Sampling on some nilpotent Lie groups

Vignon S. Ouissa

Bridgewater State University

Bridgewater, Massachusetts USA
First and foremost, many thanks go to Professor Hartmut Führ for his hospitality and for hosting me during my visit. I also thank, the entire group (Maike and René) for the wonderful company. It is really a pleasure to visit Aachen, Germany.
Sampling on the real line

How can we recover a function $f : \mathbb{R} \to \mathbb{C}$ if we only know a countable set of values

$$(f(k))_{k \in I}?$$

Formulated this way, the problem is ill-posed since there are infinitely many functions that take the same prescribed values on a given countable set. We consider the Paley-Wiener space

$$\mathcal{PW}(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : \text{supp } \hat{f} \subseteq \left[ -\frac{1}{2}, \frac{1}{2} \right] \right\}.$$ 

The Paley-Wiener space consists of equivalence classes of functions. Since the Fourier transform of these functions has compact support, each of the equivalence classes contains a continuous function.
Define the sinc-function by

\[
\text{sinc}(x) = \begin{cases} 
\frac{\sin(\pi x)}{\pi x} & \text{if } x \neq 0 \\
1 & \text{if } x = 0 
\end{cases}
\]

**Theorem 1** (Shannon’s sampling theorem) The functions \(\{\text{sinc}(x - k)\}_{k \in \mathbb{Z}}\) form an orthonormal basis for \(PW\). If \(f \in PW \cap C(\mathbb{R})\), then

\[
f(x) = \sum_{k \in \mathbb{Z}} f(k) \text{sinc}(x - k)
\]

with convergence of the symmetric partial sums in \(L^2(\mathbb{R})\) and point-wise for all \(x \in \mathbb{R}\).
Sampling on locally compact groups

Let $N$ be a locally compact group, and let $\Gamma$ be a discrete subset of $N$. Let $H$ be a left-invariant closed subspace of $L^2(N)$ consisting of continuous functions. We call $H$ a sampling space with respect to $\Gamma$ (or $\Gamma$-sampling space) if

1. The restriction map $R_{\Gamma} : H \to l^2(\Gamma)$, $R_{\Gamma} f = (f(\gamma))_{\gamma \in \Gamma}$ is an isometry.

2. There exists $S \in H$ such that for any vector $f \in H$,

$$f(x) = \sum_{\gamma \in \Gamma} f(\gamma) S(\gamma^{-1}x)$$

with convergence in the norm of $H$.

3. The vector $S$ is called a sinc-type function. Moreover, if $R_{\Gamma}$ is surjective, we say that the sampling space $H$ has the interpolation property.
Sampling on the Heisenberg group

Let $\mathbb{H}$ be the three-dimensional Heisenberg Lie group

$$\mathbb{H} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \right\}$$

and

$$\Gamma = \left\{ \begin{bmatrix} 1 & m & k \\ 0 & 1 & l \\ 0 & 0 & 1 \end{bmatrix} : \begin{bmatrix} m \\ l \\ k \end{bmatrix} \in \mathbb{Z}^3 \right\}$$

Then $\Gamma$ is a discrete subgroup of the Heisenberg group.

**Theorem 2** (H. Fuhr, 2005, B. Currey, A. Mayeli, 2009) The Heisenberg group admits sampling spaces with respect to $\Gamma$. Moreover, the Heisenberg group admits a sampling space with respect to $\Gamma$ which also has the interpolation property.
Some basic questions

• How far can we extend the results on the Heisenberg group to other non-commutative nilpotent Lie groups?
• What is a nilpotent Lie group?
  - I will give you a crash course on the topic.
• Do we have some kind of Fourier theory available for this class of groups?
  - We will brush the basic.
• Can we use such a theory?
  - To a certain extend, we have some hope.
  - The general case is still murky (at least to me)
• What do we mean by bandlimited spaces on nilpotent Lie groups?
  - We will brush the basic.
• What should $\Gamma$ be? Could $\Gamma$ be a subgroup?
  - For a large class of groups, we will see that $\Gamma$ can be taken to be a subgroup.
Basic facts about Lie groups and Lie algebra

- **Abstract definition** the notion of **Lie group** mimic the definition of a topological group. Let $G$ be a **topological group**. Roughly speaking, suppose that there is an analytic structure on the set $G$ which is compatible with its topology, converting it into an analytic manifold such that
  
  $$(x, y) \mapsto xy \text{ and } x \mapsto x^{-1}$$

  are both analytic. Then $G$ endowed with this structure is called a Lie group.

- A vector space $g$ over $\mathbb{R}$ is called a real **Lie algebra** if there is a map
  
  $$g \times g \ni (X, Y) \mapsto [X, Y] \in g$$

  such that
  
  - $(X, Y) \mapsto [X, Y]$ is bilinear
  - $[X, Y] = -[Y, X]$
  - $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity)

- If $g$ is a Lie algebra over the reals, and $\{X_1, \cdots, X_n\}$ is a basis of $g$ then there are uniquely determined **(structure) constants** $c_{ijk}$ such that
  
  $$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k.$$
• A **homomorphism** \( \varphi : g_1 \rightarrow g_2 \) of Lie algebras is a linear map satisfying \( \varphi ([X,Y]) = [\varphi (X), \varphi (Y)] \).

• A Lie algebra \( g \) is called **abelian** if \( [X,Y] = 0 \) for all \( X,Y \in g \).

