Sampling on some nilpotent Lie groups

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Sampling on the real line

How can we recover a function $f : \mathbb{R} \to \mathbb{C}$ if we only know a countable set of values

$$(f(k))_{k \in I}?$$

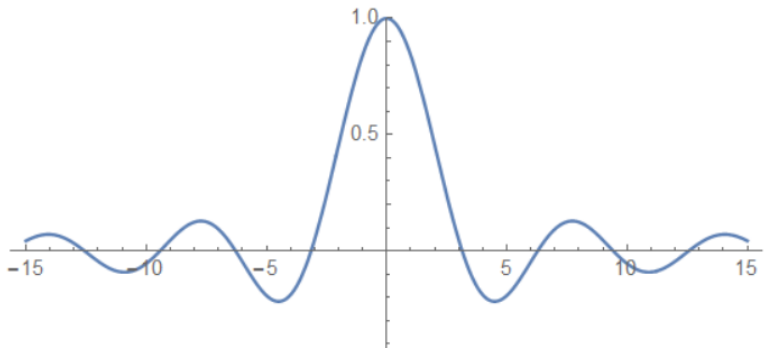
Formulated this way, the problem is ill-posed since there are infinitely many functions that take the same prescribed values on a given countable set. We consider the Paley-Wiener space

$$PW(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : \ \operatorname{supp} \widehat{f} \subseteq \left[-\frac{1}{2}, \frac{1}{2} \right] \right\}.$$

The Paley-Wiener space consists of equivalence classes of functions. Since the Fourier transform of these functions has compact support, each of the equivalence classes contains a continuous function.

Define the sinc-function by

$$\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x} \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{cases}.$$



Theorem 1 *(Shannon's sampling theorem) The functions* $\{\operatorname{sinc}(x-k)\}_{k\in\mathbb{Z}}$ *form an orthonormal basis for* PW*. If* $f \in PW \cap C(\mathbb{R})$*, then*

$$f(x) = \sum_{k\in\mathbb{Z}} f(k) \operatorname{sinc}(x-k)$$

with convergence of the symmetric partial sums in $L^2(\mathbb{R})$ *and point-wise for all* $x \in \mathbb{R}$.

Sampling on locally compact groups

Let N be a locally compact group, and let Γ be a discrete subset of N. Let $\mathbf{H}$ be a left-invariant closed subspace of $L^2(N)$ consisting of continuous functions. We call $\mathbf{H}$ a **sampling space** with respect to Γ (or Γ-sampling space) if

1. The restriction map $R_\Gamma : \mathbf{H} \to l^2(\Gamma)$, $R_\Gamma f = (f(\gamma))_{\gamma\in\Gamma}$ is an isometry.

2. There exists $S \in \mathbf{H}$ such that for any vector $f \in \mathbf{H}$,

$$f(x) = \sum_{\gamma\in\Gamma} f(\gamma)\, S\left(\gamma^{-1}x\right)$$

   with convergence in the norm of $\mathbf{H}$.

3. The vector S is called a **sinc-type** function. Moreover, if R_Γ is surjective, we say that the sampling space $\mathbf{H}$ has the **interpolation property**.

Sampling on the Heisenberg group

Let $\mathbb{H}$ be the three-dimensional Heisenberg Lie group

$$\mathbb{H} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \right\}$$

and

$$\Gamma = \left\{ \begin{bmatrix} 1 & m & k \\ 0 & 1 & l \\ 0 & 0 & 1 \end{bmatrix} : \begin{bmatrix} m \\ l \\ k \end{bmatrix} \in \mathbb{Z}^3 \right\}$$

Then Γ is a discrete subgroup of the Heisenberg group.

Theorem 2 *(H. Fuhr, 2005, B. Currey, A. Mayeli, 2009) The Heisenberg group admits sampling spaces with respect to Γ. Moreover, the Heisenberg group admits a sampling space with respect to Γ which also has the interpolation property.*

Some basic questions

- How far can we extend the results on the Heisenberg group to other non-commutative nilpotent Lie groups?
- What is a nilpotent Lie group?
  - I will give you a crash course on the topic.
- Do we have some kind of Fourier theory available for this class of groups?
  - We will brush the basic.
- Can we use such a theory?
  - To a certain extend, we have some hope.
  - The general case is still murky (at least to me)
- What do we mean by bandlimited spaces on nilpotent Lie groups?
  - We will brush the basic.
- What should Γ be? Could Γ be a subgroup?
  - For a large class of groups, we will see that Γ can be taken to be a subgroup.

Basic facts about Lie groups and Lie algebra

- **Abstract definition** the notion of **Lie group** mimic the definition of a topological group. Let G be a **topological group**. Roughly speaking, suppose that there is an analytic structure on the set G which is compatible with its topology, converting it into an analytic manifold such that
$$(x, y) \mapsto xy \text{ and } x \mapsto x^{-1}$$
are both analytic. Then G endowed with this structure is called a Lie group.

- A vector space $\mathfrak{g}$ over $\mathbb{R}$ is called a real **Lie algebr**a if there is a map
$$\mathfrak{g} \times \mathfrak{g} \ni (X, Y) \mapsto [X, Y] \in \mathfrak{g}$$
such that
  - $(X, Y) \mapsto [X, Y]$ is bilinear
  - $[X, Y] = -[Y, X]$
  - $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity)

- If $\mathfrak{g}$ is a Lie algebra over the reals, and $\{X_1, \cdots, X_n\}$ is a basis of $\mathfrak{g}$ then there are uniquely determined **(structure) constants** c_{ijk} such that
$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k.$$

