The orbit method and its applications

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Content

This presentation is brief survey of Lie theory with an emphasis on nilpotent Lie groups, and how the orbit methods is exploited for the purpose of Harmonic analysis on nilpotent Lie groups. If you are an expert in Lie theory, half of the presentation will be trivial. The other half which focuses on the orbit method will probably be new to you if you are not in representation theory. If you are not an expert in both Lie theory and representation theory, my hope is that this presentation will show you the beauty that we see in these theories, and convince you to join the rest of us in the adventure.

Acknowledgment

I would like to thank Professor Keith Taylor who was very generous with his time, and for hosting me during my visit at Dalhousie University. I have had a wonderful time. I also thank, Professor Mahya Ghandehari and Mr. Joshua MacArthur for the wonderful discussions, and for welcoming during the 3 weeks that I have spent here in Halifax, NS, Canada.

A starting point

A **real Lie group** is a set with two structures. First G is a group and G is manifold. These structures agree in the sense that multiplication and inversion maps are smooth. That is

1. $(x, y) \mapsto xy$ is a smooth map
2. $x \mapsto x^{-1}$ is a smooth map

Some notations

The **general Linear group over the real numbers** denoted $GL(n, \mathbb{R})$ is the group of all $n \times n$ invertible matrices with real entries.

The **general linear group over the complex numbers** denoted $GL(n, \mathbb{C})$ is the group of all $n \times n$ invertible matrices with complex entries entries.

The **set** of all $n \times n$ matrices with entries in some field K is called $M(n, K)$. We usually think of this set as a vector space. I will explain this later.

A precise definition of Matrix Lie groups

A **matrix Lie group** is any subgroup G of $GL(n, \mathbb{C})$ with the following property.

If $(A_m)_m$ is any sequence of matrices in G and if $A_m \to A$ then either A is in G or A is not invertible.

That is, a matrix Lie group is a **closed subgroup** of $GL(n, \mathbb{C})$.

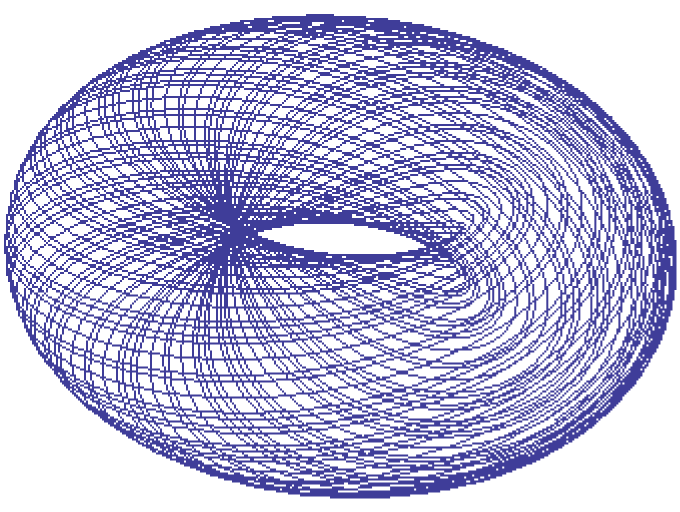
The fact that we require closeness is a technicality issue. Most subgroups that we are interested in are actually closed in the general linear groups. However, there are non closed subgroups of the general linear groups. I will give an example soon.

A non-closed subgroup of $GL(n, \mathbb{C})$

Here is an example of subgroup of $GL(n, \mathbb{C})$ which is not closed and is by definition **not a matrix Lie group**.

$$G = \left\{ \begin{bmatrix} e^{it} & 0 \\ 0 & e^{it\sqrt{2}} \end{bmatrix} : t \in \mathbb{R} \right\}$$

Clearly G is a subgroup of $GL(2, \mathbb{C})$. In fact, the closure of G is the torus $\mathbb{T} \times \mathbb{T}$.



