SINC-TYPE FUNCTIONS ON A CLASS OF NILPOTENT LIE GROUPS

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Abstract. Let N be a simply connected, connected nilpotent Lie group with the following assumptions. Its Lie algebra $\mathfrak{n}$ is an n-dimensional vector space over the reals. Moreover, $\mathfrak{n}=\mathfrak{z}\oplus\mathfrak{b}\oplus\mathfrak{a}$, $\mathfrak{z}$ is the center of $\mathfrak{n}$, $\mathfrak{z}=\mathbb{R}Z_{n-2d}\oplus\mathbb{R}Z_{n-2d-1}\oplus\cdots\oplus\mathbb{R}Z_1$, $\mathfrak{b}=\mathbb{R}Y_d\oplus\mathbb{R}Y_{d-1}\oplus\cdots\oplus\mathbb{R}Y_1$, $\mathfrak{a}=\mathbb{R}X_d\oplus\mathbb{R}X_{d-1}\oplus\cdots\oplus\mathbb{R}X_1$. Next, assume $\mathfrak{z}\oplus\mathfrak{b}$ is a maximal commutative ideal of $\mathfrak{n}$, $[\mathfrak{a},\mathfrak{b}]\subseteq\mathfrak{z}$, and $\det\left([X_i,Y_j]\right)_{1\leq i,j\leq d}$ is a non-trivial homogeneous polynomial defined over the ideal $[\mathfrak{n},\mathfrak{n}]\subseteq\mathfrak{z}$. We do not assume that $[\mathfrak{a},\mathfrak{a}]$ is generally trivial. We obtain some precise description of band-limited spaces which are sampling subspaces of $L^2(N)$ with respect to some discrete set Γ. The set Γ is explicitly constructed by fixing a strong Malcev basis for $\mathfrak{n}$. We provide sufficient conditions for which a function f is determined from its sampled values on $(f(\gamma))_{\gamma\in\Gamma}$. We also provide an explicit formula for the corresponding sinc-type functions. Several examples are also computed in the paper.

1. Introduction

Let Ω be a positive number. A function f in $L^2(\mathbb{R})$ is called Ω-band-limited if its Fourier transform: $\mathcal{F}f(\lambda)$ is equal to zero for almost every λ outside of the interval $[-\Omega,\Omega]$. According to the well-known **Shannon-Whittaker-Kotel'nikov** theorem, f is determined by its sampled values $\left(f\left(\frac{\pi n}{\Omega}\right)\right)_{n\in\mathbb{Z}}$. In fact, for any function in the Hilbert space

$$\mathbf{H}_\Omega=\left\{f\in L^2(\mathbb{R}):\ \text{support }\ (\mathcal{F}f)\subset[-\Omega,\Omega]\right\}$$

we have the following reconstruction formula

$$f(x)=\sum_{n\in\mathbb{Z}}f\left(\frac{\pi n}{\Omega}\right)\frac{\sin(\pi(x-k))}{(\pi(x-k))} \tag{1}$$

and we say that $\mathbf{H}_\Omega$ is a **sampling subspace** of $L^2(\mathbb{R})$ with respect to the lattice $\frac{\pi}{\Omega}\mathbb{Z}$. A relatively novel problem in abstract harmonic analysis has been to find analogues of (1) for other locally compact groups [6, 4, 14, 15, 12, 13]. Any attempt to generalize the given formula above leads to several obstructions.

(1) Let us recall that a unitary representation π of a locally compact group G is a factor representation if the center of the commutant algebra of π is trivial, in the sense that it consists of scalar multiples of the identity operator. Moreover, G is said to be a type I group if every factor representation of the group is a direct sum of copies of some irreducible representation. In general, harmonic

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analysis on non-type I groups is not well understood (see [3]). For example the classification (in a reasonable sense) of the unitary dual of a non-type I group is a hopeless quest. Thus, for non-type I groups, it is not clear how to define a natural notion of band-limitation.

(2) For type I groups, there exist a group Fourier transform, a Plancherel theory, and a natural notion of band-limitation. In general, the Plancherel transform intertwines the left regular representation of the group with a direct integral of unitary irreducible representations occurring with some multiplicities. Although there is a nice notion of band-limitation available, the fact that we have to deal with multiplicities (unlike the abelian case) is a serious obstruction that needs to be addressed.