• Each associated algebra is a Lie algebra when endowed with the bracket

\[
[X,Y] = XY - YX.
\]

To verify this, it is enough to check the axioms described above.

• Let \( X \) be a matrix of order \( n \). Define the power series

\[
\exp X = \sum_{k=0}^{\infty} \frac{X^m}{m!}.
\]

– For any matrix \( X \), \( \exp X \) is convergent, and \( \exp X \) is **always invertible**.

– \( \exp 0 = id, \exp (-X) = (\exp X)^{-1} \)

– In general \( \exp (X + Y) \neq \exp (X) \exp (Y) \).

\[
\exp \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e & e - 1 \\ 0 & 1 \end{bmatrix} \neq \exp \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \exp \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e & e \\ 0 & 1 \end{bmatrix}
\]
– **Example.** Let $\lambda \in \mathbb{R}$. Then

$$
\exp \begin{bmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 \\
0 & 0 & \lambda
\end{bmatrix} =
\begin{bmatrix}
e^\lambda & e^\lambda & \frac{1}{2}e^\lambda \\
e^\lambda & e^\lambda \\
e^\lambda & 0 & e^\lambda
\end{bmatrix}
$$

– **Exercise.** Let $t \in \mathbb{R}$

$$
\exp \begin{bmatrix}
0 & -t \\
t & 0
\end{bmatrix} =?
$$
Lie groups and Lie algebra

• A closed subgroup $G \subseteq GL(n, \mathbb{C})$ is called a linear Lie group.

• Define $\text{Lie}(G) = \{X \in M(n, \mathbb{C}) : \exp(tX) \in G \text{ for all } t \in \mathbb{R}\}$

$\text{Lie}(G)$ is called the Lie algebra of $G$ and is denoted $\mathfrak{g}$.

Example 3

1. $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are matrix Lie groups.

2. The Lie algebra of $GL(n, \mathbb{C})$ is the set of all real matrices of order $n$. $\text{Lie}(GL(n, \mathbb{C})) = M(n, \mathbb{R})$

3. $SL(n, \mathbb{R})$ and $SL(n, \mathbb{C})$ are matrix Lie groups.

4. The Lie algebra of $SL(n, \mathbb{R})$ is $\text{sl}(n, \mathbb{R}) = \{X : \text{trace}(X) = 0\}$.

5. (Important) The Heisenberg group

$$\mathbb{H} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$
is a matrix Lie group. Observe that

\[
\exp \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.
\]

Thus,

\[\mathfrak{h} = \left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}\]

is the Lie algebra of the Heisenberg group. Moreover, given

\[X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\]

it is not hard to verify that \([X, Y] = Z\) is the only non-trivial Lie brackets for \(\mathfrak{h}\).

6. \(N_n\) (triangular matrices with ones on the diagonal) is a matrix Lie group.
Nilpotent Lie algebra

- Let $\mathfrak{g}$ be a Lie algebra over the real. The descending central series of $\mathfrak{g}$ is defined inductively by

\[ \mathfrak{g}^{(1)} = \mathfrak{g} \]

\[ \mathfrak{g}^{(n+1)} = [\mathfrak{g}, \mathfrak{g}^{(n)}] = \mathbb{R}\text{-span}\left\{ [X, Y] : X \in \mathfrak{g}, Y \in \mathfrak{g}^{(n)} \right\} \]

- We say that $\mathfrak{g}$ is a nilpotent Lie algebra if there is an integer $n$ such that $\mathfrak{g}^{(n+1)} = \{0\}$. If additionally, $\mathfrak{g}^{(n)}$ is not the zero vector space then $\mathfrak{g}$ is called an $n$-step nilpotent Lie algebra.

- For the Heisenberg Lie algebra. Recall that $\mathfrak{h} = \mathbb{R}X_1 + \mathbb{R}X_2 + \mathbb{R}X_3$ and $[X_3, X_2] = X_1$

\[ \mathfrak{h}^{(1)} = \mathfrak{h} \]

\[ \mathfrak{h}^{(2)} = [\mathfrak{h}, \mathfrak{h}^{(1)}] = \mathbb{R}Z = \text{center} \]

\[ \mathfrak{h}^{(3)} = [\mathfrak{h}, \mathfrak{h}^{(2)}] = \{0\} \]

Thus, $\mathfrak{h}$ is a two-step nilpotent Lie algebra.
Let \( \mathfrak{r} \) be the Lie algebra spanned by \( \{ X_1, X_2, X_3, X_4, X_5 \} \) such that
\[
[X_4, X_3] = 2X_1, [X_5, X_3] = 2X_2, [X_5, X_2] = 2X_1
\]

A matrix realization of \( \mathfrak{r} \)
\[
\mathfrak{r} = \left\{ \begin{bmatrix} 0 & 2x_5 & 2x_4 & x_1 \\ 0 & 0 & 2x_5 & x_2 \\ 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \mathbb{R}^5 \right\}.
\]

**Exercise** Verify that \( \mathfrak{r} \) is a three-step nilpotent Lie algebra.
Theorem 4  If $H$ is connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{h}$ then

$$\exp : \mathfrak{h} \to H$$

is an analytic diffeomorphism. Additionally, every connected, simply connected nilpotent Lie group has a faithful embedding as a closed subgroup of $N_n$ for some $n$.