- A **homomorphism** $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ of Lie algebras is a linear map satisfying $\varphi([X,Y]) = [\varphi(X), \varphi(Y)]$.
- A Lie algebra $\mathfrak{g}$ is called **abelian** if $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$.
- Each associated algebra is a Lie algebra when endowed with the bracket

$$[X,Y] = XY - YX.$$

  To verify this, it is enough to check the axioms described above.
- Let X be a matrix of order n. Define the power series

$$\exp X = \sum_{k=0}^{\infty} \frac{X^m}{m!}.$$

  - For any matrix X, $\exp X$ is convergent, and $\exp X$ is **always invertible**.
  - $\exp 0 = id$, $\exp(-X) = (\exp X)^{-1}$
  - In general $\exp(X+Y) \neq \exp(X)\exp(Y)$.

$$\exp\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix} \neq \exp\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\exp\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e & e \\ 0 & 1 \end{bmatrix}$$

– **Example.** Let $\lambda \in \mathbb{R}$. Then

$$\exp \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} e^{\lambda} & e^{\lambda} & \frac{1}{2}e^{\lambda} \\ 0 & e^{\lambda} & e^{\lambda} \\ 0 & 0 & e^{\lambda} \end{bmatrix}$$

– **Exercise.** Let $t \in \mathbb{R}$

$$\exp \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} = ?$$

Lie groups and Lie algebra

- A closed subgroup $G \subseteq GL(n, \mathbb{C})$ is called a **linear Lie group**.
- Define
$$\mathbf{Lie}(G) = \{X \in M(n, \mathbb{C}) : \exp(tX) \in G \text{ for all } t \in \mathbb{R}\}$$
$\mathbf{Lie}(G)$ is called the Lie algebra of G and is denoted $\mathfrak{g}$.

Example 3

1. *$GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are matrix Lie groups.*
2. *The Lie algebra of $GL(n, \mathbb{C})$ is the set of all real matrices of order n.* $\mathbf{Lie}(GL(n, \mathbb{C})) = M(n, \mathbb{R})$
3. *$SL(n, \mathbb{R})$ and $SL(n, \mathbb{C})$ are matrix Lie groups.*
4. *The Lie algebra of $SL(n, \mathbb{R})$ is* $\mathfrak{sl}(n, \mathbb{R}) = \{X : \text{trace}(X) = 0\}$.
5. *(Important) The Heisenberg group*
$$\mathbb{H} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

is a matrix Lie group. Observe that

$$\exp \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus,

$$\mathfrak{h} = \left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

is the Lie algebra of the Heisenberg group. Moreover, given

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

it is not hard to verify that $[X, Y] = Z$ *is the only non-trivial Lie brackets for* $\mathfrak{h}$.

6. N_n *(triangular matrices with ones on the diagonal) is a matrix Lie group.*

Nilpotent Lie algebra

- Let $\mathfrak{g}$ be a Lie algebra over the real. The **descending central series** of $\mathfrak{g}$ is defined inductively by
  - $\mathfrak{g}^{(1)} = \mathfrak{g}$
  - $\mathfrak{g}^{(n+1)} = \left[\mathfrak{g}, \mathfrak{g}^{(n)}\right] = \mathbb{R}\text{-span}\left\{[X, Y] : X \in \mathfrak{g}, Y \in \mathfrak{g}^{(n)}\right\}$
- We say that $\mathfrak{g}$ is a **nilpotent Lie algebra** if there is an integer n such that $\mathfrak{g}^{(n+1)} = \{0\}$. If additionally, $\mathfrak{g}^{(n)}$ is not the zero vector space then $\mathfrak{g}$ is called an n-step nilpotent Lie algebra.
  - For the Heisenberg Lie algebra. Recall that $\mathfrak{h} = \mathbb{R}X_1 + \mathbb{R}X_2 + \mathbb{R}X_3$ and $[X_3, X_2] = X_1$

$$\begin{aligned}
\mathfrak{h}^{(1)} &= \mathfrak{h} \\
\mathfrak{h}^{(2)} &= \left[\mathfrak{h}, \mathfrak{h}^{(1)}\right] = \mathbb{R}Z = \text{center} \\
\mathfrak{h}^{(3)} &= \left[\mathfrak{h}, \mathfrak{h}^{(2)}\right] = \{0\}.
\end{aligned}$$

    Thus, $\mathfrak{h}$ is a two-step nilpotent Lie algebra.

– Let $\mathfrak{r}$ be the Lie algebra spanned by $\{X_1, X_2, X_3, X_4, X_5\}$ such that

$$[X_4, X_3] = 2X_1, [X_5, X_3] = 2X_2, [X_5, X_2] = 2X_1$$

A matrix realization of $\mathfrak{r}$

$$\mathfrak{r} = \left\{ \begin{bmatrix} 0 & 2x_5 & 2x_4 & x_1 \\ 0 & 0 & 2x_5 & x_2 \\ 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \mathbb{R}^5 \right\}.$$

Exercise Verify that $\mathfrak{r}$ is a three-step nilpotent Lie algebra.

Theorem 4 *If H is connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{h}$ then*

$$\exp : \mathfrak{h} \to H$$

is an analytic diffeomorphism. Additionally, every connected, simply connected nilpotent Lie group has a faithful embedding as a closed subgroup of N_n for some n.

Example 5 *Let $\mathfrak{r}$ be the Lie algebra spanned by $\{X_1, X_2, X_3, X_4, X_5\}$ such that*

$$[X_4, X_3] = 2X_1, [X_5, X_3] = 2X_2, [X_5, X_2] = 2X_1.$$

Then

$$\exp\left(\sum_{k=1}^{3} x_k X_k\right) \exp\left(\sum_{k=4}^{5} x_k X_k\right) \mapsto \begin{bmatrix} 1 & 2x_5 & 2x_5^2 + 2x_4 & x_1 \\ 0 & 1 & 2x_5 & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

defines a Lie group isomorphism between $\exp \mathfrak{r} = R$ and a closed subgroup of N_4.