Thus, there exists a sequence of matrices in G convergent to

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \notin G.$$

Thus G is not closed in $GL(2, \mathbb{C})$

The Heisenberg group

$$N = \left\{ \begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} : (x, y, z) \in \mathbb{R}^3 \right\}$$

is a matrix Lie group. I will come back to this example later. This group plays an important role in several fields such as quantum mechanic, Gabor analysis, wavelets, $\cdots$

Exponential of a matrix

The **exponential of a matrix** plays a crucial role in the theory of Lie group. The exponential enters into the definition of the Lie algebra of a matrix Lie group. I will spend some time on the exponential map. There are many ways, we can define an exponential function. One could use a more geometric approach, using left invariant vector fields, and integral curves... However, the more accessible definition is the one that I want to focus on.

Let X be an $n \times n$ real or complex matrix. We define

$$\exp X = \sum_{k=0}^{\infty} \frac{X^m}{m!}$$

In fact this series is always convergent. Indeed,

$$\begin{aligned}\left\| \sum_{k=0}^{\infty} \frac{X^m}{m!} \right\| &\leq \sum_{k=0}^{\infty} \left\| \frac{X^m}{m!} \right\| \\ &\leq \sum_{k=0}^{\infty} \frac{\|X^m\|}{m!} \\ &\leq \sum_{k=0}^{\infty} \frac{\|X\|^m}{m!} \\ &= e^{\|X\|} < \infty\end{aligned}$$

Moreover, $\exp X$ is a **continuous function** of X.

Some general facts about the exponential map

In general

1. $\exp X \exp Y \neq \exp Y \exp X$
2. $\exp(X+Y) = \lim_{n\to\infty} (\exp(X/m) \exp(Y/m))^m$
3. $\det e^X = e^{\mathrm{trace}(X)}$

The second formula above is known as the Lie product formula. There are many others well-known formulas but I will not mention them here.

Here is an example of the exponential of a given matrix.

$$\exp\left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} e & 0 & 0 \\ e & e & 0 \\ \frac{1}{2}e & e & e \end{bmatrix}.$$

The Matrix log function

We want to define a function which we wish should be the inverse (or at least local inverse) to the exponential function. For any matrix of order n we define

$$\log A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A-I)^m}{m}$$

whenever the series is convergent. In fact, it is known that the series is convergent whenever the norm of the matrix $A - I$ is less than 1. It is also clear that $\log A$ is always convergent whenever $A - I$ is nilpotent.

For example,

$$\log \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \end{bmatrix}$$

The Lie Algebra

Let G be a matrix Lie group. The **Lie algebra** of G denoted $\mathfrak{g}$ is the set of all matrices X such that

$$\exp(tX)$$

is in G for all numbers t.

Abstract Definition of Lie algebra

A finite-dimensional Lie algebra is a finite dimensional vector space $\mathfrak{g}$ together with a map $[\cdot, \cdot]$ from $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with the following properties

1. $[\cdot, \cdot]$ is bilinear
2. $[X, Y] = -[Y, X]$
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

The third identity is known as the Jacobi identity.

For example, if X is a complex matrix of order n then $\exp(tX)$ is invertible. Thus the Lie algebra of $GL(n, \mathbb{C})$ is the set of all complex matrices of order n. We write this Lie algebra as

$$\mathfrak{gl}(n, \mathbb{C}) = M(n, \mathbb{C}).$$

The Lie algebra of the Heisenberg group is

$$\left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

The Adjoint mapping

Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Then for each p in G, we define the linear map

$$Ad_p : \mathfrak{g} \to \mathfrak{g}$$

by the formula

$$Ad_p Y = pYp^{-1}$$

In fact

$$p \to Ad_p$$

is a **group homomorphism** of G into $GL(\mathfrak{g})$ and it is also easy to check that

$$Ad_p [X, Y] = [Ad_p X, Ad_p Y]$$

for all $X, Y \in \mathfrak{g}$.

The little adjoint map

Given $X \in \mathfrak{g}$, we define

$$ad_X : \mathfrak{g} \to \mathfrak{g}$$

as follows

$$ad_X Y = [X, Y]$$

and it is easy to check that

$$\exp(ad_X) = Ad_{\exp X}$$

Basic Representation Theory

Let G be a matrix Lie group. Then, a finite-dimensional real (or complex) representation of G is a **Lie group homomorphism**

$$\pi : G \to GL\left(V\right)$$

where V is a finite-dimensional real (or complex) vector space. That is

1. $\pi\left(xy\right) = \pi\left(x\right)\pi\left(y\right)$
2. π is continuous.