Example 5  Let $\mathfrak{r}$ be the Lie algebra spanned by $\{X_1, X_2, X_3, X_4, X_5\}$ such that

$$[X_4, X_3] = 2X_1, [X_5, X_3] = 2X_2, [X_5, X_2] = 2X_1.$$

Then

$$\exp \left( \sum_{k=1}^{3} x_k X_k \right) \exp \left( \sum_{k=4}^{5} x_k X_k \right) \mapsto \begin{bmatrix} 1 & 2x_5 & 2x_5^2 + 2x_4 & x_1 \\ 0 & 1 & 2x_5 & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

defines a Lie group isomorphism between $\exp \mathfrak{r} = \mathbb{R}$ and a closed subgroup of $N_4$. 
Definition 6 Let $\mathfrak{h}$ be a Lie algebra. Given $X \in \mathfrak{h}$

$$ad_X : \mathfrak{h} \to \mathfrak{h}$$

is a linear map described as follows

$$ad_X(Y) = [X, Y].$$

The map $ad : X \mapsto ad_X$ is actually a linear representation of $\mathfrak{h}$ in $\mathfrak{gl}(\mathfrak{h})$. In the situation where $\mathfrak{h}$ is a nilpotent Lie algebra then the adjoint representation is never faithful since it has a non-trivial center.

Example 7 Let $\mathfrak{h}$ be the Lie algebra spanned by $\{X_1, X_2, X_3\}$ such that $[X_3, X_2] = X_1$. Then

$$[ad_{X_1}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [ad_{X_2}] = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [ad_{X_3}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
Let $G$ be a nilpotent Lie group with Lie algebra $\mathfrak{g}$ and denote the dual of $\mathfrak{g}$ by $\mathfrak{g}^*$. Then $G$ acts on $\mathfrak{g}^*$ by the contragradient of the adjoint map (the **coadjoint** action)

$$\langle \exp (X) \cdot \lambda, Y \rangle = \langle \lambda, e^{-adX}Y \rangle = \langle (e^{-adX})^* \lambda, e^{-adX}Y \rangle$$

- (The simplest non-commutative case) Let $\mathfrak{h}$ be the Heisenberg Lie algebra. Then

$$\begin{bmatrix}
1 & x & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
\lambda \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
-x & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\lambda \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
\lambda \\
-x\lambda \\
0
\end{bmatrix}$$

$$\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & y \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
\lambda \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
y & 0 & 1
\end{bmatrix} \begin{bmatrix}
\lambda \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
\lambda \\
0 \\
y\lambda
\end{bmatrix}$$

$$\begin{bmatrix}
1 & 0 & z \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
\lambda \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\lambda \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
\lambda \\
0 \\
0
\end{bmatrix}$$

- If $\lambda \neq 0$ then corresponding coadjoint orbits are two-dimensional.
- If $\lambda = 0$ then corresponding coadjoint orbits are zero-dimensional (points)
- Let $\Omega = \{\lambda X_1^* + \alpha X_2^* + \xi X_3^* \in n^* : \lambda \neq 0\}$. Then $\Omega$ is an-invariant, algebraic set. Moreover, a cross-section for the coadjoint orbits is by $\Sigma = \{\lambda X_1^* \in n^* : \lambda \neq 0\}$
Let $\mathfrak{r}$ be the Lie algebra spanned by $\{X_1, X_2, X_3, X_4, X_5\}$ such that

\[ [X_4, X_3] = 2X_1, [X_5, X_3] = 2X_2, [X_5, X_2] = 2X_1. \]

its Lie group $R$. Then

\[
\begin{bmatrix}
0 & 2x_5 & 2x_4 & 0 \\
0 & 0 & 2x_5 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\lambda \\
0 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
-2x_5 & 1 & 0 & 0 & 0 \\
2x_5^2 - 2x_4 & -2x_5 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\lambda \\
0 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
\lambda \\
-2\lambda x_5 \\
2\lambda x_5^2 - 2\lambda x_4 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_2 & 0 \\
0 & 0 & 0 & x_3 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\lambda \\
0 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
2x_3 & 0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\lambda \\
0 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
\lambda \\
0 \\
0 \\
2\lambda x_3
\end{bmatrix}
\]

* If $\lambda \neq 0$ then corresponding coadjoint orbits are four-dimensional submanifolds.
* Let

\[ \Omega = \{\lambda X_1^* + aX_2^* + bX_3^* + cX_4^* + dX_5^* \in \mathfrak{n}^* : \lambda \neq 0\} \]

Then $\Omega$ is an-invariant, algebraic set. Moreover, a cross-section for the coadjoint orbits is by

\[ \Sigma = \{\lambda X_1^* \in \mathfrak{n}^* : \lambda \neq 0\} \]
Let \( G \) be a locally compact group. A **unitary representation** of \( G \) is a homomorphism \( \pi \) from \( G \) into the group of unitary operators on some non-zero Hilbert space which is continuous with respect to the strong operator topology. That is

\[
\pi : G \to \mathcal{U}(H_\pi)
\]

such that

1. \( \pi(xy) = \pi(x) \pi(y) \)
2. \( \pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^* \)
3. \( x \to \pi(x)u \) is continuous from \( G \) to \( H_\pi \) for any \( u \in H_\pi \).