Definition 6 *Let $\mathfrak{h}$ be a Lie algebra. Given $X \in \mathfrak{h}$*

$$ad_X : \mathfrak{h} \to \mathfrak{h}$$

is a linear map described as follows

$$ad_X(Y) = [X, Y].$$

The map $ad : X \mapsto ad_X$ is actually a ***linear representation*** *of $\mathfrak{h}$ in $\mathfrak{gl}(\mathfrak{h})$. In the situation where $\mathfrak{h}$ is a nilpotent Lie algebra then the adjoint representation is never faithful since it has a non-trivial center.*

Example 7 *Let $\mathfrak{h}$ be the Lie algebra spanned by $\{X_1, X_2, X_3\}$ such that $[X_3, X_2] = X_1$. Then*

$$[ad_{X_1}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [ad_{X_2}] = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [ad_{X_3}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Let G be a nilpotent Lie group with Lie algebra $\mathfrak{g}$ and denote the dual of $\mathfrak{g}$ by $\mathfrak{g}^*$. Then G acts on $\mathfrak{g}^*$ by the contragradient of the adjoint map (the **coadjoint** action)

$$\langle \exp(X) \cdot \lambda, Y \rangle = \left\langle \lambda, e^{-adX} Y \right\rangle = \left\langle \left(e^{-adX}\right)^* \lambda, e^{-adX} Y \right\rangle$$

  - (The simplest non-commutative case) Let $\mathfrak{h}$ be the Heisenberg Lie algebra. Then

$$\begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda \\ -x\lambda \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \\ y\lambda \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix}$$

    * If $\lambda \neq 0$ then corresponding coadjoint orbits are two-dimensional.
    * If $\lambda = 0$ then corresponding coadjoint orbits are zero-dimensional (points)
    * Let $\Omega = \{\lambda X_1^* + \alpha X_2^* + \xi X_3^* \in \mathfrak{n}^* : \lambda \neq 0\}$. Then Ω is an-invariant, algebraic set. Moreover, a cross-section for the coadjoint orbits is by $\Sigma = \{\lambda X_1^* \in \mathfrak{n}^* : \lambda \neq 0\}$

– Let $\mathfrak{r}$ be the Lie algebra spanned by $\{X_1, X_2, X_3, X_4, X_5\}$ such that

$$[X_4, X_3] = 2X_1, [X_5, X_3] = 2X_2, [X_5, X_2] = 2X_1.$$

its Lie group R. Then

$$\begin{bmatrix} 0 & 2x_5 & 2x_4 & 0 \\ 0 & 0 & 2x_5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2x_5 & 1 & 0 & 0 & 0 \\ 2x_5^2 - 2x_4 & -2x_5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda \\ -2\lambda x_5 \\ 2\lambda x_5^2 - 2\lambda x_4 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2x_3 & 0 & 0 & 1 & 0 \\ 2x_3 - 2x_2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \\ 0 \\ 2\lambda x_3 \\ 2\lambda x_3 - 2\lambda x_2 \end{bmatrix}$$

* If $\lambda \neq 0$ then corresponding coadjoint orbits are four-dimensional submanifolds.
* Let

$$\Omega = \{\lambda X_1^* + aX_2^* + bX_3^* + cX_4^* + dX_5^* \in \mathfrak{n}^* : \lambda \neq 0\}.$$

Then Ω is an-invariant, algebraic set. Moreover, a cross-section for the coadjoint orbits is by

$$\Sigma = \{\lambda X_1^* \in \mathfrak{n}^* : \lambda \neq 0\}.$$

- Let G be a locally compact group. A **unitary representation** of G is a homomorphism π from G into the group of unitary operators on some non-zero Hilbert space which is continuous with respect to the strong operator topology. That is

$$\pi : G \to \mathcal{U}\left(\mathbf{H}_\pi\right)$$

such that

1. $\pi\left(xy\right) = \pi\left(x\right)\pi\left(y\right)$
2. $\pi\left(x^{-1}\right) = \pi\left(x\right)^{-1} = \pi\left(x\right)^*$
3. $x \to \pi\left(x\right)u$ is continuous from G to $\mathbf{H}_\pi$ for any $u \in \mathbf{H}_\pi$.

- It is worth noticing that strong continuity and weak topologies are the same in $\mathcal{U}\left(\mathbf{H}_\pi\right)$. Thus, $x \mapsto \left\langle \pi\left(x\right)u, v\right\rangle$ is continuous from G to $\mathbb{C}$. Suppose that $\{T_\alpha\}$ is a net of unitary operators convergent to T. Then for any $u \in H_\pi$

$$\begin{aligned}\left\|\left(T_\alpha - T\right)u\right\|^2 &= \left\|T_\alpha u\right\|^2 + \left\|Tu\right\|^2 - 2\operatorname{Re}\left\langle T_\alpha u, Tu\right\rangle \\ &= 2\left\|u\right\|^2 - 2\operatorname{Re}\left\langle T_\alpha u, Tu\right\rangle\end{aligned}$$

Thus, $\left\|\left(T_\alpha - T\right)u\right\|^2 \to 0$.

- If π_1 and π_2 are unitary representations of G, an **intertwining operator** for π_1 and π_2 is a bounded linear operator $T : H_{\pi_1} \to H_{\pi_2}$ such that $T\pi_1(x) = \pi_2(x)T$ for all $x \in G$. We say that π_1 and π_2 are unitarily equivalent if there is a unitary operator T which is intertwining the representations.

- Now suppose that $\mathbf{K}$ is a closed subspace of $\mathbf{H}_\pi$ such that $\pi(G)\mathbf{K} \subset \mathbf{K}$. We say that $\mathbf{K}$ is a π-invariant Hilbert subspace of $\mathbf{H}_\pi$. Moreover, a representation π is irreducible if the only π-invariant subspaces are the trivial ones. That is the zero vector space and $\mathbf{H}_\pi$.

- (**The unitary dual**) Given a locally compact group G, one of the most important questions in representation theory is to classify all of its unitary irreducible representations. The set of all irreducible representations of G up to equivalence is called the unitary dual and is denoted $\widehat{G}$
$$\widehat{G} = \{[\pi] : \pi \text{ is irreducible}\}$$

- (**Orbit Method/Kirillov Method**) Let G be a simply connected nilpotent Lie group with nilpotent Lie algebra $\mathfrak{g}$. All irreducible unitary representations of G are parametrized by the set of coadjoint orbits.