(3) Even if we manage to deal with the issues related to the presence of multiplicity functions of the irreducible representations occurring in the decomposition of the left regular representation of the group, the irreducible representations which are occurring are not just characters like in the abelian case. We often have to deal with some Hilbert Schimdt operators whose actions are usually not well understood.

(4) It is not clear what the sampling sets should be in general. Actually, in many examples it turns out that requiring the sampling sets to be groups or lattices is too restrictive.

Let G be a locally compact group, and let Γ be a discrete subset of G. Let $\mathbf{H}$ be a left-invariant closed subspace of $L^2(G)$ consisting of continuous functions. According to Definition 2.51, [4], a Hilbert space $\mathbf{H}$ is a **sampling space** with respect to Γ if the following properties hold. First, the mapping

$$R_\Gamma : \mathbf{H} \longrightarrow l^2(\Gamma), \ R_\Gamma f = (f(\gamma))_{\gamma\in\Gamma}$$

is an isometry (or a scalar multiple of an isometry). In other words, for all $f \in \mathbf{H}$, $\sum_{\gamma\in\Gamma} |f(\gamma)|^2 = \|f\|_{\mathbf{H}}^2$. Secondly, there exists a vector $s \in \mathbf{H}$ such that for any vector $f \in \mathbf{H}$, we have the following expansion $f(x) = \sum_{\gamma\in\Gamma} f(\gamma) s(\gamma^{-1}x)$ with convergence in the L^2-norm of $\mathbf{H}$. The function s is called a **sinc-type** function. We remark that there are several versions of definitions of sampling spaces. The definition which is usually encountered in the literature only requires the restriction map R_Γ to be a bounded map with a bounded inverse.

Let $(\pi, \mathbf{H}_\pi)$ denote a strongly continuous unitary representation of a locally compact group G. We say that the representation $(\pi, \mathbf{H}_\pi)$ is **admissible** if and only if the map

$$V_\phi : \mathbf{H}_\pi \longrightarrow L^2(G), \ V_\phi\psi(x) = \langle \psi, \pi(x)\phi \rangle$$

defines an isometry of $\mathbf{H}_\pi$ into $L^2(G)$, and we say that ϕ is an **admissible vector** or a **continuous wavelet**. It is known that if π is the left regular representation of G, and if G is connected and type I, then π is admissible if and only if G is nonunimodular (See [4] Theorem 4.23). The following fact is proved in Proposition 2.54 in [4]. Let ϕ be an admissible vector for $(\pi, \mathbf{H}_\pi)$ such that $\pi(\Gamma)\phi$ is a Parseval frame. Then the Hilbert space $V_\phi(\mathbf{H}_\pi)$ is a sampling space, and $V_\phi(\phi)$ is the associated **sinc-type** function for $V_\phi(\mathbf{H}_\pi)$.

Since non-commutative nilpotent Lie groups are very close to commutative Lie groups in their group structures, then it seems reasonable to conjecture that (1) extends to a large class of simply connected, connected non-commutative nilpotent Lie groups, and that this class of groups admits sampling susbpaces which resemble $\mathbf{H}_\Omega$. Presently, we do not have a complete characterization of this class of nilpotent Lie groups. However, we have some partial answers which allow us say that this class of nilpotent Lie groups is larger than the class considered in [12] and [13]. The main purpose of this paper is to present the proof of this new result.

We remark that reconstruction theorems for the Heisenberg group (the simplest example of a connected, simply connected non-commutative nilpotent Lie group) were obtained by Führ [4], and Currey and Mayeli in [2]. Other relevant sources are [11, 5, 15]. In his work, Führ developed a natural concept of band-limitation on the space of square-integrable functions over the Heisenberg group. Using the fact that the Plancherel measure of the Heisenberg group is supported on $\mathbb{R}^*$, he defined a band-limited Hilbert space over the Heisenberg group to be a space of square-integrable functions whose Plancherel transforms are supported on a fixed bounded subset of $\mathbb{R}^*$. He was then able to provide characteristics of sampling spaces with respect to some integer lattices of the Heisenberg group. Furthermore, he provides an explicit Sinc type function in Theorem 6.18 [4]. For a larger class of step-two nilpotent Lie groups of the type $\mathbb{R}^{n-d} \rtimes \mathbb{R}^d$ which is properly contained in the class of groups considered in this paper, we also obtained some sampling theorems in [12] and [13]. Since we are dealing with some non-commutative groups, it is worth noticing the following. First, unlike the commutative case, the corresponding Fourier transforms of the groups are operator-valued transforms. Secondly, the left regular representations of the groups decompose into direct integrals of infinite dimensional irreducible representations, each occurring with infinite multiplicities. We will only be concerned with the multiplicity-free case in this paper.

