Sampling on nilpotent Lie groups

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Sampling on the real line

How can we recover a function $f : \mathbb{R} \to \mathbb{C}$ if we only know a countable set of values

$$(f(k))_{k \in I}?$$

Formulated this way, the problem is ill-posed since there are infinitely many functions that take the same prescribed values on a given countable set. We consider the Paley-Wiener space

$$PW := \left\{ f \in L^2(\mathbb{R}) : \ \operatorname{supp} \widehat{f} \subseteq \left[-\frac{1}{2}, \frac{1}{2} \right] \right\}.$$

The Paley-Wiener space consists of equivalence classes of functions. Since the Fourier transform of these functions has compact support. Each of the equivalence classes contains a continuous function. Define the sinc-function by

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{cases}.$$

Theorem 1 *(Shannon's sampling theorem) The functions* $\{sinc\,(x-k)\}_{k\in\mathbb{Z}}$ *form an orthonormal basis for* PW*. If* $f \in PW \cap C(\mathbb{R})$*, then*

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)\, sinc\,(x-k)$$

with convergence of the symmetric partial sums in $L^2(\mathbb{R})$ *and pointwise for all* $x \in \mathbb{R}$.

Proof. (Shannon's sampling theorem) Since $\left\{e^{2\pi ik(\cdot)}\chi_{(-1/2,1/2)}\right\}_{k\in\mathbb{Z}}$ forms an orthonormal base in $L^2(-1/2,1/2)$,

$$F\left(e^{2\pi ik(\cdot)}\chi_{(-1/2,1/2)}\right)(\gamma) = \int_{-1/2}^{1/2} e^{2\pi ikx}e^{-2\pi ix\gamma}dx = \operatorname{sinc}(\gamma - k).$$

Since the Fourier transform is unitary, then $\{\operatorname{sinc}(x-k)\}_{k\in\mathbb{Z}}$ is orthonormal as well. Now, let $f \in L^1(\mathbb{R}) \cap PW \cap C(\mathbb{R})$. Then

$$\widehat{f}(\cdot) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi ik(\cdot)}$$

where

$$c_k = \int_{-1/2}^{1/2} \widehat{f}(\gamma)\, e^{-2\pi ik\gamma}d\gamma.$$

Since the partial sums of the Fourier series converge in the norm of $L^2(-1/2,1/2)$

$$\int_{-1/2}^{1/2} \left|\widehat{f}(\gamma) - \sum_{n=-N}^{N} c_k e^{-2\pi ik\gamma}\right|^2 d\gamma \to 0 \text{ as } N \to \infty$$

since, we are dealing with finite interval then

$$\int_{-1/2}^{1/2} \left|\widehat{f}(\gamma) - \sum_{n=-N}^{N} c_k e^{-2\pi ik\gamma}\right| d\gamma \to 0 \text{ as } N \to \infty. \tag{1}$$

In fact $c_k = f(-k)$. Next, for all $x \in \mathbb{R}$, using (1)

$$\begin{aligned}
f(x) &= \int_{\mathbb{R}} \widehat{f}(\gamma)\, e^{2\pi ix\gamma}d\gamma \\
&= \int_{-1/2}^{1/2} \left(\sum_{k\in\mathbb{Z}} f(-k)\, e^{2\pi ik\gamma}\right) e^{2\pi ix\gamma}d\gamma \\
&= \sum_{k\in\mathbb{Z}} f(-k) \int_{-1/2}^{1/2} e^{2\pi ik\gamma}e^{2\pi ix\gamma}d\gamma \\
&= \sum_{k\in\mathbb{Z}} f(-k) \int_{-1/2}^{1/2} e^{2\pi i\gamma(x+k)}d\gamma \\
&= \sum_{k\in\mathbb{Z}} f(k) \operatorname{sinc}(x-k).
\end{aligned}$$

Next, to show that this series converges in $L^2(\mathbb{R})$, we use the fact that $(f(k))_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$.

$$\left\| f - \sum_{k=-N}^{N} f(k)\,\mathrm{sinc}(\cdot - k) \right\| = \left\| \sum_{|k|>N} f(k)\,\mathrm{sinc}(\cdot - k) \right\| = \sqrt{\sum_{|k|>N} |f(k)|^2} \to$$