  - In this class of groups, a cross-section is always computable
  - There is a well-known procedure which is exploited to realize the irreducible representation corresponding to every element in the cross-section.

Back to sampling

To study sampling spaces, we need the following ingredients.

- The unitary dual of the group.
- Fourier/Plancherel transform.
- A description of band-limited spaces.
- A suitable discrete set Γ.

- Consider the Heisenberg group

$$N = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\} \text{ and } \Gamma = \left\{ \begin{bmatrix} 1 & m & k \\ 0 & 1 & l \\ 0 & 0 & 1 \end{bmatrix} : k, l, m \in \mathbb{Z} \right\}.$$

  Recall that N is a nilpotent Lie group which is two step. Thus, it is not commutative.

  - The **unitary dual** of N is up to a null set parametrized by the punctured line $\mathbb{R} \setminus \{0\}$.
  - For each $\lambda \in \mathbb{R} \setminus \{0\}$, the corresponding irreducible representation π_λ is realized as acting on $L^2(\mathbb{R})$ as follows

$$\pi_\lambda(x, y, z) f(t) = e^{2\pi i \lambda z} e^{-2\pi i \lambda y t} f(t - x).$$

  - The **Plancherel measure** is the weighted Lebesgue measure $|\lambda| \, d\lambda$. There is a procedure for computing the formula.
  - The Plancherel transform

$$\mathcal{P} : L^2(N) \to L^2\left(\mathbb{R} \setminus \{0\}, L^2(\mathbb{R}) \otimes L^2(\mathbb{R}), |\lambda| \, d\lambda\right)$$

    is an operator-valued map which is weakly defined as

$$\mathcal{F}(f)(\lambda) = \int_N f(n) \, \pi_\lambda(n) \, dn.$$

Moreover,

$$\|f\|_{L^2(N)}^2 = \int_{\mathbb{R}\setminus\{0\}} \|\mathcal{P}(f)(\lambda)\|_{\mathcal{HS}}^2 \, d\mu(\lambda).$$

- We say a function $f \in L^2(N)$ is **bandlimited** if its Plancherel transform is supported on a bounded measurable subset of $\mathbb{R}\setminus\{0\}$.
- In the **simplest case**, fix a measurable field of unit vectors $\mathbf{e} = (\mathbf{e}_\lambda)_\lambda$ where $\mathbf{e}_\lambda \in L^2(\mathbb{R})$. We say a Hilbert space $\mathbf{H}(\mathbf{e})$ is a **multiplicity-free** left-invariant subspace of $L^2(N)$ if for all $f \in \mathbf{H}(\mathbf{e})$

$$\mathcal{P}(f)(\lambda) = \mathbf{u}_\lambda \otimes \mathbf{e}_\lambda \text{ (rank-one operators)}$$

Theorem 8 ***(Führ, Currey and Mayeli)*** *Let*

$$\mathbf{E}_a = \{\lambda \in \mathbb{R}^* : \lambda \neq 0, \text{ and } |\lambda| \leq a\}$$

where $a \in (0,1]$. Let $\mathbf{H}(\mathbf{e}, \mathbf{E}_a)$ be the multiplicity-free subspace of $\mathbf{H}(\mathbf{e})$ containing functions with Fourier transforms supported on $\mathbf{E}_a$. Next, let

$$W_\phi : \mathbf{H}(\mathbf{e}, \mathbf{E}_a) \to L^2(N),$$

such that $W_\phi\psi(x) = \langle \psi, L(x)\phi \rangle$. There exists $\phi \in \mathbf{H}(\mathbf{e}, \mathbf{E}_a)$ such that $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E}_a))$ is a Γ-sampling subspace of $L^2(N)$ with sinc-type function $W_\phi(\phi)$. Moreover, $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E}_a))$ has the interpolation property with respect to Γ if $a = 1$.

Theorem 9 ***(Führ)*** *In contrast with the real case, sampling space cannot have arbitrary band-limitation.*

An extension of the results above

- Let us suppose that $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{z}$, where $[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{z}$, $\mathfrak{a}, \mathfrak{b}, \mathfrak{z}$ are abelian algebras such that

$$\begin{aligned} \mathfrak{a} &= \mathbb{R}\text{-span}\left\{X_1, X_2, \cdots, X_d\right\}, \\ \mathfrak{b} &= \mathbb{R}\text{-span}\left\{Y_1, Y_2, \cdots, Y_d\right\}, \\ \mathfrak{z} &= \mathbb{R}\text{-span}\left\{Z_1, Z_2, \cdots, Z_{n-2d}\right\}, \end{aligned}$$

  $d \geq 1$, $n > 2d$ and

$$\det \begin{bmatrix} [X_1, Y_1] & [X_1, Y_2] & \cdots & [X_1, Y_d] \\ [X_2, Y_1] & [X_2, Y_2] & \cdots & [X_2, Y_d] \\ \vdots & \vdots & \cdots & \vdots \\ [X_d, Y_1] & [X_d, Y_2] & \cdots & [X_d, Y_d] \end{bmatrix} \tag{1}$$

  is a non-vanishing homogeneous polynomial in the unknowns $Z_1, \cdots, Z_{n-2d}$.

  - The determinant condition essentially ensures that square-integrability of the important irreducible representations of N modulo the center.
  - Observe that if $\mathfrak{n}$ is the Heisenberg Lie algebra then $\mathfrak{a} = \mathbb{R}X_1, \mathfrak{b} = \mathbb{R}Y_1, \mathfrak{z} = \mathbb{R}Z_1$ and $\det [X_1, Y_1] = [X_1, Y_1] = Z_1$.

