Why was the Nobel prize winner D. Gabor wrong?

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Organization of the presentation

1. Modeling sounds as mathematical functions
2. Introduction to Gabor's theory
3. Very short introduction Functional analysis
4. Bases in infinite dimensional Hilbert spaces
5. Proof of the main result (disprove Gabor's conjecture)
6. (If time allows) Connection with quantum mechanic and the famous open problem known as the HRT conjecture.

Sound

Sound travels through air which is a gaseous medium of molecules that are constantly moving and exerting forces against each other. These forces create air pressure. That is, when a large quantity of air molecules are compressed in a small space, the pressure is high and when the molecules are spread out from one another, the pressure is low. Sound is a wave of alternating high and low air pressure that travels through time and space.

Sound

Some sounds caused by variation in air pressure can be perceived by the human ear's ability to recognize variations. For example, let us suppose that you pull the string of a violin. The resulting vibrations cause the pressure of the ambient air to vibrate as well. This propagates through the air as waves moving at the speed of roughly 768 miles per hour (speed of sound) alternating high and low air pressure that travels through time and space.

Vibration

If we realize a vibration as the up or down motion of a point, a vibration can be described as a function of time. Some vibrations can be modeled by a sinusoidal function

$$f\left(t\right) = A \sin\left(2\pi f t + p\right)$$

- A **sinusoidal** vibration has two important characteristics: **amplitude** A and **frequency** f.
- The length of time required for one cycle to move past is called the **period** $\left(T\right)$ of the vibration.
- The **frequency** is the number of cycles that move past in a given amount of time. Thus, $f = \frac{1}{T}$
- The constant p is called a **phase shift**.
- When frequencies are measured as cycles per second, the corresponding unit is called **Hertz** (Hz)

Vibration

- Humans are capable of hearing frequencies in the frequency interval $[20 \text{ Hz}, \quad 20000 \text{ Hz}]$. The human brain interprets the amplitude of a sinusoidal function as the volume of a tone.
- Dogs can hear sounds with much higher frequencies

Modeling keyboard notes

The musical notes of a keyboard can be mathematically modeled as follows (thanks to David Wood)

Function	**Time**	**Freq (Hz)**	**Note**
$c_4(t) = \sin(2\pi t \times 261.6)$	$\left[0, \frac{4}{10}\right]$	261.64	C_4
$d_4(t) = \sin(2\pi t \times 293.7)$	$\left[0, \frac{4}{10}\right]$	293.68	D_4
$e_4(t) = \sin(2\pi t \times 329.6)$	$\left[0, \frac{4}{10}\right]$	329.64	E_4
$f_4(t) = \sin(2\pi t \times 349.2)$	$\left[0, \frac{4}{10}\right]$	349.24	F_4
$g_4(t) = \sin(2\pi t \times 392.0)$	$\left[0, \frac{4}{10}\right]$	392.00	G_4
$a_4(t) = \sin(2\pi t \times 440.0)$	$\left[0, \frac{4}{10}\right]$	440.00	A_4
$b_4(t) = \sin(2\pi t \times 493.9)$	$\left[0, \frac{4}{10}\right]$	493.92	B_4

For example **Ode to Joy** by the German poet, and historian Friedrich Schiller and used by Ludwig van Beethoven in the final movement of his Ninth Symphony is obtained by playing the following sequence of functions

$$(e_4(t), e_4(t), f_4(t), g_4(t), g_4(t), f_4(t), e_4(t), d_4(t), c_4(t), c_4(t), d_4(t), e_4(t), e_4(t), d_4(t), d_4(t), e_4(t), e_4(t) f_4(t), g_4(t), g_4(t), e_4(t), d_4(t), c_4(t), c_4(t), d_4(t), e_4(t), d_4(t), c_4(t), c_4(t))$$

Mariage d'amour

We shall now listen to a composition of Richard Clayderman called **Mariage d'amour**

https://www.youtube.com/watch?v=sS16Ub2U6DY

together with the decomposition of the sound in its basic components "musical notes as shown above"

Time-frequency analysis

- **Time-frequency** analysis and processing is concerned with the analysis and processing of signals with time-varying frequency content.
- With recent progress in the subject, time-frequency analysis provides powerful tools for analyzing non-stationary signals such as radar, sonar, and speech.

Motivation for the definition of the energy of a signal

Regarding a signal as an electronic oscillation in voltage with resistance 1, the instantaneous power of this signal is understood as the derivative of the energy of the signal

$$\frac{d}{dt}\left(E\left(t\right)\right) = \left[V\left(t\right)\right]^2$$

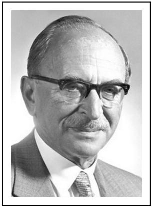
where E is the energy and V refers to the voltage. Next, the total energy expended is given by

$$\text{Total energy} = \int_{\mathbb{R}} \frac{dE\left(t\right)}{dt} dt = \int_{\mathbb{R}} \left[V\left(t\right)\right]^2 dt.$$

Atomic decomposition

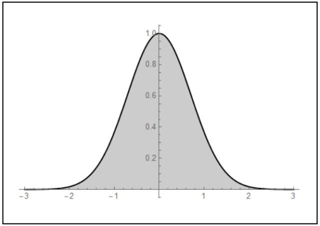
In order to analyze and describe complex phenomena, mathematicians, engineer and physicists often represent them as a superposition of simple and well-understood objects.
For example it is often of interest to be able to represent by a finite number of data the information contained in a signal.

The theory of Gabor



- Gábor Dénes (5 June 1900-9 February 1979) was a Hungarian-British electrical engineer and physicist, most notable for inventing holography, for which he later received the 1971 Nobel Prize in Physics.
- Gabor published in 1946 a major paper on the theory of communication.
- The theory provides a basis for the representation and processing of information in various sensory modalities.
- Gaussian functions can be regarded as atoms of signals of finite energy

Fixing the Gaussian function $g(t) = \exp(-t^2)$.