■

Sampling on locally compact groups

Let N be a locally compact group, and let Γ be a discrete subset of N. Let $\mathbf{H}$ be a left-invariant closed subspace of $L^2(N)$ consisting of continuous functions. We call $\mathbf{H}$ a **sampling space** with respect to Γ (or Γ-sampling space) if

1. The restriction mapping $R_\Gamma : \mathbf{H} \to l^2(\Gamma)\,,\ R_\Gamma f = (f(\gamma))_{\gamma\in\Gamma}$ is an isometry.

2. There exists a vector $S \in \mathbf{H}$ such that for any vector $f \in \mathbf{H}$, we have the following expansion

$$f(x) = \sum_{\gamma\in\Gamma} f(\gamma)\, S\left(\gamma^{-1}x\right)$$

   with convergence in the norm of $\mathbf{H}$.

The vector S is called a **sinc-type** function. Moreover, if R_Γ is surjective, we say that the sampling space $\mathbf{H}$ has the **interpolation property**.

Sampling on the Heisenberg groups

Let $\mathbb{H}$ be the three-dimensional Heisenberg Lie group with Lie algebra spanned by X, Y, Z such that $[X, Y] = Z$. We may write

$$\mathbb{H} = \exp(\mathbb{R}Z)\exp(\mathbb{R}Y)\exp(\mathbb{R}X).$$

Next, put

$$\Gamma = \exp(\mathbb{Z}Z)\exp(\mathbb{Z}Y)\exp(\mathbb{Z}X).$$

Then Γ is a discrete subgroup of the Heisenberg group.

Theorem 2 *(H. Fuhr, 2005) The Heisenberg group admits sampling spaces with respect to* Γ.

Theorem 3 *(B. Currey, A. Mayeli, 2009) The Heisenberg group admits a sampling space with respect to* Γ *which also has the interpolation property.*

Sampling on some nilpotent Lie groups

Theorem 4 *(O.) Let N be a simply connected, connected, two-step nilpotent Lie group with Lie algebra $\mathfrak{n}$ of dimension n such that $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{z}$, where $[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{z}$, $\mathfrak{a}, \mathfrak{b}, \mathfrak{z}$ are abelian algebras such that*

$$\begin{aligned} \mathfrak{a} &= \mathbb{R}\text{-}span\left\{X_1, X_2, \cdots, X_d\right\}, \\ \mathfrak{b} &= \mathbb{R}\text{-}span\left\{Y_1, Y_2, \cdots, Y_d\right\}, \\ \mathfrak{z} &= \mathbb{R}\text{-}span\left\{Z_1, Z_2, \cdots, Z_{n-2d}\right\}, \end{aligned}$$

$d \geq 1$, $n > 2d$ and

$$\det \begin{bmatrix} [X_1, Y_1] & \cdots & [X_1, Y_d] \\ \vdots & \cdots & \vdots \\ [X_d, Y_1] & \cdots & [X_d, Y_d] \end{bmatrix} \tag{2}$$

is a non-vanishing homogeneous polynomial in the unknowns $Z_1, \cdots, Z_{n-2d}$. Put

$$\Gamma = \exp\left(\sum_{k=1}^{n-2d} \mathbb{Z} Z_k\right) \exp\left(\sum_{k=1}^{d} \mathbb{Z} Y_k\right) \exp\left(\sum_{k=1}^{d} \mathbb{Z} X_k\right).$$

Then N admits sampling spaces with respect to Γ.