- The unitary dual of N is parametrized by the smooth manifold

$$\Sigma = \left\{ \begin{array}{c} \lambda \in \mathfrak{n}^* : \det\left(B\left(\lambda\right)\right) \neq 0, \lambda\left(X_1\right) = \cdots \\ = \lambda\left(X_d\right) = \lambda\left(Y_1\right) = \cdots = \lambda\left(Y_d\right) = 0 \end{array} \right\}$$

which is naturally identified with a Zariski open subset of $\mathbb{R}^{n-2d}$.

- Let $d\lambda$ be the Lebesgue measure on Σ. The Plancherel measure for the group N is supported on Σ and is equal to $d\mu(\lambda) = |\det(B(\lambda))|\, d\lambda$. Here

$$B(\lambda) = \begin{bmatrix} \lambda[X_1, Y_1] & \lambda[X_1, Y_2] & \cdots & \lambda[X_1, Y_d] \\ \lambda[X_2, Y_1] & \lambda[X_2, Y_2] & \cdots & \lambda[X_2, Y_d] \\ \vdots & \vdots & \cdots & \vdots \\ \lambda[X_d, Y_1] & \lambda[X_d, Y_2] & \cdots & \lambda[X_d, Y_d] \end{bmatrix}.$$

- The **unitary dual** of N which we denote by $\widehat{N}$ is up to a null set equal to $\{\pi_\lambda : \lambda \in \Sigma\}$ where each representation π_λ is realized as acting in $L^2(\mathbb{R}^d)$.

- Let

$$Z = \sum_{i=1}^{n-2d} z_i Z_i, Y = \sum_{i=1}^{d} y_i Y_i \text{ and } X = \sum_{i=1}^{d} x_i X_i.$$

Then

$$\begin{aligned} [\pi_\lambda \exp(Z) f](t) &= e^{2\pi i \lambda\left(\sum_{i=1}^{n-2d} z_i Z_i\right)} \cdot f(t) && \text{(characters)} \\ \pi_\lambda(\exp(Y)) f(t) &= e^{-2\pi i \langle B(\lambda) y, t\rangle} \cdot f(t) && \text{(modulations)} \\ \pi_\lambda(\exp(X)) f(t) &= f(t - x) && \text{(translations)} \end{aligned}$$

- Put
$$\Gamma = \exp\left(\sum_{i=1}^{n-2d} \mathbb{Z}Z_i\right) \exp\left(\sum_{i=1}^{d} \mathbb{Z}Y_i\right) \exp\left(\sum_{i=1}^{d} \mathbb{Z}X_i\right) \subset N.$$
- The **Plancherel transform**
$$\mathcal{P} : L^2(N) \to L^2\left(\Sigma, L^2\left(\mathbb{R}^d\right) \otimes L^2\left(\mathbb{R}^d\right) |\det(B(\lambda))|\, d\lambda\right)$$
and we have a **general Plancherel theorem**
$$\|f\|^2_{L^2(N)} = \int_\Sigma \|\mathcal{P}(f)(\lambda)\|^2_{\mathcal{HS}} \ |\det(B(\lambda))|\, d\lambda.$$
- Fix a unit vector $\mathbf{e} \in L^2\left(\mathbb{R}^d\right)$, and let
$$\mathbf{E} = \{\lambda \in \mathfrak{z}^* : |\det B(\lambda)| \neq 0, \text{ and } |\det B(\lambda)| \leq 1\}$$
  * Let $\mathbf{C} \subseteq \mathbf{E}$ such that $\mathbf{C}$ is contained in a fundamental domain of $\mathbb{Z}^{n-2d}$ and $|\mathbf{E} \cap \mathbf{C}| > 0$.
  * Define $\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}) \subset L^2(N)$
$$\mathcal{P}(\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})) = L^2\left(\mathbf{E} \cap \mathbf{C}, L^2\left(\mathbb{R}^d\right) \otimes \mathbf{e}, |\det(B(\lambda))|\, d\lambda\right)$$

Theorem 10 *(O) There exists $\phi \in \mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$ such that $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}))$ is a Γ-sampling subspace of $L^2(N)$ with sinc-type function $W_\phi(\phi)$. Moreover, $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}))$ does not generally have the interpolation property with respect to Γ.*

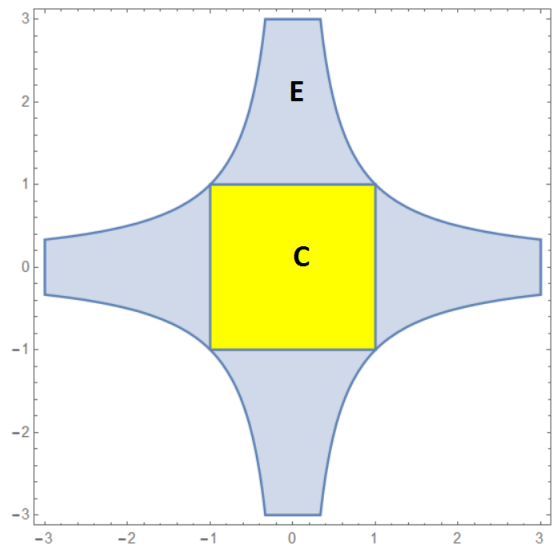
Let N be a nilpotent Lie group with Lie algebra spanned by the vectors $\{Z_1, Z_2, Y_1, Y_2, X_1, X_2\}$ with the following non-trivial Lie brackets: $[X_1, Y_1] = Z_1, [X_2, Y_2] = Z_2$. In this example, the discrete set

$$\Gamma = \exp\left(\mathbb{Z}Z_1 + \mathbb{Z}Z_2\right) \exp\left(\mathbb{Z}Y_1 + \mathbb{Z}Y_2\right) \exp\left(\mathbb{Z}X_1 + \mathbb{Z}X_2\right)$$

is actually a uniform subgroup of the Lie group N.