It is worth noticing that strong continuity and weak topologies are the same in \( \mathcal{U}(H_\pi) \). Thus, \( x \mapsto \langle \pi(x)u, v \rangle \) is continuous from \( G \) to \( \mathbb{C} \). Suppose that \( \{T_\alpha\} \) is a net of unitary operators convergent to \( T \). Then for any \( u \in H_\pi \)

\[
\| (T_\alpha - T)u \|^2 = \|T_\alpha u\|^2 + \|Tu\|^2 - 2 \Re \langle T_\alpha u, Tu \rangle \\
= 2 \|u\|^2 - 2 \Re \langle T_\alpha u, Tu \rangle
\]

Thus, \( \| (T_\alpha - T)u \|^2 \to 0. \)
• If \( \pi_1 \) and \( \pi_2 \) are unitary representations of \( G \), an \textbf{intertwining operator} for \( \pi_1 \) and \( \pi_2 \) is a bounded linear operator \( T : H_{\pi_1} \rightarrow H_{\pi_2} \) such that \( T\pi_1(x) = \pi_2(x)T \) for all \( x \in G \). We say that \( \pi_1 \) and \( \pi_2 \) are unitarily equivalent if there is a unitary operator \( T \) which is intertwining the representations.

• Now suppose that \( K \) is a closed subspace of \( H_\pi \) such that \( \pi(G)K \subset K \). We say that \( K \) is a \( \pi \)-invariant Hilbert subspace of \( H_\pi \). Moreover, a representation \( \pi \) is irreducible if the only \( \pi \)-invariant subspaces are the trivial ones. That is the zero vector space and \( H_\pi \).

• \textbf{(The unitary dual)} Given a locally compact group \( G \), one of the most important questions in representation theory is to classify all of its unitary irreducible representations. The set of all irreducible representations of \( G \) up to equivalence is called the unitary dual and is denoted \( \hat{G} \)

\[
\hat{G} = \{ [\pi] : \pi \text{ is irreducible} \}
\]

• \textbf{(Orbit Method/Kirillov Method)} Let \( G \) be a simply connected nilpotent Lie group with nilpotent Lie algebra \( g \). All irreducible unitary representations of \( G \) are parametrized by the set of coadjoint orbits.

  - In this class of groups, a cross-section is always computable
  - There is a well-known procedure which is exploited to realize the irreducible representation corresponding to every element in the cross-section.
Back to sampling

To study sampling spaces, we need the following ingredients.

- The unitary dual of the group.
- Fourier/Plancherel transform.
- A description of band-limited spaces.
- A suitable discrete set $\Gamma$. 
• Consider the Heisenberg group

\[ N = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\} \] and \[ \Gamma = \left\{ \begin{bmatrix} 1 & m & k \\ 0 & 1 & l \\ 0 & 0 & 1 \end{bmatrix} : k, l, m \in \mathbb{Z} \right\}. \]

Recall that \( N \) is a nilpotent Lie group which is two step. Thus, it is not commutative.

- The unitary dual of \( N \) is up to a null set parametrized by the punctured line \( \mathbb{R} \setminus \{0\} \).
- For each \( \lambda \in \mathbb{R} \setminus \{0\} \), the corresponding irreducible representation \( \pi_\lambda \) is realized as acting on \( L^2(\mathbb{R}) \) as follows

\[ \pi_\lambda (x, y, z) f(t) = e^{2\pi i \lambda z} e^{-2\pi i \lambda y t} f(t - x). \]

- The Plancherel measure is the weighted Lebesgue measure \( |\lambda| \, d\lambda \). There is a procedure for computing the formula.

- The Plancherel transform

\[ \mathcal{P} : L^2(N) \to L^2 \left( \mathbb{R} \setminus \{0\}, L^2(\mathbb{R}) \otimes L^2(\mathbb{R}), |\lambda| \, d\lambda \right) \]

is an operator-valued map which is weakly defined as

\[ \mathcal{F} (f) (\lambda) = \int_N f(n) \pi_\lambda (n) \, dn. \]
Moreover,
\[ \|f\|_{L^2(N)}^2 = \int_{\mathbb{R} \setminus \{0\}} \|\mathcal{P}(f)(\lambda)\|_{HS}^2 \, d\mu(\lambda). \]

- We say a function \( f \in L^2(N) \) is **bandlimited** if its Plancherel transform is supported on a bounded measurable subset of \( \mathbb{R} \setminus \{0\} \).

- In the simplest case, fix a measurable field of unit vectors \( \mathbf{e} = (\mathbf{e}_\lambda) \) where \( \mathbf{e}_\lambda \in L^2(\mathbb{R}) \). We say a Hilbert space \( \mathcal{H}(\mathbf{e}) \) is a **multiplicity-free** left-invariant subspace of \( L^2(N) \) if for all \( f \in \mathcal{H}(\mathbf{e}) \)
\[ \mathcal{P}(f)(\lambda) = u_\lambda \otimes \mathbf{e}_\lambda \] (rank-one operators)

**Theorem 8** (*Führ, Currey and Mayeli*) Let
\[ \mathbf{E}_a = \{ \lambda \in \mathbb{R}^*: \lambda \neq 0, \text{ and } |\lambda| \leq a \} \]

where \( a \in (0, 1] \). Let \( \mathcal{H}(\mathbf{e}, \mathbf{E}_a) \) be the multiplicity-free subspace of \( \mathcal{H}(\mathbf{e}) \) containing functions with Fourier transforms supported on \( \mathbf{E}_a \). Next, let
\[ W_\phi : \mathcal{H}(\mathbf{e}, \mathbf{E}_a) \rightarrow L^2(N), \]
such that \( W_\phi \psi(x) = \langle \psi, L(x) \phi \rangle \). There exists \( \phi \in \mathcal{H}(\mathbf{e}, \mathbf{E}_a) \) such that \( W_\phi(\mathcal{H}(\mathbf{e}, \mathbf{E}_a)) \) is a \( \Gamma \)-sampling subspace of \( L^2(N) \) with sinc-type function \( W_\phi(\phi) \). Moreover, \( W_\phi(\mathcal{H}(\mathbf{e}, \mathbf{E}_a)) \) has the interpolation property with respect to \( \Gamma \) if \( a = 1 \).