A toy example

Let N be a nilpotent Lie group with Lie algebra $\mathfrak{n}$ which is spanned by the following vectors

$$Z_1, Z_2, Y_1, Y_2, X_1, X_2$$

such that

$$\begin{aligned} &[X_1, Y_1] = Z_1, [X_1, Y_2] = Z_1 \\ &[X_2, Y_1] = 0, \ \ [X_2, Y_2] = Z_2. \end{aligned}$$

Then

$$\det \begin{bmatrix} Z_1 & Z_1 \\ 0 & Z_2 \end{bmatrix} = Z_1 Z_2$$

is a non-zero polynomial in the unknowns $Z_1 Z_2$. Thus, N belongs to the class of groups considered. Put

$$\Gamma = \exp\left(\mathbb{Z}Z_1 + \mathbb{Z}Z_2\right) \exp\left(\mathbb{Z}Y_1 + \mathbb{Z}Y_2\right) \exp\left(\mathbb{Z}X_1 + \mathbb{Z}X_2\right).$$

Plancherel theory

- The **unitary dual** of N is parametrized by an open subset of $\mathbb{R}^2$:
$$\Sigma = \left\{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_1 \lambda_2 \neq 0 \right\}.$$

- The **irreducible representations** of N can be realized as acting in $L^2\left(\mathbb{R}^2\right)$ as follows. For every $(\lambda_1, \lambda_2) \in \Sigma$,
$$\pi_{(\lambda_1,\lambda_2)}\left(\exp\left(z_1 Z_1 + z_2 Z_2\right)\right) f\left(t_1, t_2\right) = e^{2\pi i \lambda_1 z_1} e^{2\pi i \lambda_2 z_2} f\left(t_1, t_2\right)$$
$$\pi_{(\lambda_1,\lambda_2)}\left(\exp\left(y_1 Y_1 + y_2 Y_2\right)\right) f\left(t_1, t_2\right) = e^{-2\pi i \left\langle \left[\begin{array}{cc} \lambda_1 & \lambda_1 \\ 0 & \lambda_2 \end{array}\right] \left[\begin{array}{c} y_1 \\ y_2 \end{array}\right], \left[\begin{array}{c} t_1 \\ t_2 \end{array}\right] \right\rangle} f\left(t_1, t_2\right)$$
$$\pi_{(\lambda_1,\lambda_2)}\left(\exp\left(x_1 X_1 + x_2 X_2\right)\right) f\left(t_1, t_2\right) = f\left(t_1 - x_1, t_2 - x_2\right)$$

- Let $\mathcal{P}$ be the **Plancherel transform** on $L^2(N)$ and $\mathcal{F}$ the **Fourier transform** defined on $L^2(N) \cap L^1(N)$ by
$$\mathcal{F}(f)(\lambda) = \int_N f(n)\, \pi_{(\lambda_1,\lambda_2)}(n)\, dn.$$
Then
$$\mathcal{P} : L^2(N) \to \int_\Sigma^{\oplus} L^2\left(\mathbb{R}^2\right) \otimes L^2\left(\mathbb{R}^2\right) \ \ |\lambda_1 \lambda_2| \ d\lambda_1 d\lambda_2$$
is such that the Plancherel transform is the extension of the Fourier transform to $L^2(N)$ inducing the equality
$$\|f\|^2_{L^2(N)} = \int_\Sigma \|\mathcal{P}(f)(\lambda_1, \lambda_2)\|^2_{\mathcal{HS}} \ \ |\lambda_1 \lambda_2| \ d\lambda_1 d\lambda_2.$$

- The **Plancherel measure** here is $|\lambda_1 \lambda_2|\, d\lambda_1 d\lambda_2$.

Bandlimitation

Let L be the left regular representation of N. Put

$$\mathbf{H}=\left\{\begin{array}{c} f\in L^{2}\left(N\right):\mathcal{P}\left(f\right)\left(\lambda\right)=\left\{\begin{array}{c} u_{\lambda}\otimes\chi_{[0,1)^{2}}\text{ if }\lambda\in\left[-\frac{1}{2},\frac{1}{2}\right]^{2}\\ 0\text{ if }\lambda\notin\left[-\frac{1}{2},\frac{1}{2}\right]^{2}\end{array}\right.\text{ where}\\ u_{\lambda}\in L^{2}\left(\mathbb{R}^{2}\right)\end{array}\right\}$$

Then $\mathbf{H}$ is a bandlimited multiplicity-free, left-invariant subspace of $L^{2}\left(N\right)$.