1.1. **Overview of the Paper.** Let N be a simply connected, connected nilpotent Lie group. Let $\mathfrak{n}$ be its Lie algebra satisfying the following.

Condition 1.

1: $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{b} \oplus \mathfrak{a}$, $\mathfrak{z}$ is the center of $\mathfrak{n}$,

$$\begin{aligned} \mathfrak{z} &= \mathbb{R}Z_{n-2d} \oplus \mathbb{R}Z_{n-2d-1} \oplus \cdots \oplus \mathbb{R}Z_1, \\ \mathfrak{b} &= \mathbb{R}Y_d \oplus \mathbb{R}Y_{d-1} \oplus \cdots \oplus \mathbb{R}Y_1, \\ \mathfrak{a} &= \mathbb{R}X_d \oplus \mathbb{R}X_{d-1} \oplus \cdots \oplus \mathbb{R}X_1 \end{aligned}$$

2: $\mathfrak{z} \oplus \mathfrak{b}$ is a maximal commutative ideal of $\mathfrak{n}$

3: $[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{z}$

4: Given the square matrix of order d

$$S = \begin{bmatrix} [X_1, Y_1] & \cdots & [X_d, Y_1] \\ \vdots & \ddots & \vdots \\ [X_d, Y_1] & \cdots & [X_d, Y_d] \end{bmatrix} \tag{2}$$

the homogeneous polynomial $\det(S)$ is a non-trivial polynomial defined over the ideal $[\mathfrak{n},\mathfrak{n}] \subseteq \mathfrak{z}$.

Define the discrete set

$$\Gamma = \exp(\mathbb{Z}Z_{n-2d})\cdots\exp(\mathbb{Z}Z_1)\exp(\mathbb{Z}Y_d)\cdots\exp(\mathbb{Z}Y_1)\exp(\mathbb{Z}X_d)\cdots\exp(\mathbb{Z}X_1) \tag{3}$$

which is a subset of N and define a matrix-valued function on $\mathfrak{z}^*$ as follows

$$S(\lambda) = \begin{bmatrix} \lambda[X_1,Y_1] & \cdots & \lambda[X_1,Y_d] \\ \vdots & & \vdots \\ \lambda[X_d,Y_1] & \cdots & \lambda[X_d,Y_d] \end{bmatrix}.$$

In order to have a reconstruction formula, we will need a very specific definition of band-limitation. Indeed, we prove that there exists a fundamental domain $\mathbf{K}$ for $\mathbb{Z}^{n-2d}\cap\mathfrak{z}^*$ such that $\mathbf{I} = \mathbf{F}\cap\mathbf{K}$ is a set of positive measure in $\mathfrak{z}^*$ and

$$\mathbf{F} = \left\{\lambda\in\mathfrak{z}^* : |\det S(\lambda)| \leq 1, \det S(\lambda) \neq 0, \text{ and } \left\| S(\lambda)^{Tr}\right\|_\infty < 1\right\}.$$

Let $\{\mathbf{u}_\lambda : \lambda\in\mathbf{I}\}$ be a measurable field of unit vectors in $L^2(\mathbb{R}^d)$. Let $\mathbf{H}_{\mathbf{u},\mathbf{I}}$ be a left-invariant subspace of $L^2(N)$ such that

$$\mathbf{H}_{\mathbf{u},\mathbf{I}} = \left\{\begin{array}{c} f\in L^2(N) : \widehat{f}(\lambda) = v_\lambda\otimes\mathbf{u}_\lambda \text{ is a rank-one operator} \\ \text{in } L^2(\mathbb{R}^d)\otimes L^2(\mathbb{R}^d) \text{ and support of } \widehat{f}\subseteq\mathbf{I} \end{array}\right\}.$$

We have two main results. If $1_{d,d}$ stands for the identity matrix of order d, and $0_{d,d}$ is the zero matrix of order d, then

