\lambda - \kappa_j}, \varpi_{\lambda - \kappa_j}(\gamma_1) u_{\lambda - \kappa_j} \right\rangle \right|^2 r(\lambda - \kappa_j) d\lambda$$

$$= \int_{\Lambda} \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \sum_{j=1}^{\text{card}(P)} \left| \left\langle w_{\lambda - \kappa_j}, \varpi_{\lambda - \kappa_j}(\gamma_1) u_{\lambda - \kappa_j} \right\rangle \right|^2 r(\lambda - \kappa_j) d\lambda.$$

We would like to be able to state that for $\lambda \in E^\circ$,

$$\sum_{\gamma_1 \in \Gamma_b \Gamma_a} \sum_{j=1}^{\text{card}(P)} \left| \left\langle w_{\lambda - \kappa_j}, \varpi_{\lambda - \kappa_j}(\gamma_1) u_{\lambda - \kappa_j} \right\rangle \right|^2$$

$$= \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \sum_{j=1}^{\text{card}(P)} \left| \left\langle w_{\lambda - \kappa_j}, \varpi_{\lambda - \kappa_j}(\gamma_1) u_{\lambda - \kappa_j} \right\rangle \right|^2.$$
Indeed, letting \((b_{\gamma_1}(\lambda))_{\gamma_1 \in \Gamma_b \Gamma_a} \in l^2(\Gamma_b \Gamma_a)\) such that \((b_{\gamma_1}(\lambda))_{\gamma_1 \in \Gamma_b \Gamma_a}\) is a sum of \(\text{card}(P)\)-many sequences of the type \(\left(b_{\gamma_1}^j(\lambda)\right)_{\gamma_1 \in \Gamma_b \Gamma_a}\) such that

\[
(b_{\gamma_1}(\lambda))_{\gamma_1 \in \Gamma_b \Gamma_a} = \sum_{j=1}^{\text{card}(P)} \left(\left<w_{\lambda-\kappa_j, \pi_{\lambda-\kappa_j}}(\gamma_1) u_{\lambda-\kappa_j}\right>\right)_{\gamma_1 \in \Gamma_b \Gamma_a}
\]

we compute the norm of the sequence \((b_{\gamma_1}(\lambda))_{\gamma_1 \in \Gamma_b \Gamma_a}\) in two different ways. First,

\[
\left\| (b_{\gamma_1}(\lambda))_{\gamma_1 \in \Gamma_b \Gamma_a}\right\|^2 = \sum_{\gamma_1 \in \Gamma_b \Gamma_a} |b_{\gamma_1}(\lambda)|^2 = \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \left(\sum_{j=1}^{\text{card}(P)} \left(\left<w_{\lambda-\kappa_j, \pi_{\lambda-\kappa_j}}(\gamma_1) u_{\lambda-\kappa_j}\right>\right)_{\gamma_1 \in \Gamma_b \Gamma_a}\right)^2.
\]

Second

\[
\left\| (b_{\gamma_1}(\lambda))_{\gamma_1 \in \Gamma_b \Gamma_a}\right\|^2 = \left\| \sum_{j=1}^{\text{card}(P)} \left(b_{\gamma_1}^j(\lambda)\right)_{\gamma_1 \in \Gamma_b \Gamma_a}\right\|^2 = \sum_{j=1}^{\text{card}(P)} \left\| (b_{\gamma_1}^j(\lambda))_{\gamma_1 \in \Gamma_b \Gamma_a}\right\|^2 = \sum_{j=1}^{\text{card}(P)} \left\| \left(\left<w_{\lambda-\kappa_j, \pi_{\lambda-\kappa_j}}(\gamma_1) u_{\lambda-\kappa_j}\right>\right)_{\gamma_1 \in \Gamma_b \Gamma_a}\right\|^2 = \sum_{j=1}^{\text{card}(P)} \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \left(\left<w_{\lambda-\kappa_j, \pi_{\lambda-\kappa_j}}(\gamma_1) u_{\lambda-\kappa_j}\right>\right)^2.
\]