Lemma 5 *Let* $\phi \in \mathbf{H}$ *such that* $\mathcal{P}(f)(\lambda_1, \lambda_2) = \chi_{[0,1)^2} \otimes \chi_{[0,1)^2}$. *For every vector* $\psi \in \mathbf{H}$,

$$\sum_{\gamma \in \Gamma} |\langle \psi, L(\gamma)\phi \rangle|^2 = \|\psi\|^2_{\mathbf{H}}.$$

Proof. The computations are very formal. First, identify Γ with $\mathbb{Z}^6$.

$$\begin{aligned}
&\sum_{\gamma\in\mathbb{Z}^6} \left|\langle \psi, L(\gamma)\phi\rangle_{\mathbf{H}}\right|^2 \\
&= \sum_{\gamma\in\mathbb{Z}^6} \left| \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^2} \langle \mathcal{P}\psi(\lambda), \pi_\lambda(\gamma)\circ\mathcal{P}\phi(\lambda)\rangle_{\mathcal{HS}} |\lambda_1\lambda_2|\, d\lambda \right|^2 \\
&= \sum_{\gamma\in\mathbb{Z}^6} \left| \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^2} \left\langle \mathcal{P}\psi(\lambda), \pi_\lambda(\gamma)|\lambda_1\lambda_2|\,\chi_{[0,1)^2}\otimes\chi_{[0,1)^2}\right\rangle_{\mathcal{HS}} d\lambda \right|^2 &(3)\\
&= \sum_{\gamma_1\in\mathbb{Z}^4}\sum_{k\in\mathbb{Z}^2} \left| \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^2} e^{-2\pi i\lambda_1 k_1} e^{-2\pi i \lambda_2 k_2} \overbrace{\left\langle \mathcal{P}\psi(\lambda), \pi_\lambda(\gamma_1)|\lambda_1\lambda_2|\,\chi_{[0,1)^2}\otimes\chi_{[0,1)^2}\right\rangle_{\mathcal{HS}}}^{f_{\gamma_1}(\lambda_1,\lambda_2)} d\lambda \right|^2 &(4)\\
&= \sum_{\gamma_1\in\mathbb{Z}^4}\sum_{k\in\mathbb{Z}^2} \left| \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^2} e^{-2\pi i\lambda_1 k_1} e^{-2\pi i \lambda_2 k_2} f_{\gamma_1}(\lambda_1,\lambda_2)\, d\lambda \right|^2 \\
&= \sum_{\gamma_1\in\mathbb{Z}^4}\sum_{k\in\mathbb{Z}^2} \left|\widehat{f}_{\gamma_1}(k_1,k_2)\right|^2 = \sum_{\gamma_1\in\mathbb{Z}^4} \|f_{\gamma_1}\|^2 \text{ (Apply Plancherel theor for } L^2(\mathbb{T}^2)\text{)} \\
&= \sum_{\gamma_1\in\mathbb{Z}^4} \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^2} |f_{\gamma_1}(\lambda_1,\lambda_2)|^2\, d\lambda &(5)\\
&= \sum_{\gamma_1\in\mathbb{Z}^4} \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^2} \left|\left\langle \mathcal{P}\psi(\lambda), \pi_\lambda(\gamma_1)|\lambda_1\lambda_2|\,\chi_{[0,1)^2}\otimes\chi_{[0,1)^2}\right\rangle_{\mathcal{HS}}\right|^2 d\lambda \text{ (subs } f_{\gamma_1} \text{ back)} &(6)\\
&= \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^2} \sum_{\gamma_1\in\mathbb{Z}^4} \left|\left\langle \mathcal{P}\psi(\lambda), \pi_\lambda(\gamma_1)\left(|\lambda_1\lambda_2|^{1/2}\chi_{[0,1)^2}\right)\otimes\chi_{[0,1)^2}\right\rangle_{\mathcal{HS}}\right|^2 |\lambda_1\lambda_2|\, d\lambda &(7)
\end{aligned}$$

Now, we write

$$\mathcal{P}\psi(\lambda) = v_\lambda\otimes\chi_{[0,1)^2} \text{ for } v_\lambda\in L^2\left(\mathbb{R}^2\right).$$