Theorem 2. *The unitary dual of N is up to a null set exhausted by the set of irreducible representations $\{\pi_\lambda : \lambda\in\mathfrak{z}^*$ and $\det S(\lambda)\neq 0\}$ where, for $f\in L^2(\mathbb{R}^d)$, $\pi_\lambda(\Gamma_1) f$ is a Gabor system of the type $\mathcal{G}(f, B(\lambda)\mathbb{Z}^{2d}) = \{e^{2\pi i\langle k,x\rangle} f(x-n) : (n,k)\in B(\lambda)\mathbb{Z}^{2d}\}$, where*

$$\Gamma_1 = \exp(\mathbb{Z}Y_d)\cdots\exp(\mathbb{Z}Y_1)\exp(\mathbb{Z}X_d)\cdots\exp(\mathbb{Z}X_1),$$

$$B(\lambda) = \begin{bmatrix} 1_{d,d} & 0_{d,d} \\ -X(\lambda) & -S(\lambda)\end{bmatrix},\ S(\lambda) = (\lambda[X_i,Y_i])_{1\leq i,j\leq d}, \tag{4}$$

and $X(\lambda)$ is a strictly upper triangular matrix with entries in the dual of the vector space $[\mathfrak{a},\mathfrak{a}]$, with $X(\lambda)_{i,j} = \lambda[X_i,X_j]$ for $i<j$.

Theorem 3. *There exists a function $f\in\mathbf{H}_{\mathbf{u},\mathbf{I}}$ such that the Hilbert subspace $V_f(\mathbf{H}_{\mathbf{u},\mathbf{I}})$ of $L^2(N)$ is a sampling space with sinc-type function $V_f(f)$.*

The paper is organized around our two main results. Theorem 2 is proved in the second section and Theorem 3 is proved in the last section of the paper. Also, several interesting examples are given throughout the paper, to help the reader follow the stream of ideas presented.

2. The Unitary Dual of N and the Proof of Theorem 2

Let us start by setting up some notation. In this paper, all representations are strongly continuous and unitary. All sets are measurable. The characteristic function of a set E is written as χ_E, and L stands for the left regular representation of a given locally compact group. If $\mathfrak{a}, \mathfrak{b}$ are vector subspaces of some Lie algebra $\mathfrak{g}$, we denote by $[\mathfrak{a}, \mathfrak{b}]$ the set of linear combinations of the form $[X, Y]$ where $X \in \mathfrak{a}$ and $Y \in \mathfrak{b}$. The linear dual of a finite-dimensional vector space V is denoted by V^*, and given two equivalent representations π_1 and π_2, we write $\pi_1 \cong \pi_2$. Given a matrix M, the transpose of M is written as M^{Tr}.

2.1. **The Unitary Dual of N.** Let $\mathfrak{n}$ be a nilpotent Lie algebra of dimension n over $\mathbb{R}$ with corresponding Lie group $N = \exp \mathfrak{n}$. We assume that N is simply connected and connected. Let $\mathfrak{s}$ be a subset in $\mathfrak{n}$ and let λ be a linear functional in $\mathfrak{n}^*$. We define the corresponding sets $\mathfrak{s}^\lambda$ and $\mathfrak{s}(\lambda)$ such that

$$\mathfrak{s}^\lambda = \{Z \in \mathfrak{n} : \lambda [Z, X] = 0 \text{ for every } X \in \mathfrak{s}\}$$

and $\mathfrak{s}(\lambda) = \mathfrak{s}^\lambda \cap \mathfrak{s}$. The ideal $\mathfrak{z}$ denotes the center of the Lie algebra of $\mathfrak{n}$ and the coadjoint action on the dual of $\mathfrak{n}$ is simply the dual of the adjoint action of N on $\mathfrak{n}$. Given $X \in \mathfrak{n}, \lambda \in \mathfrak{n}^*$, the coadjoint action is defined multiplicatively as follows: $\exp X \cdot \lambda(Y) = \lambda(Ad_{\exp -X} Y)$. The following discussion describes some stratification procedure of the dual of the Lie algebra $\mathfrak{n}$ which will be used to develop a precise Plancherel theory for N. This theory is also well exposed in [1]. Let $\mathcal{B} = \{X_1, \cdots, X_n\}$ be a basis for $\mathfrak{n}$. Let

$$\mathfrak{n}_1 \subseteq \mathfrak{n}_2 \subseteq \cdots \subseteq \mathfrak{n}_{n-1} \subseteq \mathfrak{n}$$

be a sequence of subalgebras of $\mathfrak{n}$. We recall that $\mathcal{B}$ is called a **strong Malcev basis** through $\mathfrak{n}_1, \mathfrak{n}_2, \cdots, \mathfrak{n}_{n-1}, \mathfrak{n}$ if and only if the following holds.