- N is a direct product of two Heisenberg groups and satisfies all properties described above
- The Plancherel measure is $|\lambda_1\lambda_2|\, d\lambda_1 d\lambda_2$.
- $\mathbf{E} = \left\{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_1\lambda_2 \neq 0 \text{ and } |\lambda_1\lambda_2| \leq 1\right\}$.
- Let $\mathbf{C} = [-1, 1]^2$. Observe that

$$\mu\left(\mathbf{E} \cap \mathbf{C}\right) = \int_{-1}^{1}\int_{-1}^{1} |\lambda_1\lambda_2|\, d\lambda_1 d\lambda_2 = 1.$$



- Let $\phi \in \mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$ such that $\mathcal{P}\phi(\lambda) = \chi_{[0,1)^2}(t) \otimes \chi_{[0,1)^2}(t)$.
- $L(\Gamma)\phi$ is an orthonormal basis for $\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$ and $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}))$ is a Γ-sampling space with the interpolation property.
- It is perhaps worth highlighting the fact that in order to obtain a sampling space it is necessary to select the spectral set such that it is essentially contained in the blue region.

Remark 11

1. *The class of groups considered here is very small, and the proofs heavily rely on well-known results of time-frequency analysis.*

2. *In the case where* Γ *is a subgroup, it is worth pointing out that the representation theory of such groups is not well-understood and is quite wild. Thus, sampling theory on nilpotent Lie groups gives some insights into the representation theory of* Γ*. Perhaps these could be exploited to shed some light on the theory of such discrete groups.*

3. *A referee once pointed out to me that the results of Currey and Mayeli lead naturally to a central decomposition of the left regular representation of* Γ*.*

4. *In summary, the first point is discouraging but the second point seems to balance the first.*

Extension of the results to a larger class of groups

- Let $\mathfrak{n}$ be a nilpotent Lie algebra of dimension n, and let $\mathfrak{n}^*$ be the dual vector space of $\mathfrak{n}$. A **polarizing subalgebra** $\mathfrak{p}(\lambda)$ subordinated to a linear functional $\lambda \in \mathfrak{n}^*$ is a maximal algebra satisfying

$$[\mathfrak{p}(\lambda), \mathfrak{p}(\lambda)] = \text{Span-}\{[X, Y] \in \mathfrak{n} : X, Y \in \mathfrak{p}(\lambda)\} \subseteq \ker(\lambda).$$

- Recall again that the coadjoint action on the dual of $\mathfrak{n}$ is the dual of the adjoint action of $N = \exp \mathfrak{n}$ on $\mathfrak{n}$. In other words, for $X \in \mathfrak{n}$, and a linear functional $\lambda \in \mathfrak{n}^*$, the coadjoint action is defined as follows:

$$(\exp X \cdot \lambda)(Y) = \left\langle \left(e^{-ad(X)}\right)^* \lambda, Y \right\rangle = \left[\left(e^{-ad(X)}\right)^* \lambda\right](Y).$$

- $\mathfrak{p}$ is called a **constant polarization** subalgebra of $\mathfrak{n}$ if $\mathfrak{p}$ is a polarization algebra for almost all linear functionals

Condition 12 *Let us suppose that* $N = P \rtimes M = \exp(\mathfrak{p}) \rtimes \exp(\mathfrak{m})$ *is a simply connected, connected non-commutative nilpotent Lie group with Lie algebra* $\mathfrak{n} = \mathfrak{p} + \mathfrak{m}$ *such that*

1. $\mathfrak{p}$ *is a* ***constant polarization ideal*** *of* $\mathfrak{n}$ *(thus commutative)* $\mathfrak{m}$ *is commutative as well,* $p = \dim \mathfrak{p}$, $m = \dim \mathfrak{m}$ *and* $p - m > 0$.

2. *There exists a strong Malcev basis* $\{Z_1, \cdots, Z_p, A_1, \cdots, A_m\}$ *for* $\mathfrak{n}$ *such that* $\{Z_1, \cdots, Z_p\}$ *is a basis for* $\mathfrak{p}$, $\{A_1, \cdots, A_m\}$ *is a basis for* $\mathfrak{m}$ *and*

$$\Gamma = \exp\left(\mathbb{Z}Z_1 + \cdots + \mathbb{Z}Z_p\right) \exp\left(\mathbb{Z}A_1 + \cdots + \mathbb{Z}A_m\right)$$

   is a discrete uniform subgroup of N. *This is equivalent to the fact that* $\mathfrak{n}$ *has* ***rational structure constants****.*

Example 13 *The following collection contains a small list of groups meeting all conditions described above.*

1. *Let N be a simply connected, connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ of dimension four or less.*

2. *(nilpotent Lie groups of the type $\mathbb{R}^p \rtimes \mathbb{R}$) Let N be a simply connected, connected nilpotent Lie group with Lie algebra spanned by $Z_1, Z_2, \cdots, Z_p, A_1$, the vector space generated by $Z_1, Z_2, \cdots, Z_p$ is a commutative ideal, $[ad A_1]|_{\mathfrak{p}} = A$ is a nonzero rational upper triangular nilpotent matrix of order p, and $e^A \mathbb{Z}^p \subseteq \mathbb{Z}^p$.*

3. *Let N be a simply connected, connected nilpotent Lie group with Lie algebra spanned by the vectors $Z_1, Z_2, \cdots, Z_p, A_1, \cdots, A_m$ where $p = m+1$, the vector space generated by $Z_1, Z_2, \cdots, Z_p$ is a commutative ideal, the vector space generated by $A_1, \cdots A_m$ is commutative and the matrix representation of ad $(\sum_{k=1}^{m} t_k A_k)$ restricted to $\mathfrak{p}$ is given by*

$$A(t) = \left[ad \sum_{k=1}^{m} t_k A_k \right] \Bigg| \mathfrak{p} = m! \begin{bmatrix} 0 & t_1 & t_2 & \cdots & t_{m-1} & t_m \\ & 0 & t_1 & t_2 & \ddots & t_{m-1} \\ & & 0 & t_1 & \ddots & \vdots \\ & & & 0 & \ddots & t_2 \\ & & & & \ddots & t_1 \\ & & & & & 0 \end{bmatrix}.$$

- Let Σ be a parameterizing set for the unitary dual of N. In other words, Σ is a cross-section for the coadjoint orbits in an N-invariant open and dense set $\Omega \subset \mathfrak{n}^*$.