Then

$$\sum_{\gamma_1\in\mathbb{Z}^6} \left|\langle \psi, L(\gamma)\phi\rangle_{\mathbf{H}}\right|^2 = \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^2} \sum_{\gamma_1\in\mathbb{Z}^4} \left|\left\langle v_\lambda, \pi_\lambda(\gamma_1)|\lambda_1\lambda_2|^{1/2}\chi_{[0,1)^2}\right\rangle_{L^2\left(\mathbb{R}^2\right)}\right|^2 |\lambda_1\lambda_2|\, d\lambda.$$

Typically,

$$|\lambda_1\lambda_2|^{1/2}\pi_\lambda(\gamma_1)\chi_{[0,1)^2}(t_1,t_2) = |\lambda_1\lambda_2|^{1/2} e^{-2\pi i\left\langle \begin{bmatrix}\lambda_1 & \lambda_1\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix} m_1\\ m_2\end{bmatrix}, \begin{bmatrix} t_1\\ t_2\end{bmatrix}\right\rangle} \chi_{[0,1)^2}(t_1-k_1, t_2 -$$

and with well-known arguments from Gabor theory,

$$\sum_{\gamma_1 \in \Gamma_1} \left| \left\langle v_\lambda, \pi_\lambda(\gamma_1) |\lambda_1 \lambda_2|^{1/2} \chi_{[0,1)^2} \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 = \|v_\lambda\|^2_{L^2(\mathbb{R}^2)}$$

Finally

$$\sum_{\gamma \in \Gamma} |\langle \psi, L(\gamma)\phi \rangle_{\mathbf{H}}|^2 = \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^2} \|v_\lambda\|^2_{L^2(\mathbb{R}^2)} |\lambda_1\lambda_2| \, d\lambda = \|\psi\|^2_{\mathbf{H}}.$$

■

Remark 6 *We remark that*

$$
\begin{aligned}
\|\phi\|_{\mathbf{H}}^{2} &= \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^{2}} \left\|\chi_{[0,1)^{2}}\right\|_{L^{2}\left(\mathbb{R}^{2}\right)}^{2} d\lambda_{1} d\lambda_{2} \\
&= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \left|\lambda_{1}\lambda_{2}\right| d\lambda_{1} d\lambda_{2} \\
&= \frac{1}{16}.
\end{aligned}
$$

Thus $L(\Gamma)\,\phi$ *is not an orthonormal basis.*

Admissibility and sampling

Definition 7 *Let* $(\pi, \mathbf{H}_\pi)$ *denote a strongly continuous unitary representation of a locally compact group* G*. We say that the representation* $(\pi, \mathbf{H}_\pi)$ *is admissible if and only if the map* $W_\phi : \mathbf{H}_\pi \rightarrow L^2(G)$,

$$W_\phi \psi(x) = \langle \psi, \pi(x) \phi \rangle$$

defines an isometry of $\mathbf{H}$ *into* $L^2(G)$, *and we say that* ϕ *is an* ***admissible vector*** *or a* ***continuous wavelet****.*

Theorem 8 *(H. Fuhr) Let Γ be a discrete subset of G. Let ϕ be an admissible vector for $(\pi, \mathbf{H}_\pi)$ such that for all $\psi \in \mathbf{H}_\pi$*

$$\sum_{\gamma \in \Gamma} |\langle \psi, \pi(\gamma)\phi \rangle|^2 = \|\psi\|^2_{\mathbf{H}_\pi}$$

Then $\mathbf{K} = W_\phi(\mathbf{H}_\pi)$ is a Γ-sampling space, and $W_\phi(\phi)$ is the associated sinc-type function for $\mathbf{K}$.