The second equality above is due to the fact that we assume that for \(j \neq j'\),

\[
\left(\left<w_{\lambda-\kappa_j, \pi_{\lambda-\kappa_j}}(\gamma_1) u_{\lambda-\kappa_j}\right>\right)_{\gamma_1 \in \Gamma_b \Gamma_a} \perp \left(\left<w_{\lambda-\kappa_j', \pi_{\lambda-\kappa_j}}(\gamma_1) u_{\lambda-\kappa_j'}\right>\right)_{\gamma_1 \in \Gamma_b \Gamma_a} \quad \text{for } \lambda \in \Lambda.
\]
Thus, Equality 15 holds. Next, coming back to (14), we obtain

\[ \sum_{j=1}^{\text{card}(P)} \left\| \psi_j \right\|_{H(e, A_j)}^2 = \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \int_\Lambda |a_{\gamma_1}(\lambda)|^2 d\lambda \]

where

\[ a_{\gamma_1}(\lambda) = \sum_{j=1}^{\text{card}(P)} \left( \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \left\{ \prod_{\lambda - \kappa_j} (\gamma_1) \prod_{\lambda - \kappa_j} (\gamma_1) \prod_{\lambda - \kappa_j} (\gamma_1) \right\} \prod_{\lambda - \kappa_j} (\gamma_1) \right| \left\{ \prod_{\lambda - \kappa_j} (\gamma_1) \right\} \right). \]

Writing \( \lambda(m) = e^{2\pi i (\lambda, m)} \), it follows that

\[ \sum_{j=1}^{\text{card}(P)} \left\| \psi_j \right\|_{H(e, A_j)}^2 = \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \left\| a_{\gamma_1} \right\|_{L^2(\Lambda)}^2 \]

\[ = \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \left\| \tilde{a}_{\gamma_1} \right\|_{L^2(\mathbb{Z}^{n-2d})}^2 \]

\[ = \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \sum_{m \in \mathbb{Z}^{n-2d}} \left| \tilde{a}_{\gamma_1}(m) \right|^2 \]

\[ = \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \sum_{m \in \mathbb{Z}^{n-2d}} \left| \int_\Lambda a_{\gamma_1}(\lambda) \lambda(m) d\lambda \right|^2 \]

\[ = \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \sum_{m \in \mathbb{Z}^{n-2d}} \left| \sum_{j=1}^{\text{card}(P)} \int_\Lambda \left\langle w_{\lambda - \kappa_j}, qj, \gamma_1(\lambda) \right\rangle \lambda(m) d\lambda \right|^2. \]