The trivial Representation

This is very stupid but I will do it anyway. Let G be a group and define

$$\pi : G \to GL(n, \mathbb{C})$$

such that $\pi(x) = I$ for all $x \in G$.

One-Parameter Representation

Here is a representation of $\mathbb{R}$.

$$\pi : \mathbb{R} \to GL(3, \mathbb{R})$$

and

$$\pi(t) = \begin{bmatrix} e^t & 0 & 0 \\ te^t & e^t & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}.$$

A representation of the Heisenberg group

Here is a more interesting example. Let $\mathfrak{g}$ be the Lie algebra spanned by the vectors X, Y, Z such that $[X, Y] = Z$. Now, let G be its Lie group. G is the Heisenberg group. Define

$$\pi : G \rightarrow GL(3, \mathbb{R})$$

such that

$$\pi(\exp Z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\pi(\exp Y) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\pi(\exp X) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\pi(G)$ is actually isomorphic to $\mathbb{R}^2$. This is a **non-faithful representation** of the Heisenberg group

Let G be a locally compact group. A unitary representation of G is a **homomorphism** π from G into the group of **unitary operators** on some non-zero Hilbert space which is **continuous with respect to the strong operator topology**. That is

$$\pi : G \to U\left(H_\pi\right)$$

such that

1. $\pi\left(xy\right) = \pi\left(x\right)\pi\left(y\right)$
2. $\pi\left(x^{-1}\right) = \pi\left(x\right)^{-1} = \pi\left(x\right)^{*}$
3. $x \to \pi\left(x\right)u$ is continuous from G to H_π for any $u \in H_\pi$.

It is worth noticing that **strong continuity and weak topologies are the same** in $U(H_\pi)$. Thus,

$$x \mapsto \langle \pi(x) u, v \rangle$$

is continuous from G to $\mathbb{C}$.

Suppose that $\{T_\alpha\}$ is a net of unitary operators convergent to T. Then for any $u \in H_\pi$

$$\begin{aligned} \|(T_\alpha - T) u\|^2 &= \|T_\alpha u\|^2 + \|Tu\|^2 - 2 \operatorname{Re} \langle T_\alpha u, Tu \rangle \\ &= 2\|u\|^2 - 2 \operatorname{Re} \langle T_\alpha u, Tu \rangle \end{aligned}$$

Thus,

$$\|(T_\alpha - T) u\|^2 \to 0.$$

If π_1 and π_2 are unitary representations of G, an **intertwining operator** for π_1 and π_2 is a bounded linear operator $T : H_{\pi_1} \to H_{\pi_2}$ such that

$$T\pi_1(x) = \pi_2(x)\,T$$

for all $x \in G$. We say that π_1 and π_2 are **unitarily equivalent** if there is a unitary operator T which is intertwining the representations.

Now suppose that K is a closed subspace of H_π such that that

$$\pi\left(G\right)K\subset K$$

We say that K is a π**-invariant Hilbert subspace** of H_π. Moreover, a representation π is **irreducible** if the only π-invariant subspaces are the trivial ones. That is the zero vector space and H_π.

The unitary dual

Given a locally compact group G, one of the most important questions in representation theory is to classify all of its unitary irreducible representations. The set of all irreducible representations of G up to equivalence is called the unitary dual and is denoted $\widehat{G}$

$$\widehat{G} = \{[\pi] : \pi \text{ is irreducible}\}$$

Once, one knows the unitary dual of G, for a fairly large class of groups, then it is possible to do Fourier analysis. That is, one can define a **Fourier transform, a Plancherel transform and even establish some natural notion of Plancherel theory**.

The Classic stuff everyone knows

One the real line, the set of all unitary irreducible representations of $\mathbb{R}$ forms a group isomorphic to $\mathbb{R}$. In fact

$$\widehat{\mathbb{R}} = \{\pi_x = \exp(2\pi i x) \in \mathbb{T} : x \in \mathbb{R}\}$$

The object $\exp(2\pi i x)$ is regarded as an operator on $\mathbb{C}$ into $\mathbb{C}$ acting on complex numbers by rotations. That is

$$\pi_x(z) = \exp(2\pi i x)(z) = \exp(2\pi i x) \times z$$

Next, the **Fourier-transform** on $L^1(\mathbb{R})$ is defined as follows.

$$Ff(\lambda) = \int_{\mathbb{R}} f(x)\, \pi_x(\lambda)\, dx$$

Nilpotent Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra over the reals. The descending central series of $\mathfrak{g}$ is defined inductively as follows.