Proof. Since we assume that π is a strongly (thus weakly) continuous homomorphism, then it is easy to see that $\mathbf{K} = W_\phi(\mathbf{H}_\pi)$ consists of continuous functions. Now, let $f = W_\phi \psi$. Then the fact that $\sum_{\gamma \in \Gamma} |\langle \psi, \pi(\gamma)\phi \rangle|^2 = \|\psi\|^2_{\mathbf{H}_\pi}$ implies that

$$f = W_\phi\left(\sum_{\gamma \in \Gamma} \langle \psi, \pi(\gamma)\phi \rangle \pi(\gamma)\phi\right).$$

So,

$$\begin{aligned} f &= \sum_{\gamma \in \Gamma} \overbrace{\langle \psi, \pi(\gamma)\phi \rangle}^{W_\phi(\psi(\gamma)) = f(\gamma)} \overbrace{\left(W_\phi \pi(\gamma)\phi\right)}^{\langle \pi(\gamma)\phi, \pi(\cdot)\phi \rangle} \\ &= \sum_{\gamma \in \Gamma} f(\gamma) \langle \pi(\gamma)\phi, \pi(\cdot)\phi \rangle \\ &= \sum_{\gamma \in \Gamma} f(\gamma) \left\langle \phi, \pi\left(\gamma^{-1}\cdot\right)\phi \right\rangle \\ &= \sum_{\gamma \in \Gamma} f(\gamma) W_\phi \phi\left(\gamma^{-1}\cdot\right) \end{aligned}$$

The above series converges in the norm of $\mathbf{H}$ and uniformly as well. Finally, since W_ϕ is an isometry then

$$\sum_{\gamma \in \Gamma} \left|W_\phi \psi(\gamma)\right|^2 = \sum_{\gamma \in \Gamma} |\langle \psi, \pi(\gamma)\phi \rangle|^2 = \|\psi\|^2_{\mathbf{H}} = \left\|W_\phi \psi\right\|^2_{L^2(G)}.$$

This completes the proof. ∎

Finally, let

$$\mathbf{H}=\left\{\begin{array}{c} f\in L^{2}\left(N\right):\mathcal{P}\left(f\right)\left(\lambda\right)=\left\{\begin{array}{c} u_{\lambda}\otimes\chi_{[0,1)^{2}}\text{ if }\lambda\in\left[-\frac{1}{2},\frac{1}{2}\right]^{2}\\ 0\text{ if }\lambda\notin\left[-\frac{1}{2},\frac{1}{2}\right]^{2}\end{array}\right.\text{ where}\\ u_{\lambda}\in L^{2}\left(\mathbb{R}^{2}\right)\end{array}\right\}$$

Let $\phi\in\mathbf{H}$ such that $\mathcal{P}\left(f\right)\left(\lambda_{1},\lambda_{2}\right)=\chi_{[0,1)^{2}}\otimes\chi_{[0,1)^{2}}$.It is easy to show that the map $W_{\phi}:\mathbf{H}\rightarrow L^{2}\left(N\right)$,

$$W_{\phi}\psi\left(x\right)=\left\langle \psi,L\left(x\right)\phi\right\rangle$$

is an isometry. Since for all $\psi\in\mathbf{H}$

$$\sum_{\gamma\in\Gamma}\left|\left\langle \psi,L\left(\gamma\right)\phi\right\rangle \right|^{2}=\left\Vert \psi\right\Vert _{\mathbf{H}}^{2}$$

then according to the above result, the Hilbert space

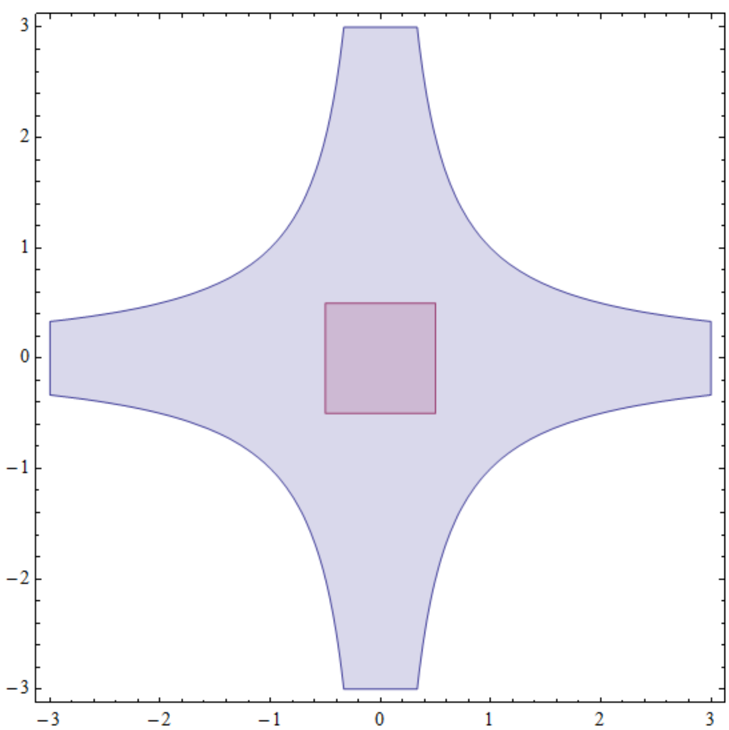
$$\mathbf{K}=W_{\phi}\left(\mathbf{H}\right)$$