- Let L be the left regular representation of N acting on $L^2(N)$ by left translations. Let

$$\mathcal{P} : L^2(N) \to L^2\left(\Sigma, L^2(\mathbb{R}^m) \otimes L^2(\mathbb{R}^m), d\mu(\lambda)\right)$$

  be the Plancherel transform which defines a unitary map on $L^2(N)$.

- The Plancherel measure $d\mu(\lambda)$ is equal to $|\mathbf{P}(\lambda)|\, d\lambda$. $\mathbf{P}(\lambda)$ is a polynomial defined over Σ and $d\lambda$ (I will give you the formula later)

- Given a μ-measurable bounded set $\mathbf{A} \subset \Sigma$, and a fixed measurable field of unit vectors $(\mathbf{u}(\lambda))_{\lambda \in \mathbf{A}}$ in $L^2(\mathbb{R}^m)$, we define the Hilbert space $\mathbf{H_A}$ as follows. $\mathbf{H_A}$ consists of vectors $f \in L^2(N)$ such that

$$\mathcal{P}f(\lambda) = \begin{cases} \mathbf{v}(\lambda) \otimes \mathbf{u}(\lambda) & \text{if } \lambda \in \mathbf{A} \\ 0 \otimes 0 & \text{if } \lambda \notin \mathbf{A} \end{cases}$$

  and $(\mathbf{v}(\lambda) \otimes \mathbf{u}(\lambda))_{\lambda \in \mathbf{A}}$ is a measurable field of rank-one operators. Consequently, $\mathbf{H_A}$ is a left-invariant multiplicity-free band-limited subspace of $L^2(N)$ which is naturally identified with the function space $L^2(\mathbf{A} \times \mathbb{R}^m)$

- The irreducible representations are given by

$$\left[\sigma_\lambda\left(\exp\left(\sum_{j=1}^{m} a_j A_j\right)\right)\mathbf{f}\right](x) = \mathbf{f}(x-a) \quad \text{(translation)}$$

$$\left[\sigma_\lambda\left(\exp\left(\sum_{j=1}^{m} z_j Z_j\right)\right)\mathbf{f}\right](x) = \exp\left(2\pi i\left\langle \lambda, e^{-ad(x_1A_1+\cdots+x_mA_m)}\sum_{j=1}^{m} z_j Z_j\right\rangle\right)\mathbf{f}(x)$$

  The second one above is a **higher order modulation**.

- Let $\mathrm{proj}_{\mathfrak{p}^*} : \mathfrak{n}^* \to \mathfrak{p}^*$ be the restriction mapping given by

$$\mathrm{proj}_{\mathfrak{p}^*}\left(\sum_{j=1}^{p}\left(\lambda_j Z_j^*\right) + \sum_{j=1}^{m}\left(\xi_j A_j^*\right)\right) = \sum_{j=1}^{n}\lambda_j Z_j^*$$

  where $\left\{Z_1^*, \cdots, Z_p^*, A_1^*, \cdots, A_m^*\right\}$ is a dual basis for $\left\{Z_1, \cdots, Z_p, A_1, \cdots, A_m\right\}$.

- Define the map $\beta : \Sigma \times \mathbb{R}^m \to \mathbb{R}^p$ by

$$\beta(\lambda, t) = \mathrm{proj}_{\mathfrak{p}^*}\left(\exp\left(\sum_{j=1}^{m} t_j A_j\right)\cdot\lambda\right) = \mathrm{proj}_{\mathfrak{p}^*}\left(\left(e^{-ad\sum_{j=1}^{m} t_j A_j}\right)^*\lambda\right)$$

  where $\cdot$ denotes the coadjoint action of N.

Lemma 14 *Let $J_\beta(\lambda, s_1, \cdots, s_m)$ be the Jacobian of the smooth map β. The Plancherel measure of N is up to multiplication by a constant equal to*

$$d\mu(\lambda) = \left|\det J_\beta(\lambda, 0)\right| d\lambda$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{R}^{\dim \Sigma}$. Consequently, $|\mathbf{P}(\lambda)| = \left|\det J_\beta(\lambda, 0)\right|$.

Theorem 15 *(O) Let $N = P \rtimes M = \exp(\mathfrak{p}) \rtimes \exp(\mathfrak{m})$ be a simply connected, connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ satisfying the conditions above. Let $\mathbf{A}$ be a μ-measurable bounded subset of Σ.*

1. *If $\beta\left(\mathbf{A} \times [0,1)^m\right)$ has positive Lebesgue measure in $\mathbb{R}^p$ and is contained in a fundamental domain of $\mathbb{Z}^p$ then there exists a vector $\eta \in \mathbf{H_A}$ such that $W_\eta(\mathbf{H_A})$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with respect to Γ.*

2. *If $\beta\left(\mathbf{A} \times [0,1)^m\right)$ is equal to a fundamental domain of $\mathbb{Z}^p$ then there exists a vector $\eta \in \mathbf{H_A}$ such that $W_\eta(\mathbf{H_A})$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with the interpolation property with respect to Γ.*

Problem 16 *Can we actually construct the set* $\mathbf{A}$ *as described above? Yes we have a procedure.*

Let $s = (s_1, s_2, \cdots, s_m)$ be an element of $\mathbb{R}^m$ and define $A(s)$ to be the restriction of the linear map $ad\left(-\sum_{j=1}^{m} s_j A_j\right)$ to the ideal $\mathfrak{p} \subset \mathfrak{n}$.