1. $\mathfrak{g}^{(1)} = \mathfrak{g}$
2. $\mathfrak{g}^{(n+1)} = \left[\mathfrak{g}, \mathfrak{g}^{(n)}\right]$

We say that $\mathfrak{g}$ is a **nilpotent Lie algebra** if there is an integer n such that

$$\mathfrak{g}^{(n+1)} = (0)\,.$$

If $\mathfrak{g}^{(n)}$ is not equal to the zero vector space then n is minimal and we say that $\mathfrak{g}$ is a nilpotent Lie algebra of n-step.

This is very trivial. However, it is worth observing that **every commutative Lie algebra is a one-step Lie algebra**.

Let us suppose that

$$\mathfrak{g} = \mathbb{R} - \operatorname{span}\{X, Y, Z\}$$

such that the only non-trivial brackets are $[X, Y] = Z$. Then

$$\begin{aligned} \mathfrak{g}^{(1)} &= \mathfrak{g} \\ \mathfrak{g}^{(2)} &= \mathbb{R}Z \\ \mathfrak{g}^{(3)} &= (0)\,. \end{aligned}$$

Thus, $\mathfrak{g}$ is a two-step nilpotent Lie group.

Let us suppose that

$$\mathfrak{g} = \mathbb{R} - \text{span}\{X, Y, Z, W\}$$

such that the only non-trivial brackets are

$$[X, Y] = Z, \ \ [W, X] = Y$$

Then

$$\begin{aligned}
\mathfrak{g}^{(1)} &= \mathfrak{g} \\
\mathfrak{g}^{(2)} &= \mathbb{R}Z \oplus \mathbb{R}Y \\
\mathfrak{g}^{(3)} &= \mathbb{R}Z \\
\mathfrak{g}^{(4)} &= 0.
\end{aligned}$$

Thus, $\mathfrak{g}$ is a three-step nilpotent Lie algebra

Nilpotent Lie group

Let G be a Lie group with **nilpotent Lie algebra $\mathfrak{g}$**. Then G is called a **nilpotent Lie group**. Assume that G is a connected, simply connected nilpotent Lie group. It is known that every connected simply connected nilpotent Lie group has a faithful embedding as a closed subgroup of group of upper triangular matrices of order n with ones on the diagonal.

What is the unitary dual of G

If G is a simply connected, connected nilpotent Lie group? This is a deep question that has kept a lot of mathematicians busy for a long time. With the work of mainly Pukansky, Dixmier and **Kirillov**, this question has been settled in the sixties.

Orbit method

Let G be a simply connected nilpotent Lie group with nilpotent Lie algebra $\mathfrak{g}$. We donote the dual of $\mathfrak{g}$ by $\mathfrak{g}^*$. The group G acts on $\mathfrak{g}^*$ by the **contragradient of the adjoint map**. That is given $l \in \mathfrak{g}^*, x \in G$ and $Y \in \mathfrak{g}$

$$x \cdot l(Y) = l\left(Ad_{x^{-1}} Y\right).$$

This action is called the **coadjoint action**. All irreducible unitary representations of G are **parametrized by the set of coadjoint orbits**. This is what the **orbit method** is all about.

Some interesting Fact

Let $l \in \mathfrak{g}^*$ and $O_l = G \cdot l$, its coadjoint orbit. $G \cdot l$ is an even-dimensional smooth manifold which has a natural symplectic form that turns the manifold into a symplectic manifold. There are some beautiful geometrical theorems about this fact, which I will not talk about.