Next, letting

\[ \lambda(m) = e^{2\pi i (\lambda, m)} = e^{-2\pi i (\lambda, m)}, \]
we obtain that
\[
\sum_{j=1}^{\text{card}(P)} \|\psi_j\|_{\mathcal{H}(e, A_j)}^2 = \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \sum_{m \in \mathbb{Z}^{d-2d}} \int_{\Lambda} \left| \sum_{j=1}^{\text{card}(P)} \left\langle \mathbf{w}_{\lambda - \kappa_j, \lambda}(m) \mathbf{q}_{j, \gamma_1}(\lambda) \right| d\lambda \right|^2
\]
\[
= \sum_{\gamma \in \Gamma} \left| \int_{E^0} \left\langle \mathcal{P} \psi(\sigma), \pi_{\sigma}(\gamma) |\sigma|^{-1/2} \mathbf{u}_{\sigma} \otimes \mathbf{e}_{\sigma} \right|_{\mathcal{H}^0} \right|^2 \sum_{\gamma \in \Gamma} \left| \int_{E^0} \left\langle \mathcal{P} \psi(\sigma), \pi_{\sigma}(\gamma) \mathcal{P} \phi(\sigma) \right|_{\mathcal{H}^0} |\sigma| d\sigma \right|^2
\]
\[
= \sum_{\gamma \in \Gamma} \left| \left\langle \psi, L(\gamma) \phi \right|_{\mathcal{H}(e, E^0)} \right|^2.
\]
Finally, we arrive to this fact:
\[
\|\psi\|_{\mathcal{H}(e, E^0)}^2 = \sum_{\gamma \in \Gamma} \left| \left\langle \psi, L(\gamma) \phi \right|_{\mathcal{H}(e, E^0)} \right|^2.
\]
Thus, \(L(\Gamma) \phi\) is a Parseval frame for \(\mathcal{H}(e, E^0)\). Now, we compute the norm of the vector \(\phi\). Since
\[
\mathcal{P} \phi(\lambda) = (\mathbf{u}_{\lambda} \otimes \mathbf{e}_{\lambda}) |\det B(\lambda)|^{-1/2}, \text{ and } \mathbf{u}_{\lambda} = |\det B(\lambda)|^{1/2} \chi_{E(\lambda)}
\]
then
\[
\mathcal{P} \phi(\lambda) = \left( |\det B(\lambda)|^{1/2} \chi_{E(\lambda)} \otimes \mathbf{e}_{\lambda} \right) |\det B(\lambda)|^{-1/2} = \chi_{E(\lambda)} \otimes \mathbf{e}_{\lambda}.
\]
Since \(E(\lambda)\) is a fundamental domain for \(\mathbb{Z}^d\), it follows that
\[
\|\phi\|_{\mathcal{H}(e, E \cap C)}^2 = 1.
\]
Finally, because \(L\) is a unitary operator, and using the fact that \(L(\Gamma) \phi\) forms a unit norm Parseval frame, then \(L(\Gamma) \phi\) forms an orthonormal basis in \(\mathcal{H}(e, E^0)\).

Now, we would like to present some simpler sufficient conditions for the statement
\[
\left( \left\langle f, \pi_{\lambda - \kappa_j}(\gamma_1) \mathbf{u}_{\lambda - \kappa_j} \right\rangle \right)_{\gamma_1 \in \Gamma_b \Gamma_a} \perp \left( \left\langle g, \pi_{\lambda - \kappa_j}(\gamma_1) \mathbf{u}_{\lambda - \kappa_j} \right\rangle \right)_{\gamma_1 \in \Gamma_b \Gamma_a}
\]
given in Proposition 25.
Proposition 26. For \( j \neq j', \lambda \in \Lambda \) and \( u_{\lambda - \kappa_j}, u_{\lambda - \kappa_{j'}} \in L^2(\mathbb{R}^d) \) as given in Theorem 25, if the following conditions are satisfied for any fixed \( m \in \mathbb{Z}^d \)

\[
\bigcup_{\kappa_s \in S} B(\lambda - \kappa_s)^{tr}(E(\lambda - \kappa_s) + m) \quad \text{is a fundamental domain of} \quad \mathbb{Z}^d
\]

and

\[
\left( B(\lambda - \kappa_j)^{tr}(E(\lambda - \kappa_j) + m) \right) \cap \left( B(\lambda - \kappa_{j'})^{tr}(E(\lambda - \kappa_{j'}) + m) \right)
\]

is a null set, then

\[
\left( \langle f, \pi_{\lambda - \kappa_j}(\gamma_1)u_{\lambda - \kappa_j} \rangle \right)_{\gamma_1 \in \Gamma_b \Gamma_a} \perp \left( \langle g, \pi_{\lambda - \kappa_{j'}}(\gamma_1)u_{\lambda - \kappa_{j'}} \rangle \right)_{\gamma_1 \in \Gamma_b \Gamma_a}
\]

for all \( f, g \in L^2(\mathbb{R}^d) \).