Gabor proposed that any signal of **finite energy** f :

$$\int_{\mathbb{R}} [f(t)]^2 \, dt < \infty$$

can be decomposed into an **infinite linear combination of time-frequency shifts of the Gaussian function**.

In other words, for integers j, k, defining

$$g_{j,k}(t) = e^{-2\pi itk} e^{-(t-j)^2}$$

there exist coefficients (scalars) $c_{k,j}$ depending on the signal f such that

$$f(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{k,j} \times g_{j,k}(t)$$

In 1932 John von Neumann conjectured without proof that the time-frequency shifts of the Gaussian span a dense subspace in the space of signals of finite energy. In 1946 Gabor conjectured that the time-frequency shifts of the Gaussian is a basis for the space of signals of finite energy. It was proved in the 1980s by Janssen that the convergence can only be understood in a sense of distribution (disregard this technicality). In colloquial terms, the expansions are numerically unstable and cannot be used in practice.

The idea to represent a function in terms of the time-frequency shifts of a single atom g originated fifteen years prior to Gabor's theory in quantum mechanics. In order to expand general functions (quantum mechanical states) with respect to states with minimal uncertainty, John von Neumann suggested to use of time-frequency shifts (parametrized by a lattice) of the Gaussian. While Gabor was awarded the Nobel Prize in Physics in 1971 for the invention of holography, his paper on the Theory of Communication was essentially ignored until the early eighties.

Definition

If $f:\mathbb{R}\to\mathbb{C}$ such that the integral

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx \text{ is convergent}$$

then we say that f is **square-integrable**.

- $|f(x)|$ stands for the modulus of the complex number $f(x)$

$$f(x) = a(x) + ib(x) \quad \Rightarrow \quad |f(x)| = \sqrt{a(x)^2 + b(x)^2}$$

- The set of all complex-valued square integrable functions is denoted by $L^2(\mathbb{R})$.
- The nonnegative quantity

$$\|f\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^2\, dx\right)^{1/2} \geq 0$$

is called the L^2-**norm or the energy of** f.

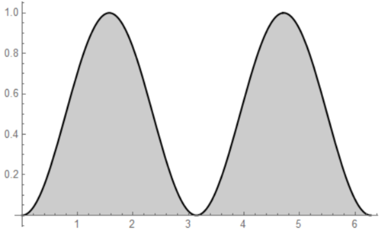
Examples of square-integrable functions

Let

$$f(x) = \begin{cases} \sin x \text{ if } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ 0 \text{ if } x \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{cases}$$

Then $\|f\|^2_{L^2(\mathbb{R})} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin x)^2 \, dx = \frac{\pi}{2}$.

Figure: Energy of $\sin(x)$



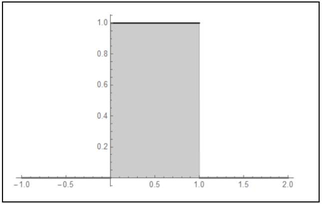
- Let $h(x)$ be a continuous function which vanishes outside of a closed interval. Then h is square-integrable.
- (The Gaussian) Put $g(x) = e^{-x^2}$. Then

$$\|g\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} \left(e^{-x^2}\right)^2 dx = \frac{\sqrt{2\pi}}{2}.$$

- If $p(x)$ is a polynomial then $g(x) = p(x)\, e^{-x^2}$ is square-integrable as well.

- $L^2(\mathbb{R})$ is an infinite-dimensional vector space. Indeed, define for every integer k,

$$\varphi_k(x) = \varphi(x-k) \text{ where } \varphi(x) = \begin{cases} 1 \text{ if } x \in [0,1) \\ 0 \text{ if } x \notin [0,1) \end{cases}.$$



- Define the collection of square-integrable functions

$$\Phi = \bigcup_{k \in \mathbb{Z}} \{\varphi(x-k)\} \subset L^2(\mathbb{R}).$$

- Then

$$V_\Phi = \left\{ \begin{array}{c} f \in L^2(\mathbb{R}) : f \text{ is a finite linear combinations} \\ \text{of elements in } \Phi \end{array} \right\}.$$

- V_Φ is an infinite-dimensional proper vector subspace of $L^2(\mathbb{R})$

Definition

Let X be a non-empty subset of $\mathbb{R}$ such that $\int_X dx > 0$. Next, let

$$L^2(X) = \left\{ \begin{array}{c} f : \mathbb{R} \to \mathbb{C} : f \text{ is supported on } X \\ \int_X |f(x)|^2 \, dx \text{ is convergent} \end{array} \right\}$$

Then $L^2(X)$ is a vector subspace of $L^2(\mathbb{R})$.

The dot product of square-integrable functions

Some important metric notions such as length, angle and energy of physical systems can be expressed in terms of inner product or dot product.

Definition

Given $f, g \in L^2(\mathbb{R})$, the **inner product** of f and g is given by

$$\langle f, g \rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx.$$

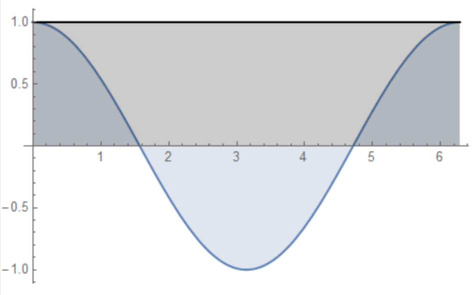
Example

Let

$$f(x) = \begin{cases} 1 \text{ if } x \in [0, 2\pi) \\ 0 \text{ if } x \notin [0, 2\pi) \end{cases}, g(x) = \begin{cases} \cos x \text{ if } x \in [0, 2\pi) \\ 0 \text{ if } x \notin [0, 2\pi) \end{cases}.$$