The unitary dual of the Heisenberg group

Let us consider the Heisenberg group.

$$G = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

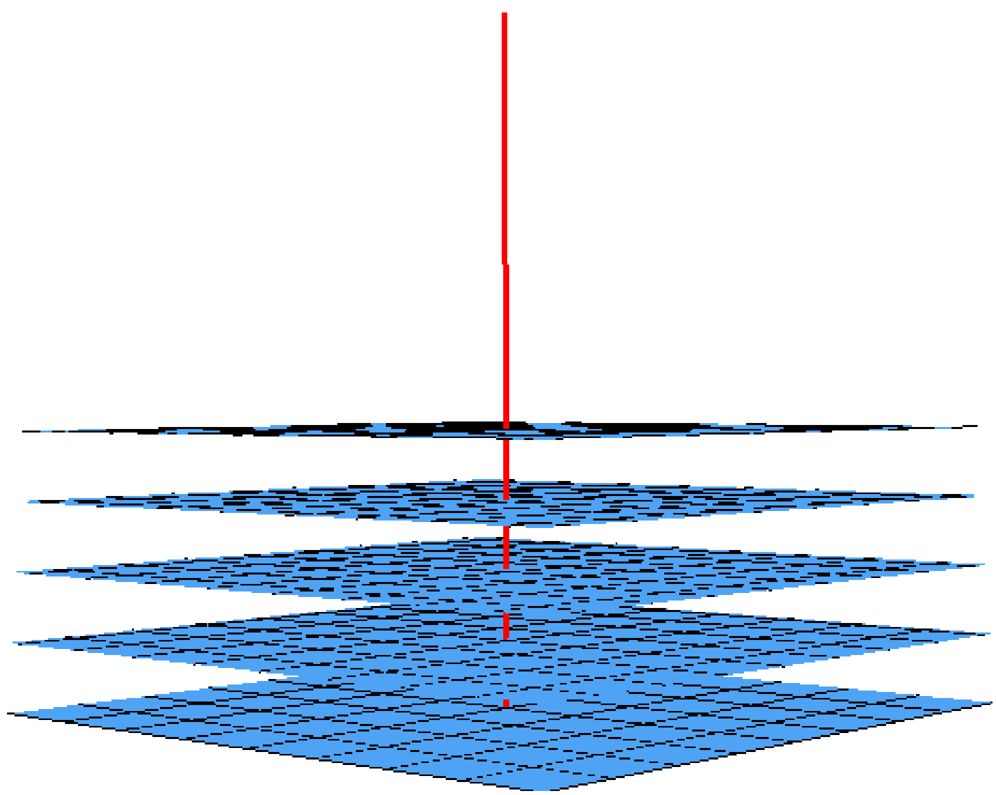
$$\mathfrak{g} = \left\{ \begin{bmatrix} 0 & X & Z \\ 0 & 0 & Y \\ 0 & 0 & 0 \end{bmatrix} : Z, Y, X \in \mathbb{R} \right\},$$

and the dual vector space

$$\mathfrak{g}^* = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \lambda & \beta & 0 \end{bmatrix} : \lambda, \alpha, \beta \in \mathbb{R} \right\}.$$

Then

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \lambda & \beta & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \alpha + y\lambda & 0 & 0 \\ \lambda & \beta - x\lambda & 0 \end{bmatrix}$$



Notice that if $\lambda \neq 0$, then the coadjoint orbit of

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}$$

is two-dimensional plane

$$\left\{ \begin{bmatrix} 0 & 0 & 0 \\ t & 0 & 0 \\ \lambda & r & 0 \end{bmatrix} : (t, r) \in \mathbb{R}^2 \right\}$$

Next, the coadjoint orbit of

$$\begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & \beta & 0 \end{bmatrix}$$

is the linear functional itself.

Thus, the unitary dual of N is parametrized by the following set

$$\left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix} : \lambda \neq 0 \right\} \dot{\cup} \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & \beta & 0 \end{bmatrix} : (\alpha, \beta) \in \mathbb{R}^2 \right\}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix} \mapsto \text{infinite dimensional unitary irr}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & \beta & 0 \end{bmatrix} \mapsto \text{one-dimensional unitary irr rep}$$

The unitary dual of a step-three nilpotent Lie group

Let $\mathfrak{g} = \mathbb{R}Z \oplus \mathbb{R}Y \oplus \mathbb{R}X \oplus \mathbb{R}W$ such that the only non-trivial Lie brackets are given by

$$[X, Y] = Z, [W, X] = Y$$

Then there is **faithful matrix representation** of G such that

$$G = \left\{ \begin{bmatrix} 0 & x & -y & 0 & 3z \\ 0 & 0 & w & -x & 2y \\ 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & w \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} : x, y, z, w \in \mathbb{R} \right\}.$$

We may identify $\mathfrak{g}$ and its dual (vector space of linear functionals) with $\mathbb{R}^4$. This identification will make the discussion easier to present.

