Band-limited Spaces on Some 2-step Nilpotent Lie Groups With One Parseval Frame Generator

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Definition

Let $\mathcal{H}$ be a Hilbert space, and let $\{\phi_i : i \in I\}$ be a countable sequence of vectors in $\mathcal{H}$. $\{\phi_i : i \in I\}$ forms a Parseval frame (PF) if and only if for any $f \in \mathcal{H}$, $\sum_{i \in I} |\langle f, \phi_i \rangle|^2 = \|f\|_{\mathcal{H}}^2$.

Example

Any ONB in $\mathcal{H}$ is a PF. The sequence $\left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}$ is a PF for $\mathbb{C}$.

Fact

(Expansion property) If $\{\phi_i : i \in I\}$ forms a PF for $\mathcal{H}$,

$$f = \sum_{n \in I} \langle f, \phi_n \rangle \phi_n.$$
Definition

Let \( G \) be a locally compact connected type I group endowed with a left Haar measure. Let \( \mathcal{H} \) be a closed left-invariant subspace for \( L^2(G) \), and let \( \Gamma \) be a discrete set in \( G \). We say a function \( \phi \) is a \textbf{PF generator} if and only if the set of all translates of \( \phi \) by elements of \( \Gamma \) forms a PF for \( \mathcal{H} \).

Example

(Paley-Wiener space) Let \( G = \mathbb{R} \), \( \Gamma = \mathbb{Z} \), \( \phi(x) = \text{sinc}(x) \) and define the \textbf{band-limited} subspace of \( L^2(\mathbb{R}) \).

\[
\mathcal{H} = \left\{ f \in L^2(\mathbb{R}) : \text{supp} \hat{f} \subseteq [-1/2, 1/2] \right\},
\]

\( \{\phi(\cdot - k) : k \in \mathbb{Z}\} \) forms a PF and an ONB.
Let $\mathbb{H}$ be the Heisenberg group with Lie algebra $\mathfrak{n}$ spanned by $\{X, Y, Z\}$, $[X, Y] = Z$.

$L$ is the Left regular representation of $\mathbb{H}$.

The Plancherel Transform is supported on $\mathbb{R}^*$ with Plancherel measure $|\lambda| \, d\lambda$.

$$\mathcal{P} : L^2(\mathbb{H}) \to \int_{\mathbb{R}^*}^{\oplus} L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) |\lambda| \, d\lambda.$$

$\mathbb{H}$ is the simplest nilpotent Lie group which is step two.

**Theorem**

*(Hartmut Fuhr)* Let $\mathbf{H}$ be a closed left invariant mult-free band-limited subspace of $L^2(\mathbb{H})$ with

$$\mathcal{P}(\mathbf{H}) = \int_{[-0.5,0.5]}^{\oplus} \left( L^2(\mathbb{R}) \otimes \mathbb{C} \mathbf{u} \right) |\lambda| \, d\lambda,$$

where $\mathbf{u}$ is a unit vector. There exists a vector $\mathbf{\phi} \in \mathbf{H}$ and a lattice $\Gamma \subset \mathbf{H}$ such that $L(\Gamma) \mathbf{\phi}$ forms a PF for $\mathbf{H}$. 
Problem

Let $N$ be a simply connected, connected step two nilpotent Lie group. Let $L$ be the left regular representation of $N$. Let $H$ be a closed left invariant, band-limited subspace of $L^2(N)$. How do we pick a discrete subset $\Gamma \subset N$ and a function $\phi$ in $H$ such that $L(\Gamma)\phi$ forms a PF or an ONB in $H$?

- (Non nilpotent groups) Dooley (1989) studied band-limited subspaces defined on motion groups of the type $\mathbb{R}^k \rtimes K$, $K$ is compact matrix group.
- Fuhr and Grochenig (2005). General results for band-limited spaces on stratified nilpotent Lie groups. They used sub-Laplacians but obtained no PF.