Then

$$\langle f, g \rangle_{L^2(\mathbb{R})} = \int_0^{2\pi} \cos(x)\, dx = 0.$$

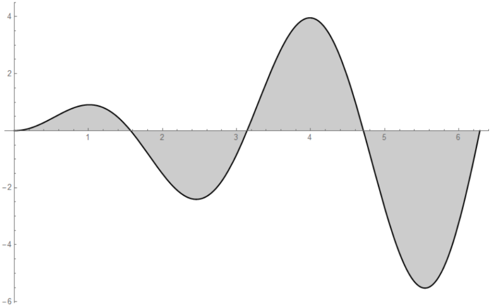


Let

$$f(x) = \begin{cases} \sin(2x) & \text{if } x \in [0, 2\pi) \\ 0 & \text{if } x \notin [0, 2\pi) \end{cases} \quad \text{and } g(x) = \begin{cases} x \text{ if } x \in [0, 2\pi) \\ 0 \text{ if } x \notin [0, 2\pi) \end{cases}$$

Then

$$\langle f, g \rangle_{L^2(\mathbb{R})} = \int_0^{2\pi} x \sin(2x)\, dx = -\pi$$



Definition

A sequence $\{f_k\}_{k=1}^{\infty}$ is an **orthonormal set** for $L^2(\mathbb{R})$ if

$$\langle f_m, f_n \rangle_{L^2(\mathbb{R})} = \delta_{mn} = \begin{cases} 1 \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases}.$$

Theorem

For any $f, g, h \in L^2(\mathbb{R})$ *and* $\lambda \in \mathbb{C}$

1. $\langle f, g + h \rangle_{L^2(\mathbb{R})} = \langle f, g \rangle_{L^2(\mathbb{R})} + \langle f, h \rangle_{L^2(\mathbb{R})}$
2. $\langle \lambda f, g \rangle_{L^2(\mathbb{R})} = \lambda \langle f, g \rangle_{L^2(\mathbb{R})}$
3. $\langle f, \lambda g \rangle_{L^2(\mathbb{R})} = \overline{\lambda} \langle f, g \rangle_{L^2(\mathbb{R})}$
4. $\langle f, g \rangle_{L^2(\mathbb{R})} = \langle h, g \rangle_{L^2(\mathbb{R})}$ *for all* $g \in L^2(\mathbb{R})$ *then* $f = h$.
5. *(Cauchy-Schwarz inequality) For any* $f, g \in L^2(\mathbb{R})$

$$\left| \langle f, g \rangle_{L^2(\mathbb{R})} \right| \leq \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}.$$

In other words,

$$\left| \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx \right| \leq \left(\int_{\mathbb{R}} |f(x)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} |g(x)|^2 dx \right)^{1/2}.$$

We need to address convergence issues in L^2-norm

Finite sums

Let $\mathcal{F} = \{f_k : 1 \leq k \leq n\} \subset L^2(\mathbb{R})$ be a finite subset. Intuitively, we believe that

$$f = \sum_{k=1}^{n} a_k f_k \text{ where } a_k \in \mathbb{C}$$

is square-integrable. Indeed,

$$\left\| \sum_{k=1}^{n} a_k f_k \right\|_{L^2(\mathbb{R})}^2 = \left\langle \sum_{k=1}^{n} a_k f_k, \sum_{j=1}^{n} a_j f_j \right\rangle = \sum_{k=1}^{n} \sum_{j=1}^{n} a_k \overline{a_j} \langle f_k, f_j \rangle < \infty$$

We cannot form infinite linear combinations unless we have some notion of what it means to **converge** in $L^2(\mathbb{R})$. Generally, given a sequence of square integrable functions $(f_k)_{k=1}^{\infty}$, the infinite series

$$\sum_{k=1}^{\infty} f_k$$

need not converge. If it does converge then there are additional issues with the convergence that need to be considered. For example, one needs to address the **stability of the convergence.** It is possible that the convergence actually **depends on the ordering** of the f_k. Another aspect of convergence that we need to address is **absolute convergence.**

Definition

Let $(f_k)_{k=1}^{\infty}$ be a sequence of vectors in $L^2(\mathbb{R})$. We say that $\sum_{k=1}^{\infty} f_k$ is **convergent in** $L^2(\mathbb{R})$ if the partial sums

$$S_N = \sum_{k=1}^{N} f_k$$

converges to some function $f \in L^2(\mathbb{R})$ in the following sense. For every $\epsilon > 0$ there exists $N_0 > 0$ such that for all $N \geq N_0$

$$\left\| \left(\sum_{k=1}^{N} f_k \right) - f \right\|_{L^2(\mathbb{R})} = \left(\int_{-\infty}^{\infty} \left| \sum_{k=1}^{N} f_k(x) - f(x) \right|^2 dx \right)^{1/2} < \epsilon.$$

In other words,

$$\lim_{N \to \infty} \left\| \sum_{k=1}^{N} f_k - f \right\|_{L^2(\mathbb{R})} = 0.$$

Definition

(see A Basis Theory Primer by Chris Heil) Let $(f_k)_{k=1}^{\infty}$ be a sequence of vectors in $L^2(\mathbb{R})$.

1. The series $\sum_{k=1}^{\infty} f_k$ is **unconditionally convergent** if

$$\sum_{k=1}^{\infty} f_{\sigma(k)}$$

   is convergent to f in $L^2(\mathbb{R})$ for every permutation σ of $\mathbb{N}$ (for every bijection $\sigma : \mathbb{N} \to \mathbb{N}$)

2. The series $\sum_{k=1}^{\infty} f_k$ is **absolutely convergent** if the series

$$\sum_{k=1}^{\infty} \|f_k\|_{L^2(\mathbb{R})}$$

   is convergent.

Convergence of classical series vs convergence in the L^2-norm

It is worth noting that one should not confuse convergence results of real or complex series with convergence results in $L^2(\mathbb{R})$. F*or example, if* $(c_n)_{n\in\mathbb{N}}$ *is a sequence of real or complex numbers then* $\sum_{n\in\mathbb{N}} c_n$ *converges unconditionally if and only if it converges absolutely.* Those results cannot be extended in the settings of convergence in the $L^2(\mathbb{R})$-norm.