Let $\lambda \in \mathfrak{g}^*$. Writing $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$,

$$G \cdot \lambda = \left\{ \begin{array}{c} \left(\lambda_1, \lambda_2 - s\lambda_1, \lambda_3 + r\lambda_1 - u\lambda_2 + \frac{1}{2}su\lambda_1, \lambda_2 s - \frac{1}{2}\lambda_1 s^2 + \lambda_4\right) : \\ r, s, u \in \mathbb{R} \end{array} \right\}$$

1. If $\lambda_1 \neq 0$ the coadjoint orbits are two dimensional and are parametrized by

$$\Sigma_1 = \{(\lambda_1, 0, 0, \lambda_4) \in \mathfrak{g}^* : \lambda_1 \neq 0, \lambda_4 \in \mathbb{R}\}$$

2. If $\lambda_1 = 0$ and $\lambda_2 \neq 0$ then

$$\Sigma_2 = \{(0, \lambda_2, 0, 0) \in \mathfrak{g}^* : \lambda_2 \neq 0\}$$

3. If $\lambda_1 = 0$ and $\lambda_2 = 0$ then the coadjoint orbits are two dimensional as well, and they are parametrized by

$$\Sigma_3 = \{(0, 0, \lambda_3, \lambda_4) \in \mathfrak{g}^*\}$$

4. The unitary dual of G is then parametrized by

$$\Sigma = \bigcup_{k=1}^{3} \Sigma_k$$

Exponential Solvable Lie groups

Let $G = \mathbb{R}^n \rtimes H$ where

$$H = \exp\left(\mathbb{R}A_1 \oplus \cdots \oplus \mathbb{R}A_m\right) < GL\left(n, \mathbb{R}\right)$$

where the A_k are upper triangular matrices with non purely imaginary complex numbers. The group multiplication in G is

$$\left(v_1, M_1\right)\left(v_2, M_2\right) = \left(v_1 + M_1 v_2, M_1 M_2\right).$$

Then G is an **exponenial solvable Lie group** with Lie algebra $\mathfrak{g} = \mathbb{R}^n \oplus \mathfrak{h}$. The orbit method works for this class of groups as well.

An example of Solvable Lie group

Here is a concrete example $G = \mathbb{R}^3 \rtimes H$ where $H = \exp(\mathbb{R}A_1 \oplus \mathbb{R}A_2)$ and