**Theorem 9** (*Führ*) In contrast with the real case, sampling space cannot have arbitrary band-limitation.
An extension of the results above

- Let us suppose that \( n = a \oplus b \oplus \mathfrak{z} \), where \([a, b] \subseteq \mathfrak{z}\), \(a, b, \mathfrak{z}\) are abelian algebras such that

\[
\begin{align*}
    a &= \mathbb{R}\text{-span} \{X_1, X_2, \ldots, X_d\}, \\
    b &= \mathbb{R}\text{-span} \{Y_1, Y_2, \ldots, Y_d\}, \\
    \mathfrak{z} &= \mathbb{R}\text{-span} \{Z_1, Z_2, \ldots, Z_{n-2d}\},
\end{align*}
\]

\(d \geq 1, n > 2d\) and

\[
\text{det} \begin{bmatrix}
    [X_1, Y_1] & [X_1, Y_2] & \cdots & [X_1, Y_d] \\
    [X_2, Y_1] & [X_2, Y_2] & \cdots & [X_2, Y_d] \\
    \vdots & \vdots & \cdots & \vdots \\
    [X_d, Y_1] & [X_d, Y_2] & \cdots & [X_d, Y_d]
\end{bmatrix}
\]

is a non-vanishing homogeneous polynomial in the unknowns \(Z_1, \ldots, Z_{n-2d}\).

- The determinant condition essentially ensures that square-integrability of the important irreducible representations of \(N\) modulo the center.
- Observe that if \(n\) is the Heisenberg Lie algebra then \(a = \mathbb{R}X_1, b = \mathbb{R}Y_1, \mathfrak{z} = \mathbb{R}Z_1\) and \(\text{det}[X_1, Y_1] = [X_1, Y_1] = Z_1\).

- The unitary dual of \(N\) is parametrized by the smooth manifold

\[
\Sigma = \left\{ \lambda \in \mathfrak{n}^* : \text{det}(B(\lambda)) \neq 0, \lambda(X_1) = \cdots = \lambda(X_d) = \lambda(Y_1) = \cdots = \lambda(Y_d) = 0 \right\}
\]
which is naturally identified with a Zariski open subset of $\mathbb{R}^{n-2d}$.

- Let $d\lambda$ be the Lebesgue measure on $\Sigma$. The Plancherel measure for the group $N$ is supported on $\Sigma$ and is equal to $d\mu(\lambda) = |\det(B(\lambda))| \, d\lambda$. Here

$$B(\lambda) = \begin{bmatrix}
\lambda [X_1,Y_1] & \lambda [X_1,Y_2] & \cdots & \lambda [X_1,Y_d] \\
\lambda [X_2,Y_1] & \lambda [X_2,Y_2] & \cdots & \lambda [X_2,Y_d] \\
\vdots & \vdots & \ddots & \vdots \\
\lambda [X_d,Y_1] & \lambda [X_d,Y_2] & \cdots & \lambda [X_d,Y_d]
\end{bmatrix}.$$

- The unitary dual of $N$ which we denote by $\hat{N}$ is up to a null set equal to $\{\pi_\lambda : \lambda \in \Sigma\}$ where each representation $\pi_\lambda$ is realized as acting in $L^2(\mathbb{R}^d)$.

- Let

$$Z = \sum_{i=1}^{n-2d} z_i Z_i, \quad Y = \sum_{i=1}^d y_i Y_i \quad \text{and} \quad X = \sum_{i=1}^d x_i X_i.$$

Then

$$[\pi_\lambda \exp(Z)f](t) = e^{2\pi i \lambda (\sum_{i=1}^{n-2d} z_i Z_i)} \cdot f(t) \quad \text{(characters)}$$

$$\pi_\lambda(\exp(Y))f(t) = e^{-2\pi i (B(\lambda)y,t)} \cdot f(t) \quad \text{(modulations)}$$

$$\pi_\lambda(\exp(X))f(t) = f(t - x) \quad \text{(translations)}$$
- Put
  \[ \Gamma = \exp\left(\sum_{i=1}^{n-2d} ZZ_i\right) \exp\left(\sum_{i=1}^{d} ZY_i\right) \exp\left(\sum_{i=1}^{d} ZX_i\right) \subset N. \]
- The Plancherel transform
  \[ \mathcal{P} : L^2(N) \to L^2\left(\Sigma, L^2\left(\mathbb{R}^d\right) \otimes L^2\left(\mathbb{R}^d\right) |\det (B (\lambda))| d\lambda\right) \]
  and we have a general Plancherel theorem
  \[ \|f\|_{L^2(N)}^2 = \int_{\Sigma} \|\mathcal{P} (f) (\lambda)\|_{\mathcal{H}_s}^2 \ |\det (B (\lambda))| d\lambda. \]
- Fix a unit vector \( e \in L^2(\mathbb{R}^d) \), and let
  \[ E = \{ \lambda \in \mathbb{R}^d : |\det B (\lambda)| \neq 0, \ \text{and} \ |\det B (\lambda)| \leq 1 \} \]
  * Let \( C \subseteq E \) such that \( C \) is contained in a fundamental domain of \( \mathbb{Z}^{n-2d} \) and \( |E \cap C| > 0 \).
  * Define \( H (e, E \cap C) \subset L^2(N) \)
  \[ \mathcal{P} (H (e, E \cap C)) = L^2\left( E \cap C, L^2\left(\mathbb{R}^d\right) \otimes e, |\det (B (\lambda))| d\lambda\right) \]