Theorem

(Lemma 3.5 in Basis theory primer of C. Heil) If $\sum_{k=1}^{\infty} f_k$ is absolutely convergent then it converges unconditionally.

Theorem

Given an **orthonormal sequence** $\{e_k\}_{k=1}^{\infty}$ *in* $L^2(\mathbb{R})$ *the following statements are equivalent*

1. $\sum_{k=1}^{\infty} a_k e_k$ *converges*
2. $\sum_{k=1}^{\infty} a_k e_k$ *converges unconditionally*
3. $(a_k)_{k=1}^{\infty}$ *is square-summable in the sense that* $\sum_{k=1}^{\infty} |a_k|^2 < \infty$.

Note

$$\text{unconditional convergence} \;\not\Rightarrow\; \text{absolute convergence}$$

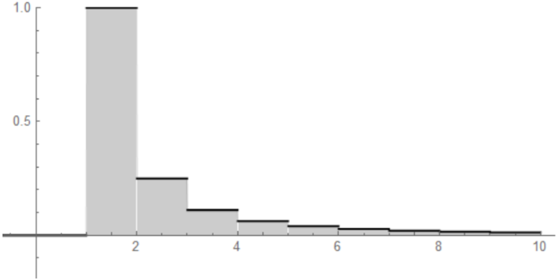
To see this, let $\{f_k\}_{k=1}^{\infty}$ be an orthonormal sequence such that

$$f_k(x) = f(x-k) \text{ where } f(x) = \begin{cases} 1 \text{ if } x \in [0,1) \\ 0 \text{ if } x \notin [0,1) \end{cases}.$$

From our previous results, $\sum_{k=1}^{\infty} a_k f_k$ is convergent if and only if

$$\sum_{k=1}^{\infty} |a_k|^2 < \infty \quad \text{(square summable)}$$

Figure: Infinite sum



Thus $\sum_{k=1}^{\infty} \frac{1}{k} f_k$ is **unconditionally convergent** since

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

However, $\sum_{k=1}^{\infty} \frac{1}{k} f_k$ is **not absolutely convergent**. Indeed,

$$\sum_{k=1}^{\infty} \left\| \frac{1}{k} f_k \right\|_{L^2(\mathbb{R})} = \sum_{k=1}^{\infty} \frac{1}{k} \underbrace{\| f_k \|_{L^2(\mathbb{R})}}_{=1} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

Theorem

*(**Orlicz's Theorem**) If $(f_k)_{k\in\mathbb{N}}$ is a sequence in a Hilbert space and $\sum_{k=1}^{\infty} f_k$ is unconditionally convergent then*

$$\sum_{k=1}^{\infty} \|f_k\|_{L^2(\mathbb{R})}^2 < \infty \text{ (note the square in the summand)}$$

This result is proved in Theorem 3.27 in the Monograph *A Basis theory primer* by Chris Heil.

Bases in $L^2(\mathbb{R})$

- A countable sequence $\{f_k\}_{k=1}^{\infty}$ in the vector space of square-integrable functions is a **basis** for $L^2(\mathbb{R})$ if for any $f \in L^2(\mathbb{R})$ there exist **unique scalars** $a_k(f) \in \mathbb{C}$ such that

$$f = \sum_{k=1}^{\infty} a_k(f) \times f_k.$$

- The series above is called the **basis expansion** with respect to $\{f_k\}_{k=1}^{\infty}$. The convergence above is to be understood in the L^2-norm. In other words, given any $\epsilon > 0$ there exists $N > 0$ such that as long as $n \geq N$,

$$\left\| f - \left(\sum_{k=1}^{n} (a_k(f) \times f_k) \right) \right\|_{L^2(\mathbb{R})} < \epsilon.$$

- A sequence $\{f_k\}_{k=1}^{\infty}$ is a **complete sequence** sequence if every element $f \in L^2(\mathbb{R})$ can be written as the limit of a sequence of finite linear combinations of the elements f_k. More precisely, given $\epsilon > 0$ there exists some $N > 0$ and

$$c_1(f,\epsilon), \cdots, c_N(f,\epsilon) \in \mathbb{C}$$

such that

$$\left\| f - \sum_{k=1}^{n} c_k(f,\epsilon) f_k \right\|_{L^2(\mathbb{R})} < \epsilon.$$

- Note that the coefficients $c_k(f,\epsilon)$ depends on both ϵ and f.

Theorem

Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of square-integrable functions. The following are equivalent

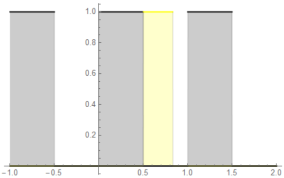
1. $\{f_k\}_{k=1}^{\infty}$ *is complete*
2. *The only element $f \in L^2(\mathbb{R})$ which satisfies $\langle f, f_k \rangle_{L^2(\mathbb{R})} = 0$ for every k is $f = 0$ (the zero function)*

The sequence $\{f_k\}_{k=1}^{\infty}$ given by $f_k(x) = f(x-k)$ where

$$f(x) = \begin{cases} 1 \text{ if } x \in \left[0, \frac{1}{2}\right) \\ 0 \text{ if } x \notin \left[0, \frac{1}{2}\right) \end{cases}$$

is **not a complete system** in $L^2(\mathbb{R})$. For example, let $\epsilon > 0$ such that $\epsilon < \frac{1}{2}$

$$g(x) = \begin{cases} 1 \text{ if } x \in \left[\frac{1}{2}, \frac{1}{2}+\epsilon\right) \\ 0 \text{ if } x \notin \left[\frac{1}{2}, \frac{1}{2}+\epsilon\right) \end{cases}.$$



Clearly g $\forall k \in \mathbb{N}$, $\langle g, f_k \rangle_{L^2(\mathbb{R})} = 0$.