is a Γ-sampling space, and $W_{\phi}\left(\phi\right)$ is the associated sinc-type function for $\mathbf{K}$.

We remark that unlike in the case of the real line, not every choice of bounded subset of Σ leads to the existence of sampling spaces for the group we consider in this example. One necessary condition for a left-invariant subspace H to be a sampling space is that the Fourier transforms of vectors in H are supported on a set E which is contained in

$$\left\{(x_1, x_2) \in \mathbb{R}^2 : |x_1 x_2| \neq 0, |x_1 x_2| \leq 1\right\}.$$

In the example illustrated above this is clearly the case as shown in this picture:



The general case

Put

$$B(\lambda) = \begin{bmatrix} \lambda [X_1, Y_1] & \cdots & \lambda [X_1, Y_d] \\ \vdots & \ddots & \vdots \\ \lambda [X_d, Y_1] & \cdots & \lambda [X_d, Y_d] \end{bmatrix},$$

and define

$$\Sigma = \left\{ \begin{array}{c} \lambda \in \mathfrak{z}^* = \mathbb{R}^{n-2d} : \det(B(\lambda)) \neq 0, \lambda(X_1) = \cdots \\ = \lambda(X_d) = \lambda(Y_1) = \cdots = \lambda(Y_d) = 0 \end{array} \right\}.$$

We say a function $f \in L^2(N)$ is **bandlimited** if its Plancherel transform is supported on a bounded measurable subset of Σ. Fix a measurable field of unit vectors $\mathbf{e} = \{\mathbf{e}_\lambda\}_{\lambda \in \Sigma}$ where $\mathbf{e}_\lambda \in L^2(\mathbb{R}^d)$. We say a Hilbert space is a multiplicity-free left-invariant subspace of $L^2(N)$ if

$$\mathbf{H}(\mathbf{e}) = \mathcal{P}^{-1}\left(\int_{\Sigma}^{\oplus} L^2\left(\mathbb{R}^d\right) \otimes \mathbf{e}_\lambda \, d\mu(\lambda) \right).$$

Next, we define

$$\mathbf{E} = \{\lambda \in \mathfrak{z}^* : |\det B(\lambda)| \neq 0, \text{ and } |\det B(\lambda)| \leq 1\}. \tag{8}$$

For any given bounded set $\mathbf{A} \subset \Sigma$, we define the corresponding multiplicity-free, bandlimited, left-invariant Hilbert subspace $\mathbf{H}(\mathbf{e}, \mathbf{A})$ as follows

$$\mathbf{H}(\mathbf{e}, \mathbf{A}) = \mathcal{P}^{-1}\left(\int_{\mathbf{A}}^{\oplus} L^2\left(\mathbb{R}^d\right) \otimes \mathbf{e}_\lambda \, |\det B(\lambda)| \, d\lambda \right). \tag{9}$$

Now, let $\mathbf{C} \subset \mathbf{I} \subset \mathbb{R}^{n-2d}$ such that the collection $\left\{\mathbf{I} + k : k \in \mathbb{Z}^{n-2d}\right\}$ forms a measurable partition of $\mathbb{R}^{n-2d}$.

Theorem 9 *Let N be a connected, simply connected nilpotent Lie group satisfying the conditions given. There exists $\phi \in \mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$ such that $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}))$ is a Γ-sampling subspace of $L^2(N)$ with sinc-type function $W_\phi(\phi)$. Moreover, $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}))$ does not generally have the interpolation property with respect to Γ.*