$$A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

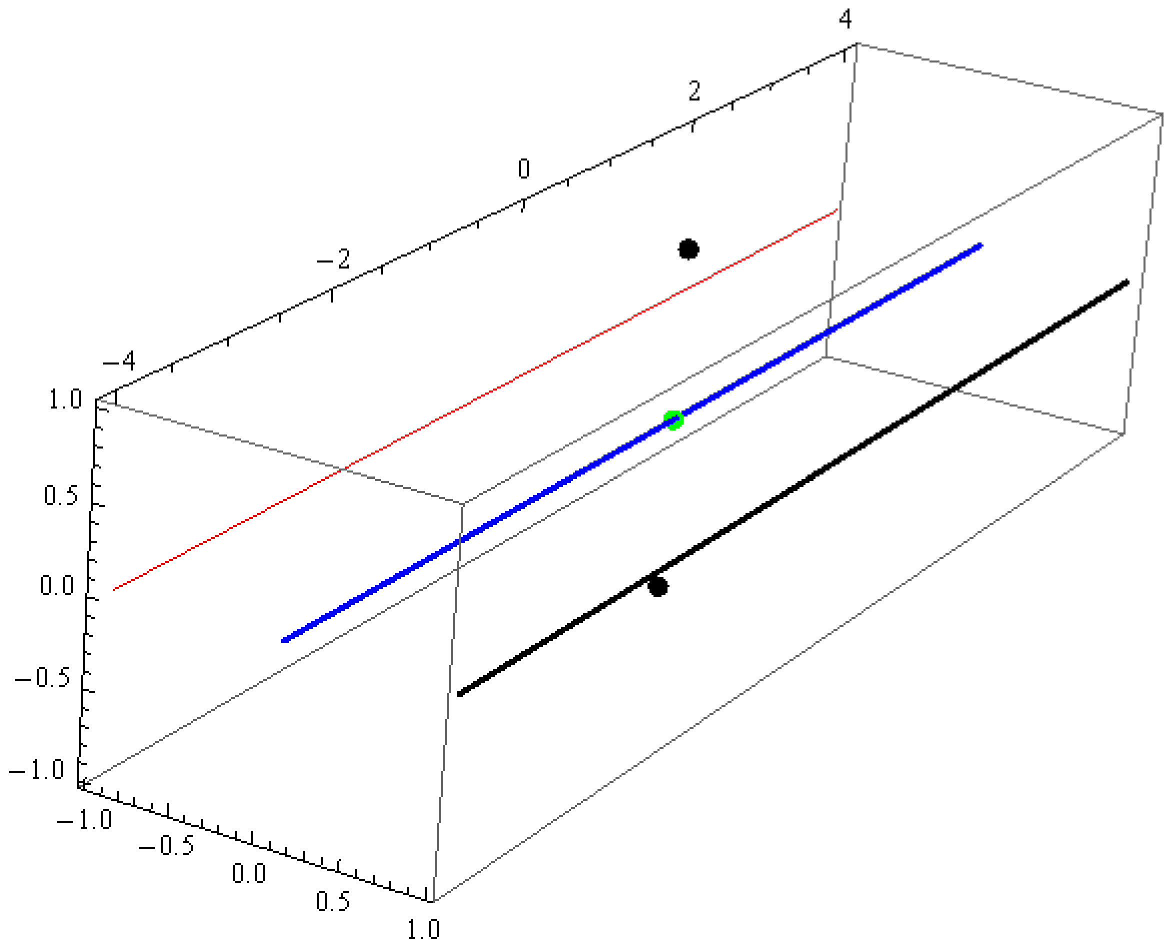
$$\mathfrak{h} = \left\{ \begin{bmatrix} t_2 & 0 & t_1 \\ 0 & t_2 & t_2 \\ 0 & 0 & t_2 \end{bmatrix} : (t_1, t_2) \in \mathbb{R} \right\}$$

$$H = \left\{ \begin{bmatrix} e^{t_2} & 0 & t_1 e^{t_2} \\ 0 & e^{t_2} & t_2 e^{t_2} \\ 0 & 0 & e^{t_2} \end{bmatrix} : (t_1, t_2) \in \mathbb{R} \right\}$$

Then the unitary dual of G is parametrized by

$$\Sigma = \bigcup_{k=1}^{4} \Sigma_k$$

$$\Sigma_1 = \left\{ (v_1, v_2, 0) \in \mathbb{R}^3 : |v_1| = 1 \right\}$$
$$\Sigma_2 = \left\{ (0, v_2, 0) \in \mathbb{R}^3 : v_2 \neq 0 \right\}$$
$$\Sigma_3 = \left\{ (0, 0, v_3) \in \mathbb{R}^3 : |v_3| = 1 \right\}$$
$$\Sigma_4 = \{(0, 0, 0)\}$$



What do we do with all of this stuff?

There is more to the story on how to construct unitary irreducible representations from a paramet rization of the coadjoint orbits. Unfortunately, I do not have enough time to give a thorough exposure of the theory. This presentation is only a brief overview. In fact the orbit method generalizes to a larger class of Lie groups called exponential solvable Lie groups.

With this theory available, then we have a nice Plancherel and Fourier theory which is well understood. Then one could attempt to answer the following (modern analysis) questions.

Some modern active area of research

1. Can we construct wavelets on non-commutative Nilpotent Lie groups?
2. How far can we go with the explicit construction of wavelets on non-commutative nilpotent Lie groups?
3. Could we generalize the theory of Paley-Wiener spaces?
4. Can we talk about sampling theory, and reconstruction of bandlimited spaces on nilpotent Lie groups
5. Linear Independence of translates on the Nilpotent Lie groups. In the case of the Heisenberg group, this question is connected to the well-known **HRT conjecture** and Gabor analysis

If you are interested in learning more about Lie groups especially nilpotent Lie groups here are some nice resources.

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If you are interested in more recent research activities related to the questions above, please refer to

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