**Theorem 10** (O) There exists \( \phi \in H (e, E \cap C) \) such that \( W_\phi (H (e, E \cap C)) \) is a \( \Gamma \)-sampling subspace of \( L^2(N) \) with sinc-type function \( W_\phi (\phi) \). Moreover, \( W_\phi (H (e, E \cap C)) \) does not generally have the interpolation property with respect to \( \Gamma \).
Let $N$ be a nilpotent Lie group with Lie algebra spanned by the vectors $\{Z_1, Z_2, Y_1, Y_2, X_1, X_2\}$ with the following non-trivial Lie brackets: $[X_1, Y_1] = Z_1, [X_2, Y_2] = Z_2$. In this example, the discrete set

$$\Gamma = \exp (ZZ_1 + ZZ_2) \exp (ZY_1 + ZY_2) \exp (ZX_1 + ZX_2)$$

is actually a uniform subgroup of the Lie group $N$.

- $N$ is a direct product of two Heisenberg groups and satisfies all properties described above.
- The Plancherel measure is $|\lambda_1 \lambda_2| d\lambda_1 d\lambda_2$.
- $E = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_1 \lambda_2 \neq 0 \text{ and } |\lambda_1 \lambda_2| \leq 1\}$.
- Let $C = [-1, 1]^2$. Observe that

$$\mu(E \cap C) = \int_{-1}^{1} \int_{-1}^{1} |\lambda_1 \lambda_2| d\lambda_1 d\lambda_2 = 1.$$
• Let \( \phi \in H(e, E \cap C) \) such that \( P\phi(\lambda) = \chi_{[0,1]^2}(t) \otimes \chi_{[0,1]^2}(t) \).

• \( L(\Gamma)\phi \) is an orthonormal basis for \( H(e, E \cap C) \) and \( W_\phi(H(e, E \cap C)) \) is a \( \Gamma \)-sampling space with the interpolation property.

• It is perhaps worth highlighting the fact that in order to obtain a sampling space it is necessary to select the spectral set such that it is essentially contained in the blue region.
Remark 11

1. The class of groups considered here is very small, and the proofs heavily rely on well-known results of time-frequency analysis.

2. In the case where $\Gamma$ is a subgroup, it is worth pointing out that the representation theory of such groups is not well-understood and is quite wild. Thus, sampling theory on nilpotent Lie groups gives some insights into the representation theory of $\Gamma$. Perhaps these could be exploited to shed some light on the theory of such discrete groups.

3. A referee once pointed out to me that the results of Currey and Mayeli lead naturally to a central decomposition of the left regular representation of $\Gamma$.

4. In summary, the first point is discouraging but the second point seems to balance the first.
Extension of the results to a larger class of groups

• Let \( n \) be a nilpotent Lie algebra of dimension \( n \), and let \( n^* \) be the dual vector space of \( n \). A **polarizing subalgebra** \( p(\lambda) \) subordinated to a linear functional \( \lambda \in n^* \) is a maximal algebra satisfying

\[
[p(\lambda), p(\lambda)] = \text{Span}\{[X, Y] \in n : X, Y \in p(\lambda)\} \subseteq \ker(\lambda).
\]

• Recall again that the coadjoint action on the dual of \( n \) is the dual of the adjoint action of \( N = \exp n \) on \( n \). In other words, for \( X \in n \), and a linear functional \( \lambda \in n^* \), the coadjoint action is defined as follows:

\[
(\exp X \cdot \lambda)(Y) = \left\langle \left( e^{-ad(X)} \right)^* \lambda, Y \right\rangle = \left[ \left( e^{-ad(X)} \right)^* \lambda \right](Y).
\]

• \( p \) is called a **constant polarization** subalgebra of \( n \) if \( p \) is a polarization algebra for almost all linear functionals
**Condition 12** Let us suppose that $N = P \rtimes M = \exp(p) \rtimes \exp(m)$ is a simply connected, connected non-commutative nilpotent Lie group with Lie algebra $\mathfrak{n} = \mathfrak{p} + \mathfrak{m}$ such that

1. $\mathfrak{p}$ is a **constant polarization ideal** of $\mathfrak{n}$ (thus commutative) $\mathfrak{m}$ is commutative as well, $p = \dim \mathfrak{p}, m = \dim \mathfrak{m}$ and $p - m > 0$.

2. There exists a strong Malcev basis $\{Z_1, \cdots, Z_p, A_1, \cdots, A_m\}$ for $\mathfrak{n}$ such that $\{Z_1, \cdots, Z_p\}$ is a basis for $\mathfrak{p}$, $\{A_1, \cdots, A_m\}$ is a basis for $\mathfrak{m}$ and

$$\Gamma = \exp(ZZ_1 + \cdots + ZZ_p) \exp(ZA_1 + \cdots + ZA_m)$$

is a discrete uniform subgroup of $N$. This is equivalent to the fact that $\mathfrak{n}$ has **rational structure constants**.
Example 13 The following collection contains a small list of groups meeting all conditions described above.