Proof. We compute the inner product of the sequences

\[
\left( \langle f, \pi_{\lambda - \kappa_j}(\gamma_1)u_{\lambda - \kappa_j} \rangle \right)_{\gamma_1 \in \Gamma_b \Gamma_a}, \left( \langle g, \pi_{\lambda - \kappa_{j'}}(\gamma_1)u_{\lambda - \kappa_{j'}} \rangle \right)_{\gamma_1 \in \Gamma_b \Gamma_a} \in l^2(\Gamma_b \Gamma_a)
\]

as follows:

\[
\sum_{\gamma_1 \in \Gamma_b \Gamma_a} \left( \langle f, \pi_{\lambda - \kappa_j}(\gamma_1)u_{\lambda - \kappa_j} \rangle \overline{\langle g, \pi_{\lambda - \kappa_{j'}}(\gamma_1)u_{\lambda - \kappa_{j'}} \rangle} \right).
\]

Moreover,

\[
\langle f, \pi_{\lambda - \kappa_j}(\gamma_1)u_{\lambda - \kappa_j} \rangle = \int_{\mathbb{R}} f(t) \overline{\pi_{\lambda - \kappa_j}(\gamma_1)u_{\lambda - \kappa_j}(t)} dt
\]

\[
= \int_{\mathbb{R}} f(t) e^{2\pi i \langle l, B(\lambda - \kappa_j)^{tr}t \rangle} u_{\lambda - \kappa_j}(t - m) dt.
\]

Put \( s = B(\lambda - \kappa_j)^{tr}t \). We recall that \( u_\lambda = |\det B(\lambda)|^{1/2} \chi_{E(\lambda)} \). So,

\[
\langle f, \pi_{\lambda - \kappa_j}(\gamma_1)u_{\lambda - \kappa_j} \rangle
\]

\[
= \int_{B(\lambda - \kappa_j)^{tr}(E(\lambda - \kappa_j) + m)} f(B(\lambda - \kappa_j)^{-tr}s) \left| \det B(\lambda - \kappa_j) \right|^{1/2} e^{-2\pi i \langle l, s \rangle} ds.
\]

Similarly,

\[
\langle g, \pi_{\lambda - \kappa_{j'}}(\gamma_1)u_{\lambda - \kappa_{j'}} \rangle
\]

\[
= \int_{B(\lambda - \kappa_{j'})^{tr}(E(\lambda - \kappa_{j'}) + m)} g(B(\lambda - \kappa_{j'})^{-tr}s) \left| \det B(\lambda - \kappa_{j'}) \right|^{1/2} e^{-2\pi i \langle l, s \rangle} ds.
\]
If for \( \lambda - \kappa_j \in A_j \),

\[
\bigcup_{\kappa_s \in S} B(\lambda - \kappa_s)^{tr} \left(E(\lambda - \kappa_s) + m\right)
\]

is a subset of a fundamental domain of \( \mathbb{Z}^d \) for distinct \( \kappa_j, \kappa_j' \in S \), and if

\[
B(\lambda - \kappa_j)^{tr} \left(E(\lambda - \kappa_j) + m\right) \cap B(\lambda - \kappa_j')^{tr} \left(E(\lambda - \kappa_j') + m\right)
\]

is a null set then

\[
\left( \langle f, \pi_{\lambda - \kappa_j} (\gamma_1) u_{\lambda - \kappa_j} \rangle \right)_{\gamma_1 \in \Gamma_b \Gamma_a} \perp \left( \langle g, \pi_{\lambda - \kappa_j'} (\gamma_1) u_{\lambda - \kappa_j'} \rangle \right)_{\gamma_1 \in \Gamma_b \Gamma_a};
\]

because

\[
\left( \langle f, \pi_{\lambda - \kappa_j} (\gamma_1) u_{\lambda - \kappa_j} \rangle \right)_{\gamma_1 \in \Gamma_b \Gamma_a}, \text{ and } \left( \langle g, \pi_{\lambda - \kappa_j'} (\gamma_1) u_{\lambda - \kappa_j'} \rangle \right)_{\gamma_1 \in \Gamma_b \Gamma_a}
\]

are Fourier inverses of the following orthogonal functions

(17)

\[
\theta_{f,\lambda,j,m} (s) = \chi_{B(\lambda - \kappa_j)^{tr} \left(E(\lambda - \kappa_j) + m\right)} (s) \frac{f \left( B (\lambda - \kappa_j)^{-tr} s \right)}{|\det B (\lambda - \kappa_j)|^{1/2}}, \text{ and}
\]