(1) The real span of $\{X_1, \cdots, X_k\}$ is equal to $\mathfrak{n}_k$.
(2) Each $\mathfrak{n}_k$ is an ideal of $\mathfrak{n}$.

We start by fixing a strong Malcev basis $\{Z_i\}_{i=1}^n$ for $\mathfrak{n}$ and we define an increasing sequence of ideals: $\mathfrak{n}_k = \mathbb{R}\text{-span}\{Z_i\}_{i=1}^k$. Given any linear functional $\lambda \in \mathfrak{n}^*$, we construct the following skew-symmetric matrix:

$$M(\lambda) = [\lambda [Z_i, Z_j]]_{1 \leq i,j,n}. \tag{5}$$

It is easy to see that $\mathfrak{n}(\lambda) = \text{nullspace}(M(\lambda))$. It is also well-known that all coadjoint orbits have a natural symplectic smooth structure, and therefore are even-dimensional manifolds. Also, for each $\lambda \in \mathfrak{n}^*$ there is a corresponding set $\mathbf{e}(\lambda) \subset \{1, 2, \cdots, n\}$ of **jump indices** defined by

$$\mathbf{e}(\lambda) = \{1 \leq j \leq n : \mathfrak{n}_k \not\subseteq \mathfrak{n}_{k-1} + \mathfrak{n}(\lambda)\}.$$

Naively speaking, the set $\mathbf{e}(\lambda)$ collects all basis elements

$$\{B_1, \cdots, B_{2d}\} \subset \{Z_1, Z_2, \cdots, Z_{n-1}, Z_n\}$$

in the Lie algebra $\mathfrak{n}$ such that $\exp(\mathbb{R}B_1) \cdots \exp(\mathbb{R}B_{2d}) \cdot \lambda = G \cdot \lambda$. For each subset $\mathbf{e} \subseteq \{1, 2, \cdots, n\}$, the set $\Omega_{\mathbf{e}} = \{\lambda \in \mathfrak{n}^* : \mathbf{e}(\lambda) = \mathbf{e}\}$ is algebraic and N-invariant.

Moreover, letting $\xi = \{\mathbf{e} \subseteq \{1, 2, \cdots, n\} : \Omega_{\mathbf{e}} \neq \emptyset\}$ then

$$\mathfrak{n}^* = \bigcup_{\mathbf{e} \in \xi} \Omega_{\mathbf{e}}.$$

The union of all **non-empty layers** defines a 'stratification' of $\mathfrak{n}^*$. It is known that there is a total ordering $\prec$ on the stratification for which the minimal element is Zariski open and consists of orbits of maximal dimension. Let $\mathbf{e}$ be a subset of $\{1, 2, \cdots, n\}$ and define $M_{\mathbf{e}}(\lambda) = [\lambda [Z_i, Z_j]]_{i,j \in \mathbf{e}}$. The set $\Omega_{\mathbf{e}}$ is also given as follows:

$$\Omega_{\mathbf{e}} = \{\lambda \in \mathfrak{n}^* : \det M_{\mathbf{e}'}(\lambda) = 0 \text{ for all } \mathbf{e}' \prec \mathbf{e} \text{ and } \det M_{\mathbf{e}}(\lambda) \neq 0 \ \}. \tag{6}$$

Let us now fix an open and dense layer $\Omega = \Omega_{\mathbf{e}} \subset \mathfrak{n}^*$. The following is standard. We define a polarization subalgebra associated with the linear functional λ by $\mathfrak{p}(\lambda)$. $\mathfrak{p}(\lambda)$ is a maximal subalgebra subordinated to λ such that $\lambda([\mathfrak{p}(\lambda), \mathfrak{p}(\lambda)]) = 0$ and $\chi(\exp X) = e^{2\pi i \lambda(X)}$ defines a character on $\exp(\mathfrak{p}(\lambda))$. It is well-known that $\dim(\mathfrak{n}(\lambda)) = n - 2d$ and $\dim(\mathfrak{n}/\mathfrak{p}(\lambda)) = d$.