We are interested in PF, ONB and explicit description of $\Gamma$ and $\phi$. 
Theorem

Let $N$ be a non commutative simply connected, connected step two nilpotent Lie group such that the Lie algebra of the center is $\mathfrak{z} = \mathbb{R}Z_1 \cdots \mathbb{R}Z_{n-2d}$, and

$$n = \mathfrak{z} \oplus \mathbb{R}Y_1 \cdots \mathbb{R}Y_d \oplus \mathbb{R}X_1 \cdots \mathbb{R}X_d.$$ 

Assume $\mathfrak{p}$ is a commutative ideal, $\mathfrak{m}$ is commutative subalgebra of $n$ and $\exp \mathfrak{m} < \text{Aut} (\exp \mathfrak{p})$. Let $H$ be a multiplicity free subspace of $L^2(N)$ with bounded spectrum. There exists a quasi-lattice $\Gamma \subset N$ and a function $\phi \in H$ such that $L(\Gamma)\phi$ forms a Parseval frame for $H$. 

• The Heisenberg group with Lie algebra spanned by $X, Y, Z$ with $[X, Y] = Z$.

• $n = \mathbb{R} \text{-span}\{Z_1, Z_2, Y_1, Y_2, X_1, X_2\}$

  $$[X_k, Y_k] = Z_1$$  
  $$[X_i, Y_j] = Z_2, \text{ for } i \neq j.$$  

• $n = \mathbb{R} \text{-span}\{Z_1, \ldots, Z_{2d}, Y_1, \ldots, Y_d, X_1, \ldots, X_d\}$

  $$[X_j, Y_i] = Z_{i+j} \text{ for } 1 \leq i, j \leq d.$$
Almost all of the irreducible representations of $N$ act in $L^2(\mathbb{R}^d)$. \textcolor{red}{\textbf{Orbit method}}

Irred rep act by multiplication of characters, \textbf{translations and modulations}.

Translations + modulations = multivariate Gabor systems.

Dual of $N$ is parametrized by a Zariski open subset of $\mathfrak{z}^*$

$$\hat{N} = \{ \pi_\lambda : \lambda \in \mathfrak{z}^* \}.$$
Let $\mathcal{P}$ be the Plancherel transform, and the support of the Plancherel measure is $\Sigma \subset \mathfrak{g}^*$. 

$$\Sigma = \{ \lambda \in \mathfrak{g}^* : \det \mathbf{B}(\lambda) \neq 0 \}$$

- $\mathcal{P}(\mathbf{H}) = \int_{\Sigma}^{\oplus} (L^2(\mathbb{R}^d) \otimes \mathbb{C}u) \mathbf{P}(\lambda) d\lambda$, $\mathbf{H}$ is left-invariant, mult-free, band-limited closed subspace of $L^2(\mathbb{N})$, and
  $$\|u\|_{L^2(\mathbb{R}^d)} = 1.$$ 

- $\mathbf{P}(\lambda) d\lambda$ is the Plancherel measure, $\mathbf{P}(\lambda) = |\det \mathbf{B}(\lambda)|$ and 

$$\mathbf{B}(\lambda) = \begin{pmatrix} 
\lambda [X_1, Y_1] & \cdots & \lambda [X_1, Y_d] \\
\vdots & \ddots & \vdots \\
\lambda [X_d, Y_1] & \cdots & \lambda [X_d, Y_d] 
\end{pmatrix}.$$
The spectrum of $\mathbf{H}$ is bounded i.e.

$$S \subseteq \prod_{k=1}^{n-2d} \left[ -\frac{a_k}{2}, \frac{a_k}{2} \right] \cap \Sigma.$$ 

Let $m = \sup \{ P(\lambda) : \lambda \in S \}$. Define the quasi-lattice

$$\Gamma = \exp \left( \frac{\mathbb{Z}Y_1 \cdots + \mathbb{Z}Y_d}{m^{1/d}} \right) \exp \left( \frac{\mathbb{Z}Z_1}{a_1} + \cdots + \frac{\mathbb{Z}Z_{n-2d}}{a_{n-2d}} \right).$$
• There exists \( \phi \in H \) such that \( L(\Gamma)\phi \) forms a Parseval frame in \( H \).

\[
\mathcal{P}\phi(\lambda) = \frac{g(\lambda)}{\sqrt{a_1 \cdots a_{n-2d} |\det B(\lambda)|}} \otimes u,
\]

where the Gabor system

\[
\mathcal{G}(g(\lambda), \text{Diag}(m^{-1/d}, \ldots, m^{-1/d})) \mathcal{Z}^d \times B(\lambda)\mathcal{Z}^d
\]

forms a Parseval Frame for almost every \( \lambda \in S \). (The existence of \( g(\lambda) \) is due to the density condition of multivariate gabor systems).