(18)

\[
\theta_{g,\lambda,j',m} (s) = \chi_{B(\lambda - \kappa_j')^{tr} \left(E(\lambda - \kappa_j') + m\right)} (s) \frac{g \left( B (\lambda - \kappa_j')^{-tr} s \right)}{|\det B (\lambda - \kappa_j')|^{1/2}} \text{ respectively.}
\]

In fact, we think of the functions above (17), (18) as being elements of \( L^2(I_{m,\lambda}) \) such that \( I_{m,\lambda} \) is a fundamental domain for \( \mathbb{Z}^d \). Combining the
observations made above, we obtain that for any \( f, g \in L^2(\mathbb{R}^d) \),
\[
\sum_{\gamma_1 \in \Gamma_b \Gamma_a} \langle f, \pi_{\lambda - \kappa_j}(\gamma_1) u_{\lambda - \kappa_j} \rangle \langle g, \pi_{\lambda - \kappa_j}(\gamma_1) u_{\lambda - \kappa_j} \rangle = \sum_{m \in \mathbb{Z}^d \setminus \mathbb{Z}^d} \sum_{l \in \mathbb{Z}^d} \left( \int_{I_{m,\lambda}} \theta_{f,\lambda,j,m}(s) e^{-2\pi i(l,s)} ds \right) \left( \int_{I_{m,\lambda}} \theta_{g,\lambda,j,m}(s) e^{-2\pi i(l,s)} ds \right)
\]
\[
= \sum_{m \in \mathbb{Z}^d \setminus \mathbb{Z}^d} \sum_{l \in \mathbb{Z}^d} \left( \frac{\theta_{f,\lambda,j,m}(l) \theta_{g,\lambda,j,m}(l)}{\theta_{f,\lambda,j,m}(l) \theta_{g,\lambda,j,m}(l)} \right) = 0
\]
\[
= 0.
\]

Thus
\[
\left( \langle f, \pi_{\lambda - \kappa_j}(\gamma_1) u_{\lambda - \kappa_j} \rangle \right)_{\gamma_1 \in \Gamma_b \Gamma_a} \perp \left( \langle g, \pi_{\lambda - \kappa_j}(\gamma_1) u_{\lambda - \kappa_j} \rangle \right)_{\gamma_1 \in \Gamma_b \Gamma_a}.
\]

This concludes the proof. \( \square \)

In light of Theorem 25 and 26, the following holds true.

**Corollary 27.** If \( \bigcup_{\kappa_s \in S} B(\lambda - \kappa_s)^{tr}(E(\lambda - \kappa_s) + m) \) is a subset of a fundamental domain of \( \mathbb{Z}^d \) and if
\[
\left( B(\lambda - \kappa_j)^{tr}(E(\lambda - \kappa_j) + m) \right) \cap \left( B(\lambda - \kappa_{j'})^{tr}(E(\lambda - \kappa_{j'}) + m) \right)
\]
is a null set for \( \lambda \in \Lambda \) for \( m \in \mathbb{Z}^d \) and for distinct \( \kappa_j, \kappa_{j'} \in S \) then \( L(\Gamma)(\phi) \) is an orthonormal basis for the Hilbert space \( H(e, E^c) \).

**Definition 28.** Let \( (\pi, H_\pi) \) denote a strongly continuous unitary representation of a locally compact group \( G \). We say that the representation \( (\pi, H_\pi) \) is admissible if and only if the map \( W_\phi : H \rightarrow L^2(G) \), \( W_\phi \phi = \langle \psi, \pi(x) \phi \rangle \) defines an isometry of \( H \) into \( L^2(G) \), and we say that \( \phi \) is an admissible vector or a continuous wavelet.

**Proposition 29.** Let \( \phi \) be an admissible vector for \( (\pi, H_\pi) \) such that \( \pi(\Gamma) \phi \) is a Parseval frame for \( H_\pi \). Then \( K = W_\phi(H_\pi) \) is a sampling space, and \( W_\phi(\phi) \) is the associated sinc-type function for \( K \).