According to the orbit method [1], all irreducible representations of N are parametrized by the a set of coadjoint orbits. In order to describe the unitary dual of N and its Plancherel measure, we need to construct a smooth cross-section Σ which is homeomorphic to Ω/N. Using standard techniques described in [1], we obtain

$$\Sigma = \{\lambda \in \Omega : \lambda(Z_k) = 0 \text{ for all } k \in \mathbf{e} \ \}.$$

Defining for each linear functional λ in the generic layer, a character of $\exp(\mathfrak{p}(\lambda))$ such that $\chi_\lambda(\exp X) = e^{2\pi i \lambda(X)}$, we realize almost all unitary irreducible representations of N by induction as follows.

$$\pi_\lambda = \mathrm{Ind}_{\exp(\mathfrak{p}(\lambda))}^{N}(\chi_\lambda)$$

and π_λ acts in the Hilbert completion of the space

$$\mathbf{H}_\lambda = \left\{ \begin{array}{c} f : N \longrightarrow \mathbb{C} : f(xy) = \chi_\lambda(y)^{-1} f(x) \text{ for } y \in \exp \mathfrak{p}(\lambda), \\ \text{and } \int_{\frac{N}{\exp(\mathfrak{p}(\lambda))}} |f(x)|^2 \, d\overline{x} < \infty \end{array} \right\} \tag{7}$$

endowed with the following inner product:

$$\langle f, f' \rangle = \int_{\frac{N}{\exp(\mathfrak{p}(\lambda))}} f(n) \overline{f'(n)} d\overline{n}.$$

In fact, there is an obvious identification between the completion of $\mathbf{H}_\lambda$ and the Hilbert space $L^2\left(\frac{N}{\exp(\mathfrak{p}(\lambda))}\right)$. We will come back to this later.

We will now focus on the class of nilpotent Lie groups that we are concerned with in this paper.

Example 4. *Let N be a nilpotent Lie group with Lie algebra spanned by $Z_1, Z_2, Y_1, Y_2, X_1, X_2$ such that*

$$[X_1, X_2] = Z_1, [X_1, Y_1] = Z_1$$
$$[X_2, Y_2] = Z_1, [X_1, Y_2] = Z_2$$
$$[X_2, Y_1] = Z_2.$$

Then $\det(S) = Z_1^2 - Z_2^2$ *and it is clear that* N *belongs to the class of groups considered here.*

Example 5. *Let* N *be a nilpotent Lie group with Lie algebra spanned by the vectors*

$$\{Z_3, Z_2, Z_1, Y_3, Y_2, Y_1, X_3, X_2, X_1\}$$

and the following non-trivial Lie brackets

$$\begin{aligned}
&[X_1, X_2] = Z_1, [X_1, X_3] = Z_2, [X_2, X_3] = Z_3\\
&[X_1, Y_1] = Z_1, [X_1, Y_2] = Z_2 - Z_3, [X_1, Y_3] = Z_1 + Z_2\\
&[X_2, Y_1] = Z_2, [X_2, Y_2] = Z_1 - Z_2, [X_2, Y_3] = Z_2 - Z_3\\
&[X_3, Y_1] = Z_3, [X_3, Y_2] = Z_1 + Z_2, [X_3, Y_3] = Z_3.
\end{aligned}$$

Then $\mathfrak{a}$ *does not commute,* $\mathfrak{z} \oplus \mathfrak{b}$ *is a maximal commutative ideal of* $\mathfrak{n}$, $[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{z}$,

$$S = \begin{bmatrix} Z_1 & Z_2 - Z_3 & Z_1 + Z_2 \\ Z_2 & Z_1 - Z_2 & Z_2 - Z_3 \\ Z_3 & Z_1 + Z_2 & Z_3 \end{bmatrix}$$

and

$$\det(S) = Z_1^2 Z_3 + Z_1 Z_2^2 + Z_2^3 + Z_2^2 Z_3 - Z_2 Z_3^2 + Z_3^3 \neq 0.$$

Let us define

$$B_1 = Z_{n-2d}, B_2 = Z_{n-2d-1} \cdots, B_{n-2d} = Z_1, B_{n-2d+1} = Y_d,$$
$$B_{n-2d+2} = Y_{d-1} \cdots, B_{n-d} = Y_1,\ B_{n-d+1} = X_d, B_{n-d+2} = X_{d-1}, \cdots, \text{ and } B_n = X_1.$$

Lemma 6. *Let* $\lambda \in \mathfrak{z}^*$. *If* $\det(S)$ *is a non-vanishing polynomial then* $\mathfrak{n}(\lambda) = \mathfrak{z}$ *for a.e.* $\lambda \in \mathfrak{z}^*$.