Theorem

The following conditions are equivalent

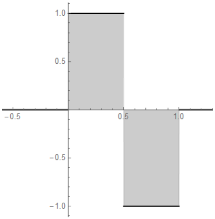
1. $\{e_k : k \in \mathbb{N}\}$ *is a maximal orthonormal set.*
2. $f \in L^2(\mathbb{R}) \Rightarrow f = \sum_{k\in\mathbb{N}} \langle f, e_k \rangle_{L^2(\mathbb{R})} e_k$
3. $f \in L^2(\mathbb{R}) \Rightarrow \|f\|^2_{L^2(\mathbb{R})} = \sum_{k\in\mathbb{N}} \left| \langle f, e_k \rangle_{L^2(\mathbb{R})} \right|^2$

Sets satisfying any of conditions above are called **orthonormal bases**

Let

$$\psi(x) = \begin{cases} 1 \text{ if } x \in \left[0, \frac{1}{2}\right) \\ -1 \text{ if } x \in \left[\frac{1}{2}, 1\right) \\ 0 \text{ otherwise} \end{cases}$$

and



define

$$\psi_{n,k}(x) = 2^{n/2}\psi(2^n x - k).$$

For any ordering of $\psi_{n,k}$, the sequence of functions $\psi_{n,k}$ is a complete system. In fact, it is an orthonormal basis for $L^2(\mathbb{R})$. This basis is known as the **Haar basis**.

Hermite functions

Let $a > 0$ be fixed and define the hermite functions

$$f_0(x) = e^{-ax^2}$$
$$f_1(x) = -4axe^{-ax^2}$$
$$\vdots$$
$$f_n(x) = e^{ax^2}\frac{d^n}{dx^n}\left(e^{-2ax^2}\right)$$

Next define

$$h_n(x) = c_n f_n(x) \text{ where } c_n = \left(4^n a^n n!\right)^{-1/2}\left(\frac{\pi}{2a}\right)^{-1/4}$$

Then the functions h_n form an orthonormal basis for $L^2(\mathbb{R})$

Definition

Given a vector space V over the complex, a norm is a function

$$\mu : V \to \mathbb{R}$$

satisfying the following properties. For all, $u, v \in V$ and $a \in \mathbb{C}$ the following holds true.

1. (absolute homogeneity or absolute scalability). $\mu(av) = |a| \mu(v)$
2. (triangle inequality) $\mu(u+v) \leq \mu(u) + \mu(v)$
3. If $\mu(v) = 0$ then v is the zero vector (the norm separates points)

Let X be a vector space endowed with a norm μ. The pair (X, μ) is called a **normed vector space**. Generally, we shall write

$$\mu(v) = \|v\|_X .$$

Continuous and Bounded Operators

- Assume that X, Y which are normed vector spaces. Next, let $T : X \to Y$ be an operator. T is **continuous** if $f_n \to f$ implies that $T(f_n) \to T(f)$. In other words,

$$\lim_{n\to\infty} \|f - f_n\|_X = 0 \Rightarrow \lim_{n\to\infty} \|Tf - Tf_n\|_Y = 0$$

- We say that T is **bounded** if there is a finite real number C such that

$$\|Tf\|_Y \leq C \|f\|_X$$

for every $f \in X$.

- The **operator norm** of T is given by

$$\|T\|_{\text{Operator}} = \sup_{f \neq 0} \frac{\|Tf\|_Y}{\|f\|_X} = \sup_{\|f\|_X = 1} \|Tf\|_Y$$

- For linear operators, $(T(af + bg) = aTf + bTg)$ there is a fundamental result which states that

$$T \text{ is continuous } \Leftrightarrow T \text{ is bounded.}$$

An example of a bounded operator

- Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for the Hilbert space of all square-integrable functions.
- Define the operator

$$T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

such that

$$Tf = \sum_{k=1}^{\infty} \frac{\langle f, e_k \rangle}{k} e_k.$$

- T is a bounded operator. Indeed,

$$\|Tf\|_{L^2(\mathbb{R})}^2 = \left\| \sum_{k=1}^{\infty} \frac{\langle f, e_k \rangle}{k} e_k \right\|_{L^2(\mathbb{R})}^2 = \sum_{k=1}^{\infty} \left| \frac{\langle f, e_k \rangle}{k} \right|^2$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^2} |\langle f, e_k \rangle|^2 \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \|f\|_{L^2(\mathbb{R})}^2$$

- It can be shown that $\|T\|_{\text{Operator}} = \frac{\pi}{6^{1/2}}$

An example of an unbounded operator

- T be the Laplace operator defined as $Tf = -f''$ for suitable f.
- Let $f_k(x) = e^{-k|x|}$.
- Then
$$\|f_k\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} e^{-2k|x|} dx = \frac{1}{k}$$
- $\|Tf_k\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} k^4 e^{-2k|x|} dx = k^3$
- Consequently,
$$\lim_{k\to\infty} \frac{\|Tf_k\|_{L^2(\mathbb{R})}}{\|f_k\|_{L^2(\mathbb{R})}} = \lim_{k\to\infty} \frac{k^{3/2}}{\frac{1}{k^{1/2}}} = \lim_{k\to\infty} k^2 = \infty$$
- $\sup_{f\neq 0} \frac{\|Tf\|_{L^2(\mathbb{R})}}{\|f\|_{L^2(\mathbb{R})}} = \infty$
- T is an unbounded operator on $L^2(\mathbb{R})$.

Linear functional

Definition

Let

$$T : L^2(\mathbb{R}) \to \mathbb{C}$$

We say that T is a linear functional if T is a linear map.

Unconditional bases and absolutely convergent bases

- Let $\{f_k\}_{k=1}^{\infty}$ be a countable sequence in the vector space of square-integrable functions which is also a basis for $L^2(\mathbb{R})$.
- $\{f_k\}_{k=1}^{\infty}$ is called an **unconditional basis** if the series

$$f = \sum_{k=1}^{\infty} \left(a_k(f) \cdot f_k\right)$$

  is unconditionally convergent for every $f \in L^2(\mathbb{R})$.
- $\{f_k\}_{k=1}^{\infty}$ is called an **absolutely convergent basis** if the series

$$f = \sum_{k=1}^{\infty} \left(a_k(f) \cdot f_k\right)$$

  converge absolutely for every $f \in L^2(\mathbb{R})$.