See Proposition 2.54 in [6].
Proposition 30. If $\phi$ satisfies all the conditions given in Theorem 25, then
\[ \|P(\phi)(\lambda)\|_{HS} = 1 \]
for $\lambda \in E^\circ$ and $\phi$ is an admissible vector for the representation $(L, H(e, E^\circ))$.

Proof. For any given $\lambda \in E^\circ$,
\[ \|P(\phi)(\lambda)\|_{HS}^2 = \left\| u_\lambda \otimes e_\lambda |\det B(\lambda)|^{-1/2} \right\|_{HS}^2 \]
\[ = |\det B(\lambda)|^{-1} \|u_\lambda\|^2_{L^2(\mathbb{R}^d)} = 1. \]

Since $N$ is unimodular, and since $\mu(E^\circ) < \infty$ then from [6], $(L, H(e, E^\circ))$ is an admissible representation of $N$. Also, using results coming from the monograph [6] Page 127, $\phi$ is an admissible vector for the representation $(L, H(e, E^\circ))$. \qed

Finally, we arrive to our main result

Theorem 31. Let $N$ be a connected, simply connected nilpotent Lie group satisfying Condition 1. The following holds:

1. There exists $\phi \in H(e, E \cap C)$ such that $W_\phi(H(e, E \cap C))$ is a $\Gamma$-sampling subspace of $L^2(N)$.

2. In general, $W_\phi(H(e, E \cap C))$ does not have the interpolation property. However if $W_\phi(H(e, A))$ is a $\Gamma$-sampling space and $\mu(A) = 1$ then $W_\phi(H(e, A))$ has the interpolation property.

3. There exist precise sufficient conditions related to the structure constants of the Lie algebra $n$ for which $W_\phi(H(e, A))$ is a $\Gamma$-sampling space which has the interpolation property.

Proof. The proof follows directly from Proposition 29, Proposition 30 and Theorem 25. \qed

We will now conclude this paper with two examples.

Example 32. This example was first discovered by Currey and Mayeli in [2]. We will present it because it is the simplest example for the class of groups considered here. Let $N$ be the Heisenberg group with Lie algebra spanned by the vectors $Z, Y, X$ such that $[X, Y] = Z$, and
\[ E^\circ = \{ \lambda \in [-1, 1] : \lambda \neq 0 \}. \]

Define $\phi_1, \phi_2 \in H(e, E^\circ)$ such that
\[ P\phi_1(\lambda) = \chi_{(0,1]}(\lambda) \left( \chi_{[\frac{1}{2} - 1, \frac{1}{2}]}(t) \otimes \chi_{[\frac{1}{2} - 1, \frac{1}{2}]}(t) \right) \]
\[ P\phi_2(\lambda) = \chi_{[-1,0]}(\lambda) \left( \chi_{[-1,0]}(t) \otimes \chi_{[-1,0]}(t) \right). \]
Then for a fixed $\lambda \in (0,1]$ and $m \in \mathbb{Z}$,

$$
\left[ \lambda \left( \frac{1}{\lambda} - 1 + m \right), \lambda \left( \frac{1}{\lambda} + m \right) \right] \cup \left[ (\lambda - 1) m, (\lambda - 1) (-1 + m) \right]
$$

$$
= [m\lambda - \lambda + 1, m\lambda + 1] \cup [m\lambda - m, m\lambda - \lambda - m + 1]
$$

is a subset of a fundamental domain for $\mathbb{Z}$ since

$$
([m\lambda - m, m\lambda - \lambda - m + 1] + m) \cup [m\lambda - \lambda + 1, m\lambda + 1] = [m\lambda, m\lambda + 1]
$$

Next,

$$
[m\lambda - \lambda + 1, m\lambda + 1] \cap [m\lambda - m, m\lambda - \lambda - m + 1]
$$

is a null set. Applying Corollary 27 then $L(\Gamma) (\phi_1 + \phi_2)$ is an orthonormal basis in $\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)$ and $\mathcal{W}_{\phi_1 + \phi_2} (\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ))$ is a $\Gamma$-sampling space with the interpolation property.