Proof. First, let

$$S(\lambda) = \begin{bmatrix} \lambda[X_1, Y_1] & \cdots & \lambda[X_1, Y_d] \\ \vdots & \ddots & \vdots \\ \lambda[X_d, Y_1] & \cdots & \lambda[X_d, Y_d] \end{bmatrix}.$$

We recall the definition of S from (2). Clearly for a.e. $\lambda \in \mathfrak{z}^*$, S and $S(\lambda)$ have the same rank, which is equal to d on a dense open subset of the linear dual of the central ideal of the Lie algebra. In fact, we call d the generic rank of the matrix $S(\lambda)$. Also, we recall that $\mathfrak{n}(\lambda)$ is the null-space of $M(\lambda)$ which is defined in (5) as follows

$$M(\lambda) = [\lambda[B_i, B_j]]_{1 \leq i,j \leq n} = \begin{bmatrix} 0_{n-2d,n-2d} & 0_{n-2d,d} & 0_{n-2d,d} \\ 0_{d,n-2d} & 0_{d,d} & S'(\lambda) \\ 0_{d,n-2d} & -S'(\lambda) & R(\lambda) \end{bmatrix};$$

$0_{p,q}$ stands for the $p \times q$ zero matrix,

$$S'(\lambda) = \begin{bmatrix} \lambda[Y_d, X_d] & \cdots & \lambda[Y_d, X_1] \\ \vdots & \ddots & \vdots \\ \lambda[Y_1, X_d] & \cdots & \lambda[Y_1, X_1] \end{bmatrix},$$

and

$$R(\lambda)=\begin{bmatrix}\lambda[X_d,X_d] & \cdots & \lambda[X_d,X_1]\\ \vdots & \ddots & \vdots\\ \lambda[X_1,X_d] & \cdots & \lambda[X_1,X_1]\end{bmatrix}$$

Since the first $n-2d$ columns of the matrix $M(\lambda)$ are zero vectors, and since the remaining $2d$ columns are linearly independent then the nullspace of $M(\lambda)$ is equal to the center of the algebra $\mathfrak{n}$ which is a vector space spanned by $n-2d$ vectors. □

Fix $\mathbf{e}=\{n-2d+1,n-2d+2,\cdots,n\}$. It is not too hard to see that the corresponding layer $\Omega=\Omega_{\mathbf{e}}=\{\lambda\in\mathfrak{n}^*:\det(S(\lambda))\neq 0\}$ is a Zariski open and dense set in $\mathfrak{n}^*$. Next, the manifold

$$\Sigma=\{\lambda\in\Omega:\lambda(\mathfrak{b}\oplus\mathfrak{a})=0\} \tag{8}$$

gives us an almost complete parametrization of the unitary dual of N since it is a cross-section for the coadjoint orbits in the layer Ω. Moreover, we observe that Σ is homeomorphic with a Zariski open subset of $\mathfrak{z}^*$. In order to obtain a realization of the irreducible representation corresponding to each linear functional in Σ, we will need to construct a corresponding polarization subalgebra [1].

The following lemma is in fact the first step toward a precise computation of the unitary dual of N. Put $\mathfrak{p}=\mathfrak{z}\oplus\mathfrak{b}$.

Lemma 7. *For every $\lambda\in\Sigma$, a corresponding polarization subalgebra is given by the ideal $\mathfrak{p}$*

Proof. Since $\mathfrak{p}$ is a commutative algebra then clearly $\lambda[\mathfrak{p},\mathfrak{p}]=\{0\}$. In order to prove the lemma, it suffices to show that $\mathfrak{p}$ is a maximal algebra such that $\lambda[\mathfrak{p},\mathfrak{p}]=\{0\}$. Let us suppose by contradiction that it is not. There exists a non-zero vector $A\in\mathfrak{a}$ such that $\mathfrak{p}\subsetneq\mathfrak{p}\oplus\mathbb{R}A$ and $[\mathfrak{p}\oplus\mathbb{R}A,\mathfrak{p}\oplus\mathbb{R}A]$ is a zero vector space. However, $[\mathfrak{p}\oplus\mathbb{R}A,\mathfrak{p}\oplus\mathbb{R}A]=[\mathfrak{b}\oplus\mathbb{R}A,\mathfrak{b}\oplus\mathbb{R}A]=\{0\}$. So there exists an element of $\mathfrak{a}$ which commutes with all the vectors $Y_k, 1\leq k\leq d$. This contradicts the fourth assumption in Condition 1. □