1. Let $N$ be a simply connected, connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ of dimension four or less.

2. (nilpotent Lie groups of the type $\mathbb{R}^p \times \mathbb{R}$) Let $N$ be a simply connected, connected nilpotent Lie group with Lie algebra spanned by $Z_1, Z_2, \ldots, Z_p, A_1$, the vector space generated by $Z_1, Z_2, \ldots, Z_p$ is a commutative ideal, $[adA_1]|_p = A$ is a nonzero rational upper triangular nilpotent matrix of order $p$, and $e^A Z^p \subseteq Z^p$.

3. Let $N$ be a simply connected, connected nilpotent Lie group with Lie algebra spanned by the vectors $Z_1, Z_2, \ldots, Z_p, A_1, \ldots, A_m$ where $p = m + 1$, the vector space generated by $Z_1, Z_2, \ldots, Z_p$ is a commutative ideal, the vector space generated by $A_1, \ldots, A_m$ is commutative and the matrix representation of $ad (\sum_{k=1}^{m} t_k A_k)$ restricted to $p$ is given by

$$A(t) = \left[ ad \sum_{k=1}^{m} t_k A_k \right] \bigg|_p = m! \begin{bmatrix} 0 & t_1 & t_2 & \cdots & t_{m-1} & t_m \\ 0 & t_1 & t_2 & \cdots & t_{m-1} & \vdots \\ 0 & t_1 & \ddots & \vdots & \vdots & \vdots \\ 0 & \ddots & \ddots & t_2 & \vdots & \vdots \\ 0 & \ddots & \ddots & t_1 & \ddots & \vdots \\ 0 & & & & & 0 \end{bmatrix}.$$
• Let $\Sigma$ be a parameterizing set for the unitary dual of $N$. In other words, $\Sigma$ is a cross-section for the coadjoint orbits in an $N$-invariant open and dense set $\Omega \subset \mathfrak{n}^*$.

• Let $L$ be the left regular representation of $N$ acting on $L^2(N)$ by left translations. Let

$$
\mathcal{P} : L^2(N) \rightarrow L^2\left(\Sigma, L^2(\mathbb{R}^m) \otimes L^2(\mathbb{R}^m), d\mu(\lambda)\right)
$$

be the Plancherel transform which defines a unitary map on $L^2(N)$.

• The Plancherel measure $d\mu(\lambda)$ is equal to $|P(\lambda)| d\lambda$. $P(\lambda)$ is a polynomial defined over $\Sigma$ and $d\lambda$ (I will give you the formula later)

• Given a $\mu$-measurable bounded set $A \subset \Sigma$, and a fixed measurable field of unit vectors $(u(\lambda))_{\lambda \in A}$ in $L^2(\mathbb{R}^m)$, we define the Hilbert space $H_A$ as follows. $H_A$ consists of vectors $f \in L^2(N)$ such that

$$
\mathcal{P} f(\lambda) = \begin{cases} 
v(\lambda) \otimes u(\lambda) & \text{if } \lambda \in A \\ 0 & \text{if } \lambda \not\in A \end{cases}
$$

and $(v(\lambda) \otimes u(\lambda))_{\lambda \in A}$ is a measurable field of rank-one operators. Consequently, $H_A$ is a left-invariant multiplicity-free band-limited subspace of $L^2(N)$ which is naturally identified with the function space $L^2(A \times \mathbb{R}^m)$.
• The irreducible representations are given by

\[
\begin{align*}
\sigma_\lambda \left( \exp \left( \sum_{j=1}^{m} a_j A_j \right) \right) f (x) &= f (x - a) \quad \text{(translation)} \\
\sigma_\lambda \left( \exp \left( \sum_{j=1}^{m} z_j Z_j \right) \right) f (x) &= \exp \left( 2 \pi i \left\langle \lambda, e^{-ad \sum_{j=1}^{m} t_j A_j} \sum_{j=1}^{m} z_j Z_j \right\rangle \right) f (x)
\end{align*}
\]

The second one above is a higher order modulation.

• Let \( \text{proj}_{p^*} : n^* \to p^* \) be the restriction mapping given by

\[
\text{proj}_{p^*} \left( \sum_{j=1}^{p} \left( \lambda_j Z_j^* \right) + \sum_{j=1}^{m} \left( \xi_j A_j^* \right) \right) = \sum_{j=1}^{n} \lambda_j Z_j^*
\]

where \( \{ Z_1^*, \cdots, Z_p^*, A_1^*, \cdots, A_m^* \} \) is a dual basis for \( \{ Z_1, \cdots, Z_p, A_1, \cdots, A_m \} \).

• Define the map \( \beta : \Sigma \times \mathbb{R}^m \to \mathbb{R}^p \) by

\[
\beta (\lambda, t) = \text{proj}_{p^*} \left( \exp \left( \sum_{j=1}^{m} t_j A_j \right) \cdot \lambda \right) = \text{proj}_{p^*} \left( \left( e^{-ad \sum_{j=1}^{m} t_j A_j} \right)^* \lambda \right)
\]

where \( \cdot \) denotes the coadjoint action of \( N \).
Lemma 14 Let $J_{\beta}(\lambda, s_1, \ldots, s_m)$ be the Jacobian of the smooth map $\beta$. The Plancherel measure of $N$ is up to multiplication by a constant equal to

$$d\mu(\lambda) = |\det J_{\beta}(\lambda, 0)| \, d\lambda$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{R}^{\dim \Sigma}$. Consequently, $|P(\lambda)| = |\det J_{\beta}(\lambda, 0)|$.