**Example 33.** Let $N$ be a nilpotent Lie group with Lie algebra spanned by the vectors $Z_1, Z_2, Y_1, Y_2, X_1, X_2$ with the following non-trivial Lie brackets:

$$
[X_1, Y_1] = Z_1, [X_2, Y_2] = Z_2.
$$

In this example, the discrete set

$$
\Gamma = \exp(ZZ_1 + ZZ_2) \exp(ZY_1 + ZY_2) \exp(ZX_1 + ZX_2)
$$

is actually a uniform subgroup of the Lie group $N$. We define

$$
\mathbf{E}^\circ = \left\{ (\lambda_1, \lambda_2) \in [-1,1]^2 : \lambda_1 \lambda_2 \neq 0 \right\}.
$$

Let $\phi_1, \cdots, \phi_4 \in \mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)$ and define $u(t) = \chi_{[0,1]^2}(t) \otimes \chi_{[0,1]^2}(t)$. Next put

$$
\mathcal{P}\phi_1(\lambda) = \chi_{[0,1]^2}(\lambda) u(t), \mathcal{P}\phi_2(\lambda) = \chi_{[-1,0) \times (0,1]}(\lambda) u(t)
$$

$$
\mathcal{P}\phi_3(\lambda) = \chi_{[-1,0) \times [-1,0]}(\lambda) u(t), \text{ and } \mathcal{P}\phi_4(\lambda) = \chi_{(0,1] \times [-1,0]}(\lambda) u(t).
$$

For a fixed $\lambda \in (0,1]^2$ and for $m \in \mathbb{Z}^2$ define

$$
M_{\lambda,1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, M_{\lambda,2} = \begin{bmatrix} \lambda_1 - 1 & 0 \\ 0 & \lambda_2 - 1 \end{bmatrix}
$$

$$
M_{\lambda,3} = \begin{bmatrix} \lambda_1 - 1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, M_{\lambda,4} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 - 1 \end{bmatrix}
$$

$$
S_{\lambda,1,m} = M_{\lambda,1} \left( [0,1]^2 + m \right), S_{\lambda,2,m} = M_{\lambda,2} \left( [0,1]^2 + m \right)
$$

$$
S_{\lambda,3,m} = M_{\lambda,3} \left( [0,1]^2 + m \right), S_{\lambda,4,m} = M_{\lambda,4} \left( [0,1]^2 + m \right).
$$
We observe that for all \( m \in \mathbb{Z}^2 \), \( \bigcup_{k=1}^{4} S_{\lambda,k,m} \) is a proper subset of a fundamental domain for \( \mathbb{Z}^2 \) and \( S_{\lambda,i,m} \cap S_{\lambda,j,m} \) is a null set for distinct \( i, j \) for \( \lambda \in (0, 1]^2 \). Thus, by Corollary 27 \( L(\Gamma)(\phi_1 + \cdots + \phi_4) \) is an orthonormal basis for \( H(e, E^\circ) \) and \( W_{\phi_1 + \cdots + \phi_4}(H(e, E^\circ)) \) is a \( \Gamma \)-sampling space with the interpolation property. In fact, in order to show that \( L(\Gamma)(\phi_1 + \cdots + \phi_4) \) is an orthonormal system, we may use a more elementary approach. That is, we can directly check that given \( \phi = \phi_1 + \cdots + \phi_4 \),

\[
\langle \phi, L(\gamma) \phi \rangle = \begin{cases} 
1 & \text{if } \gamma \text{ is the identity} \\
0 & \text{otherwise} 
\end{cases}.
\]

To see this, let us define the function

\[
\beta : \mathbb{Z}^6 \to \mathbb{C}
\]

such that

\[
\begin{align*}
(19) \quad & \beta(m_1, m_2, l_1, l_2, k_1, k_2) \\
(20) \quad & = \langle \phi, L(\exp(m_1Z_1 + m_2Z_2) \exp(l_1Y_1 + l_2Y_2) \exp(k_1X_1 + k_2X_2)) \phi \rangle.
\end{align*}
\]