Lemma 17 *Let* $[A(s)]$ *be the matrix representation of the linear map* $A(s)$ *with respect to the basis* $\{Z_1, \cdots, Z_p\}$. *Let* ε *be a positive real number satisfying*

$$\varepsilon \leq \delta = \frac{1}{2}\left(\sup\left\{\left\|e^{[A(s)]^T}\right\|_\infty : s \in [0,1)^m\right\}\right)^{-1} < \infty.$$

Then $\beta\left((-\varepsilon,\varepsilon)^{n-2m} \times [0,1)^m\right)$ *has positive Lebesgue measure and is contained in a fundamental domain of* $\mathbb{Z}^p$.

Theorem 18 ***(The δ-condition)*** *Let* $N = PM = \exp(\mathfrak{p})\exp(\mathfrak{m})$ *be a simply connected, connected nilpotent Lie group with Lie algebra* $\mathfrak{n}$ *satisfying the conditions above. For any positive number* ε *such that*

$$\varepsilon \leq \delta = \frac{1}{2}\left(\sup\left\{\left\|e^{[A(s)]^T}\right\|_\infty : s \in [0,1)^m\right\}\right)^{-1}$$

there exists a band-limited vector $\eta = \eta_\varepsilon$ *in the Hilbert space* $\mathbf{H}_{(-\varepsilon,\varepsilon)^{n-2m}}$ *such that* $W_\eta\left(\mathbf{H}_{(-\varepsilon,\varepsilon)^{n-2m}}\right)$ *is a left-invariant subspace of* $L^2(N)$ *which is a sampling space with respect to* Γ.

Example 19 *Assume that* $\mathfrak{n}$ *is a four-dimensional* Z_1, Z_2, Z_3, A_1 *such that* $[A_1, Z_2] = 2Z_1, [A_1, Z_3] = 2Z_2$. *With respect to the ordered basis* Z_1, Z_2, Z_3, *we have*

$$[adA_1]|\,\mathfrak{p} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \exp\,[adA_1]|\,\mathfrak{p} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

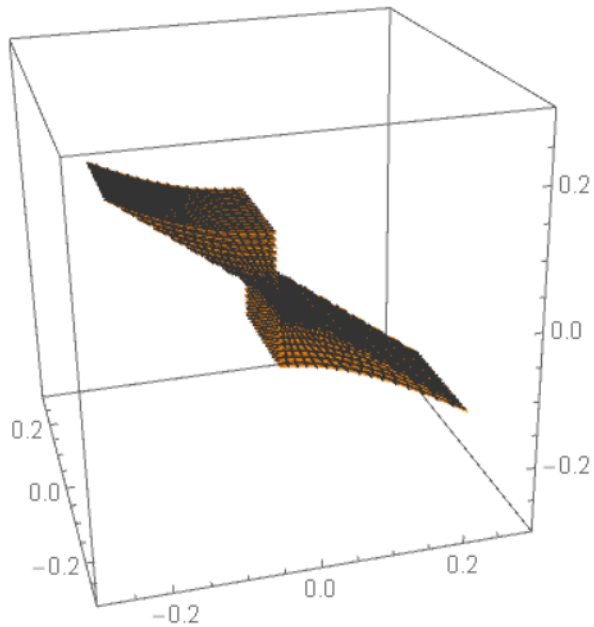
1. *Then*

$$\begin{aligned} \delta &= \left(2 \sup \left\{ \left\| \begin{bmatrix} 1 & 0 & 0 \\ -2s & 1 & 0 \\ 2s^2 & -2s & 1 \end{bmatrix} \right\|_\infty : s \in [0,1) \right\} \right)^{-1} \\ &= \frac{1}{2} \left(\max \left\{ 1, 1 + 2\,|s|\,, 1 + 2\,|s| + 2\,|s|^2 : s \in [0,1) \right\} \right)^{-1} = \frac{1}{10} \end{aligned}$$

2. *The set*

$$\beta \left(\left(-\frac{1}{10}, \frac{1}{10} \right)^2 \times [0,1) \right) \subset \left(-\frac{1}{2}, \frac{1}{2} \right)^3$$

is contained in a fundamental domain of $\mathbb{Z}^3$.



3. *Thus, there exists a band-limited vector* $\eta \in \mathbf{H}_{\left(-\frac{1}{10},\frac{1}{10}\right)}$ *such that* $W_\eta\left(\mathbf{H}_{\left(-\frac{1}{10},\frac{1}{10}\right)}\right)$ *is a sampling space with respect to*

$$\Gamma = \exp\left(\mathbb{Z}Z_1 + \mathbb{Z}Z_2 + \mathbb{Z}Z_3\right)\exp\left(\mathbb{Z}A_1\right).$$

Concluding remarks and self-criticism

1. The δ-condition is sufficient not necessary.

2. In general (for higher dimensions)

   - it is quite difficult to verify the statement: $\beta\left(\mathbf{A} \times [0,1)^m\right)$ *is equal to a fundamental domain of* Z^p.
   - The class of groups being handled is large but, our techniques do not work for all nilpotent Lie groups. For example, let $\mathfrak{n}$ be a ten-dimensional nilpotent Lie algebra such that
   $$\mathfrak{n} = \mathbb{R}\text{-span } \{X_1, X_2, X_3, X_4, X_{12}, X_{13}, X_{14}, X_{23}, X_{24}, X_{34}\}$$
   and for $i < j$ we have $\left[X_i, X_j\right] = X_{ij}$. The main problem here is that the **constant polarization property fails**.

3. Our technique seems to work with some non-unimodular Lie groups which are not nilpotent but are solvable. For example, our technique works for some solvable Lie groups. Let G be a solvable Lie group with Lie algebra
$$\mathfrak{g} = \mathbb{R}\text{-span } \{X_1, X_2, A\}$$
with non-trivial Lie brackets given by
$$[A, X_1] = X_1 \text{ and } [A, X_2] = -X_2.$$
Recent computations show, such a group can be handled as well.

Thank you very much for your attention.