Schauder bases

Note that the uniqueness of the coefficients in the expansion above implies that the map a_k given by $f \mapsto a_k(f)$ is a linear functional on $L^2(\mathbb{R})$.

Definition

Let $\{f_k\}_{k=1}^{\infty}$ be a basis. We say that $\{f_k\}_{k=1}^{\infty}$ is a **Schauder basis** if each a_k is a continuous linear functional (a bounded linear operator.)

- If $\{f_k\}_{k=1}^{\infty}$ is an **orthonormal basis** for $L^2(\mathbb{R})$ then any square-integrable function f admits the expansion

$$f = \sum_{k=1}^{\infty} \langle f, f_k \rangle_{L^2(\mathbb{R})} f_k$$

  and the linear functional

$$f \mapsto \langle f, f_k \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x) \overline{f_k(x)} dx$$

  is a bounded and thus continuous linear operator.
- An orthonormal basis is also a Schauder basis.

Riesz bases

- Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in $L^2(\mathbb{R})$. We say that $\{f_k\}_{k=1}^{\infty}$ is a **Riesz basis** if there exists a bounded bijective map

$$T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

  such that $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis and $Te_k = f_k$.
- A Riesz basis is the image of an orthonormal basis under a bounded bijective map.

Riesz bases are bounded bases

- A Riesz basis is necessarily bounded.
- Let $\{f_k\}_{k=1}^{\infty}$ be a Riesz basis for $L^2(\mathbb{R})$. Then there exists an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ and a bounded invertible operator T such that $Te_k = f_k$.
- $$\|f_k\|_{L^2(\mathbb{R})} = \|Te_k\|_{L^2(\mathbb{R})} \leq \|T\|_{\text{Operator}} \, \|e_k\|_{L^2(\mathbb{R})} = \|T\|_{\text{Operator}}$$
- Next,
$$1 = \|e_k\|_{L^2(\mathbb{R})} = \left\|T^{-1}f_k\right\|_{L^2(\mathbb{R})} \leq \left\|T^{-1}\right\|_{\text{Operator}} \, \|f_k\|_{L^2(\mathbb{R})}$$
and for all $k \in \mathbb{N}$, the following holds true
$$\frac{1}{\|T^{-1}\|_{\text{Operator}}} \leq \|f_k\|_{L^2(\mathbb{R})} \leq \|T\|_{\text{Operator}}$$

Riesz bases vs Schauder bases

Theorem

(Köthe, Lorch) **Every Riesz basis is a bounded unconditional basis** *for* $L^2(\mathbb{R})$. *Moreover, a Schauder basis* $\{f_k\}_{k=1}^{\infty}$ *is a Riesz basis if and only if* $\{f_k\}_{k=1}^{\infty}$ *is an unconditional basis and there exist* $A, B > 0$ *such that*

$$A \leq \|f_k\|_{L^2(\mathbb{R})} \leq B \text{ for all } k \in \mathbb{N}.$$

A complete characterization of Riesz bases

Theorem

The following are equivalent

1. $\{f_k\}_{k=1}^{\infty}$ *is a Riesz basis for* $L^2(\mathbb{R})$.
2. *The sequence* $\{f_k\}_{k=1}^{\infty}$ *is complete in* $L^2(\mathbb{R})$ *and there exist positive constants* A, B *such that for any scalars* $c_1, \cdots, c_n$ *one has*

$$A\sum_{i=1}^{n}|c_i|^2 \leq \left\|\sum_{i=1}^{n} c_i f_i\right\|_{L^2(\mathbb{R})}^2 \leq B\sum_{i=1}^{n}|c_i|^2.$$

Dual Riesz bases

Theorem

If $\{f_k : k \in \mathbb{N}\}$ is a Riesz basis for $L^2(\mathbb{R})$, there exists a unique a sequence $\{g_k : k \in \mathbb{N}\}$ in $L^2(\mathbb{R})$ such that

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle_{L^2(\mathbb{R})} f_k$$

for every $f \in L^2(\mathbb{R})$. In this case the sequence $\{g_k : k \in \mathbb{N}\}$ is also a Riesz basis.

Zak Transform

- Let

$$\mathbf{Q} = \left\{ f : \left[-\frac{1}{2}, \frac{1}{2} \right)^2 \to \mathbb{C} \;\middle|\; \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} |f(x,\xi)|^2 \, dx d\xi < \infty \right\}$$

We define formally the **Zak transform** as follows $\mathcal{Z} : L^2(\mathbb{R}) \to \mathbf{Q}$ and

$$\mathcal{Z} f(x,\xi) = \sum_{k \in \mathbb{Z}} f(x+k) \, e^{-2\pi i k \xi}.$$

- Note that since the Zak transform is defined in term of an infinite sum. As such, *there is no reason to believe that this definition actually makes sense.*

Okay, so why does this definition make sense?