Then, applying the Plancherel Theorem,

\[
\begin{align*}
(21) \quad & \beta(m_1, m_2, l_1, l_2, k_1, k_2) \\
& = \int_{-1}^{1} \int_{-1}^{1} \int_{0}^{1} \int_{0}^{1} |\lambda_1, \lambda_2| \exp(-2\pi i (\lambda_1 m_1 + \lambda_2 m_2 - \lambda_1 l_1 t_1 - \lambda_2 l_2 t_2)) \\
& \times \chi_{[0,1]}(t_1 - k_1, t_2 - k_2) dt_1 dt_2 d\lambda_1 d\lambda_2.
\end{align*}
\]

If \((k_1, k_2) \neq (0, 0)\) then clearly

\[
\beta(m_1, m_2, l_1, l_2, k_1, k_2) = 0.
\]
If \((k_1, k_2) = (0, 0)\) and \((m_1, m_2, l_1, l_2) \neq (0, 0, 0, 0)\) then
\[
\beta (m_1, m_2, l_1, l_2, 0, 0)
= \int_{-1}^{1} \int_{-1}^{1} \int_{0}^{1} \int_{0}^{1} \exp (-2\pi i (\lambda_1 m_1 + \lambda_2 m_2 - \lambda_1 l_1 t_1 - \lambda_2 l_2 t_2))
\times |\lambda_1, \lambda_2| dt_1 dt_2 d\lambda_1 d\lambda_2
= \int_{-1}^{1} \int_{-1}^{1} |\lambda_1, \lambda_2| \exp (-2\pi i (\lambda_1 m_1 + \lambda_2 m_2))
\times \left( \frac{\exp (\lambda_1 l_1 t_1) - 1}{2\pi i l_1} \right) \left( \frac{\exp (\lambda_2 l_2 t_2) - 1}{2\pi i l_2} \right)
d\lambda_1 d\lambda_2
\]
where
\[
F (\lambda_1, \lambda_2, 0, 0) = 1
\]
\[
F (\lambda_1, \lambda_2, l_1, 0) = -i \left( \frac{e^{2\pi i l_1 \lambda_1} - 1}{2\pi i l_1 \lambda_1} \right) \text{ for } l_1 \neq 0
\]
\[
F (\lambda_1, \lambda_2, 0, l_2) = -i \left( \frac{e^{2\pi i l_2 \lambda_2} - 1}{2\pi i l_2 \lambda_2} \right) \text{ for } l_2 \neq 0
\]
\[
F (\lambda_1, \lambda_2, l_1, l_2) = -\left( \frac{e^{2\pi i l_2 \lambda_2} - 1}{2\pi i l_1 \lambda_1} \right) \left( \frac{e^{2\pi i l_1 \lambda_1} - 1}{2\pi i l_2 \lambda_2} \right) \text{ for } l_1 \neq 0 \text{ and } l_2 \neq 0.
\]
With some further calculations, it is not too hard to for \((m_1, m_2, l_1, l_2) \neq (0, 0, 0, 0)\), we have that
\[
\beta (m_1, m_2, l_1, l_2, 0, 0) = 0.
\]
Finally, if \((k_1, k_2) = (0, 0)\) and \((m_1, m_2, l_1, l_2) = (0, 0, 0, 0)\) then
\[
\beta (0, 0, 0, 0, 0, 0) = \int_{-1}^{1} \int_{-1}^{1} \int_{0}^{1} \int_{0}^{1} |\lambda_1, \lambda_2| dt_1 dt_2 d\lambda_1 d\lambda_2 = 1.
\]
In summary, if
\[
\exp (m_1 Z_1 + m_2 Z_2) \exp (l_1 Y_1 + l_2 Y_2) \exp (k_1 X_1 + k_2 X_2)
\]
is not equal to the identity then (19) is equal to zero. Otherwise (19) is equal to one.

References