Theorem 15 (O) Let $N = P \rtimes M = \exp(p) \rtimes \exp(m)$ be a simply connected, connected nilpotent Lie group with Lie algebra $n$ satisfying the conditions above. Let $A$ be a $\mu$-measurable bounded subset of $\Sigma$.

1. If $\beta(A \times [0, 1]^m)$ has positive Lebesgue measure in $\mathbb{R}^p$ and is contained in a fundamental domain of $\mathbb{Z}^p$ then there exists a vector $\eta \in H_A$ such that $W_{\eta}(H_A)$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with respect to $\Gamma$.

2. If $\beta(A \times [0, 1]^m)$ is equal to a fundamental domain of $\mathbb{Z}^p$ then there exists a vector $\eta \in H_A$ such that $W_{\eta}(H_A)$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with the interpolation property with respect to $\Gamma$. 

35
Problem 16  *Can we actually construct the set $A$ as described above?* Yes we have a procedure.

Let $s = (s_1, s_2, \ldots, s_m)$ be an element of $\mathbb{R}^m$ and define $A(s)$ to be the restriction of the linear map $\text{ad} \left(- \sum_{j=1}^m s_j A_j\right)$ to the ideal $p \subset n$.

**Lemma 17** Let $[A(s)]$ be the matrix representation of the linear map $A(s)$ with respect to the basis $\{Z_1, \cdots, Z_p\}$. Let $\epsilon$ be a positive real number satisfying

$$
\epsilon \leq \delta = \frac{1}{2} \left( \sup \left\{ \| e^{[A(s)]^T} \|_\infty : s \in [0, 1]^m \right\} \right)^{-1} < \infty.
$$

Then $\beta \left( (-\epsilon, \epsilon)^{n-2m} \right)$ has positive Lebesgue measure and is contained in a fundamental domain of $\mathbb{Z}^p$.

**Theorem 18** *(The $\delta$-condition)* Let $N = PM = \exp(p) \exp(m)$ be a simply connected, connected nilpotent Lie group with Lie algebra $n$ satisfying the conditions above. For any positive number $\epsilon$ such that

$$
\epsilon \leq \delta = \frac{1}{2} \left( \sup \left\{ \| e^{[A(s)]^T} \|_\infty : s \in [0, 1]^m \right\} \right)^{-1}
$$

there exists a band-limited vector $\eta = \eta_\epsilon$ in the Hilbert space $H_{(-\epsilon, \epsilon)^{n-2m}}$ such that $W_\eta \left( H_{(-\epsilon, \epsilon)^{n-2m}} \right)$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with respect to $\Gamma$.  

36
Example 19 Assume that $\mathfrak{n}$ is a four-dimensional $Z_1, Z_2, Z_3, A_1$ such that $[A_1, Z_2] = 2Z_1, [A_1, Z_3] = 2Z_2$. With respect to the ordered basis $Z_1, Z_2, Z_3$, we have

$$[adA_1]|_p = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \exp [adA_1]|_p = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$  

1. Then

$$\begin{align*}
\delta &= \left(2 \sup \left\{ \left\| \begin{bmatrix} 1 & 0 & 0 \\ -2s & 1 & 0 \\ 2s^2 & -2s & 1 \end{bmatrix} \right\|_\infty : s \in [0,1] \right\} \right)^{-1} \\
&= \frac{1}{2} \left( \max \left\{ 1, 1 + 2|s|, 1 + 2|s| + 2|s|^2 : s \in [0,1] \right\} \right)^{-1} = \frac{1}{10}
\end{align*}$$

2. The set

$$\beta \left( \left( -\frac{1}{10}, \frac{1}{10} \right)^2 \times [0,1) \right) \subset \left( -\frac{1}{2}, \frac{1}{2} \right)^3$$

is contained in a fundamental domain of $Z^3$. 

37
3. Thus, there exists a band-limited vector $\eta \in \mathbf{H}_{\left(\frac{1}{m}, \frac{\pi}{m}\right)}$ such that $W_\eta \left(\mathbf{H}_{\left(\frac{1}{m}, \frac{\pi}{m}\right)}\right)$ is a sampling space with respect to

$$\Gamma = \exp (ZZ_1 + ZZ_2 + ZZ_3) \exp (ZA_1).$$
Concluding remarks and self-criticism

1. The $\delta$-condition is sufficient not necessary.

2. In general (for higher dimensions)
   - it is quite difficult to verify the statement: $\beta (A \times [0,1)^m)$ is equal to a fundamental domain of $Z^p$.
   - The class of groups being handled is large but, our techniques do not work for all nilpotent Lie groups. For example, let $\mathfrak{n}$ be a ten-dimensional nilpotent Lie algebra such that
     \[ \mathfrak{n} = \mathbb{R}\text{-span} \{X_1, X_2, X_3, X_4, X_{12}, X_{13}, X_{14}, X_{23}, X_{24}, X_{34}\} \]
     and for $i < j$ we have $[X_i, X_j] = X_{ij}$. The main problem here is that the constant polarization property fails.

3. Our technique seems to work with some non-unimodular Lie groups which are not nilpotent but are solvable. For example, our technique works for some solvable Lie groups. Let $G$ be a solvable Lie group with Lie algebra
     \[ g = \mathbb{R}\text{-span} \{X_1, X_2, A\} \]
     with non-trivial Lie brackets given by
     \[ [A, X_1] = X_1 \text{ and } [A, X_2] = -X_2. \]
     Recent computations show, such a group can be handled as well.
Thank you very much for your attention.