$$\begin{aligned}
\langle f,g\rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} f(x)\overline{g(x)}dx \\
&= \int_{-1/2}^{1/2}\sum_{\ell\in\mathbb{Z}} f(x+\ell)\overline{g(x+\ell)}dx \\
&= \int_{-1/2}^{1/2}\left(\sum_{\ell\in\mathbb{Z}} f(x+\ell)\overline{g(x+\ell)}\right)dx \\
&= \int_{-1/2}^{1/2}\sum_{\ell\in\mathbb{Z}}\sum_{\kappa\in\mathbb{Z}} f(x+\ell)\overline{g(x+\kappa)}\left(\int_{-1/2}^{1/2} e^{2\pi i(\ell-\kappa)}d\xi\right)dx
\end{aligned}$$

$$
\begin{aligned}
\langle f, g\rangle_{L^2(\mathbb{R})} &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\sum_{\ell\in\mathbb{Z}}\sum_{\kappa\in\mathbb{Z}} f\left(x+\ell\right)e^{2\pi i(\ell)}\overline{g\left(x+\kappa\right)}e^{2\pi i(-\kappa)}d\xi dx \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\left(\sum_{\ell\in\mathbb{Z}} f\left(x+\ell\right)e^{2\pi i\ell}\right)\left(\overline{\sum_{\kappa\in\mathbb{Z}} g\left(x+\kappa\right)e^{2\pi i\kappa}}\right)d\xi dx \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\mathcal{Z}f\left(x,\xi\right)\overline{\mathcal{Z}g\left(x,\xi\right)}d\xi dx \\
&= \langle \mathcal{Z}f, \mathcal{Z}g\rangle_{L^2\left(\left[-\frac{1}{2},\frac{1}{2}\right)^2\right)}.
\end{aligned}
$$

- The Zak transform was studied by J. Zak in connection with solid state physics. Recently the Zak transform has been studied by Janssen as well.
- Here are some references.

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The Zak transform is a unitary map

In light of the computations above, the Zak transform maps $L^2(\mathbb{R})$ isometrically into the space of square-integrable functions over the torus $\left[-\frac{1}{2}, \frac{1}{2}\right)^2$. That is, the map

$$\mathcal{Z} : L^2(\mathbb{R}) \to L^2\left(\left[-\frac{1}{2}, \frac{1}{2}\right)^2\right)$$

given by

$$f \mapsto \mathcal{Z}f(x, \xi) = \sum_{k \in \mathbb{Z}} f(x+k)\, e^{-2\pi i k \xi}$$

is a norm or energy preserving function.

Definition

Define the **translation** and **modulation** operators respectively as follows. For integers κ, ℓ and $f \in L^2(\mathbb{R})$

$$T_\kappa f(x) = f(x - \kappa) \text{ and } M_\ell f(x) = e^{2\pi i \ell \cdot x} f(x).$$

The group generated by T, M endowed with composition of operation is an abelian group which is isomorphic to $(\mathbb{Z}^2, +)$. Indeed, it can be shown that for every square-integrable function f

$$TMf = MTf.$$

Thus, $\mathbb{Z}^2$ acts on square-integrable functions by time-frequency shifts and the system of vectors

$$\left\{ e^{2\pi i \ell x} f(x - \kappa) : (\kappa, \ell) \in \mathbb{Z}^2 \right\}$$

can be regarded as the orbit of f under a $\mathbb{Z}^2$-action. Gabor theory's conjecture can then be reformulated as follows. The orbit of the Gaussian function under the action of $\mathbb{Z}^2$ (via time-frequency shifts) is some kind of basis for $L^2(\mathbb{R}^2)$

- Put

$$\chi_{[0,1)}(x) = \begin{cases} 1 \text{ if } x \in [0,1) \\ 0 \text{ if } x \notin [0,1) \end{cases}$$

- (Fact) The collection

$$\left\{ T_\kappa M_\ell \chi_{[0,1)} : (\kappa, \ell) \in \mathbb{Z}^2 \right\}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

- It is not hard to verify that

$$\left[\mathcal{Z} T_\kappa M_\ell \chi_{[0,1)} \right](x, \xi) = e^{2\pi i \kappa \cdot \xi} e^{2\pi i \ell \cdot x}$$

is an orthonormal basis of $L^2\left(\left[-\frac{1}{2}, \frac{1}{2}\right)^2\right)$. Thus, the Zak transform maps orthonormal bases to orthonormal bases.

Theorem

$\mathcal{Z}$ is a surjective isometry. In other words, the Zak transform is a unitary map. Moreover, the Zak transform maps Riesz bases to Riesz bases, and its inverse which is also a unitary map (surjective isometry) maps Riesz bases to Riesz bases.

We shall next need the following result.

Theorem

(E. Hernandez, H Sikic, G. Weiss, E. Wilson) Let $f \in L^2(\mathbb{R})$. *The collection* $\left\{ T_\kappa M_\ell f : (\kappa, \ell) \in \mathbb{Z}^2 \right\}$ *is a Riesz basis for* $L^2(\mathbb{R})$ *if and only if there positive constants* A, B *such that*

$$0 < A \leq |\mathcal{Z} f(x, \xi)|^2 \leq B$$

for $(x, \xi) \in \left[-\frac{1}{2}, \frac{1}{2} \right)^2$.

Theorem

(E. Hernandez, H Sikic, G. Weiss, E. Wilson) Let $f \in L^2(\mathbb{R})$. *Let*

$$\Omega_f = \left\{ (x, \xi) \in \left[-\frac{1}{2}, \frac{1}{2} \right)^2 : |\mathcal{Z} f (x, \xi)|^2 = 0 \right\}$$

The collection $\left\{ T_\kappa M_\ell f : (\kappa, \ell) \in \mathbb{Z}^2 \right\}$ *is complete in* $L^2(\mathbb{R})$ *if and only if*

$$\int_{\Omega_f} dx d\xi = 0.$$

- Note that

$$\left|\mathcal{Z}f\left(x,\xi\right)\right|^{2}=\left|\sum_{k\in\mathbb{Z}}f\left(x+k\right)e^{-2\pi ik\xi}\right|^{2}.$$

- To prove that $\left\{T_{\kappa}M_{\ell}f:\left(\kappa,\ell\right)\in\mathbb{Z}^{2}\right\}$ is a Riesz basis for $L^{2}\left(\mathbb{R}\right)$, it is enough to verify that for all $\left(x,\xi\right)\in\left[-\frac{1}{2},\frac{1}{2}\right)^{2}$

$$\inf_{(x,\xi)\in\left[-\frac{1}{2},\frac{1}{2}\right)^{2}}\left|\sum_{k\in\mathbb{Z}}f\left(x+k\right)e^{-2\pi ik\xi}\right|^{2}>0$$

and

$$\sup_{(x,\xi)\in\left[-\frac{1}{2},\frac{1}{2}\right)^{2}}\left|\sum_{k\in\mathbb{Z}}f\left(x+k\right)e^{-2\pi ik\xi}\right|^{2}\in\mathbb{R}.$$

Next, we shall prove the following results

Theorem

The following holds true

1. *The time-frequency shifts of the Gaussian is complete in* $L^2(\mathbb{R})$.
2. *The time-frequency shifts of the Gaussian is not a Riesz basis.*

The first result was conjectured by von Neumann in the 1930s and the second result is related to Gabor's conjecture.

Definition

(The third Elliptic Theta function)

$$\mathrm{EllipticTheta}\,(3, u, q) = 1 + 2\sum_{k=1}^{\infty} (q)^{k^2} \cos(2ku).$$

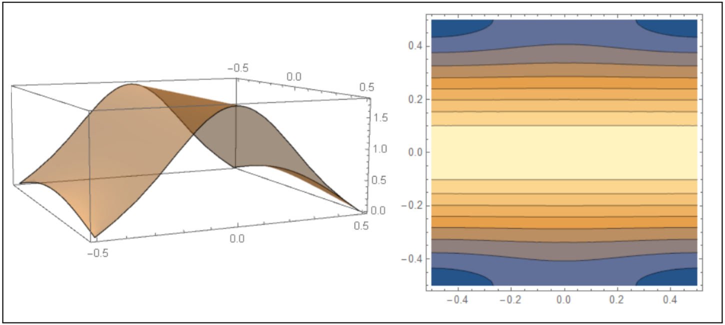
Let

$$g(x) = e^{-k^2}$$

be the Gaussian. Then

$$\begin{aligned}[\mathcal{Z}g](x,\xi) &= \sum_{k\in\mathbb{Z}} g(x+k)\, e^{-2\pi ik\xi} \\ &= e^{-\pi\xi(-2ix+\pi\xi)}\pi^{1/2} \times \mathrm{EllipticTheta}\left(3, \pi(x+i\pi\xi), e^{-\pi^2}\right) \\ &= e^{-\pi\xi(-2ix+\pi\xi)}\pi^{1/2}\left(1+2\sum_{k=1}^{\infty}\left(e^{-\pi^2}\right)^{k^2}\cos\left(2k\left(\pi\left(x+i\pi\xi\right)\right)\right)\right)\end{aligned}$$

The modulus of the Zak transform



Next, $[\mathcal{Z}g]\left(\frac{1}{2},\frac{1}{2}\right)$ is equal to

$$e^{-\pi\frac{1}{2}\left(-2i\frac{1}{2}+\pi\frac{1}{2}\right)}\pi^{1/2}\left(1+2\sum_{k=1}^{\infty}\left(e^{-\pi^2}\right)^{k^2}\cos\left(2k\left(\pi\left(\frac{1}{2}+i\pi\frac{1}{2}\right)\right)\right)\right)$$

$$=ie^{-\frac{\pi^2}{4}}\pi^{1/2}\left(1+2\sum_{k=1}^{\infty}\left(e^{-\pi^2}\right)^{k^2}\cos\left(2k\left(\pi\left(\frac{1}{2}+i\pi\frac{1}{2}\right)\right)\right)\right)$$

$$=ie^{-\frac{\pi^2}{4}}\pi^{1/2}\left(1+2\sum_{k=1}^{\infty}\left(e^{-\pi^2}\right)^{k^2}\cos\left(k\left(\pi\left(1+i\pi\right)\right)\right)\right)$$

$$=ie^{-\frac{\pi^2}{4}}\pi^{1/2}\left(1+2\sum_{k=1}^{\infty}\left(e^{-\pi^2}\right)^{k^2}\cos\left(k\pi\right)\cosh\left(k\pi^2\right)\right).$$

However,

$$\sum_{k=1}^{\infty}\left(e^{-\pi^2}\right)^{k^2}\cos\left(k\pi\right)\cosh\left(k\pi^2\right)=-\frac{1}{2}.$$

Consequently,

$$\begin{aligned}[\mathcal{Z}g]\left(\frac{1}{2},\frac{1}{2}\right)&=ie^{-\frac{\pi^2}{4}}\pi^{1/2}\left(1+2\left(-\frac{1}{2}\right)\right)\\&=ie^{-\frac{\pi^2}{4}}\pi^{1/2}\left(1-1\right)\\&=ie^{-\frac{\pi^2}{4}}\pi^{1/2}\times 0\\&=0.\end{aligned}$$

Thus

$$\inf_{(x,\xi)\in\left[-\frac{1}{2},\frac{1}{2}\right)^2}\left|e^{-\pi\xi(-2ix+\pi\xi)}\pi^{1/2}\left(1+2\sum_{k=1}^{\infty}\left(e^{-\pi^2}\right)^{k^2}\cos\left(2k\left(\pi\left(x+i\pi\xi\right)\right)\right)\right)\right.$$

is equal to zero, and the collection

$$\left\{e^{2\pi i\ell\cdot x}g\left(x-\kappa\right):\left(\kappa,\ell\right)\in\mathbb{Z}^2\right\}$$

is **not a Riesz basis**.

- Next

$$[\mathcal{Z}g](x,\xi) \Leftrightarrow (x,\xi) \in \Omega_g = \left\{ \begin{array}{c} \left(-\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},-\frac{1}{2}\right), \\ \left(-\frac{1}{2},-\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right) \end{array} \right\}$$

- Consequently,

$$\int_{\Omega_g} dx d\xi = 0.$$

- It follows that the collection

$$\left\{ e^{2\pi i \ell \cdot x} g(x-\kappa) : (\kappa,\ell) \in \mathbb{Z}^2 \right\}$$

is complete in $L^2(\mathbb{R})$.

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Thanks for your attention