We recall the definition of the discrete set Γ given in (3) and we define the discrete set

$$\Gamma_1=\exp\mathbb{Z}Y_d\cdots\exp\mathbb{Z}Y_1\exp\mathbb{Z}X_d\cdots\exp\mathbb{Z}X_1\subset N.$$

We observe that Γ_1 is not a group but is naturally identified with the set $\mathbb{Z}^{2d}$.

Next, since N is a non-commutative group, the following remark is in order. Let A be a set, and $\operatorname{Sym}(A)$ be the group of permutation maps of A.

Remark 8. *Let $\sigma\in\operatorname{Sym}(\{1,\cdots,d\})$ be a permutation map. Since $\mathfrak{a}$ is not commutative, it is clear that in general*

$$\Gamma\neq\exp(\mathbb{Z}Z_{n-2d})\cdots\exp(\mathbb{Z}Z_1)\exp(\mathbb{Z}Y_d)\cdots\exp(\mathbb{Z}Y_1)\exp\left(\mathbb{Z}X_{\sigma(d)}\right)\cdots\exp\left(\mathbb{Z}X_{\sigma(1)}\right)$$

However, for arbitrary permutation maps σ_1,σ_2 such that $\sigma_1\in\operatorname{Sym}(\{1,\cdots,n-2d\})$ and $\sigma_2\in\operatorname{Sym}(\{1,\cdots,d\})$, the following holds true:

$$\Gamma=\exp\left(\mathbb{Z}Z_{\sigma_1(n-2d)}\right)\cdots\exp\left(\mathbb{Z}Z_{\sigma_1(1)}\right)\exp\left(\mathbb{Z}Y_{\sigma_2(d)}\right)\cdots\exp\left(\mathbb{Z}Y_{\sigma_2(1)}\right)\exp(\mathbb{Z}X_d)\cdots\exp(\mathbb{Z}X_1).$$

Now, let $x=(x_1,x_2,\cdots,x_d)\in\mathbb{R}^d$.

Proposition 9. *The unitary dual of* N *is given by*

$$\left\{\pi_\lambda = \mathrm{Ind}_{\exp(\mathfrak{z}\oplus\mathfrak{b})}^{N}(\chi_\lambda) : \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{n-2d}, 0, \cdots, 0), \text{ and } \det(S(\lambda)) \neq 0\right\}$$

which we realize as acting in $L^2(\mathbb{R}^d)$ *as follows.*

$$\pi_\lambda(\exp(z_{n-2d}Z_{n-2d})\cdots\exp(z_1Z_1)\exp(l_dY_d)\cdots\exp(l_1Y_1)\exp(m_dX_d)\cdots\exp(m_1X_1))\,\phi(x)$$
$$= e^{2\pi i\sum_{j=1}^{d}s_j\lambda Z_j}e^{-2\pi i\sum_{j=1}^{d}\sum_{k=1}^{d}x_kl_j\lambda[X_k,Y_j]}e^{-2\pi i\lambda\left(\sum_{j=2}^{d}\sum_{r=1}^{j-1}m_jx_r[X_r,X_j]\right)}f(x_1-m_1,\cdots,x_d-m_d).$$

Proof. We recall that π_λ acts in the Hilbert completion of

$$(9)\quad \mathbf{H}_\lambda = \left\{\begin{array}{c} f : N \longrightarrow \mathbb{C} \text{ such that } f(xy) = \chi_\lambda(y)^{-1}f(x) \text{ for } y \in \exp(\mathfrak{z}\oplus\mathfrak{b}) \\ \text{and } x \in N/\exp(\mathfrak{z}\oplus\mathfrak{b}) \text{ and } \int_{N/\exp(\mathfrak{z}\oplus\mathfrak{b})}|f(x)|^2\,d\overline{x} < \infty \end{array}\right\}$$

as follows

$$\pi_\lambda(\exp W)\,\phi\left(\prod_{k=1}^{d}\exp(x_kX_k)\right) = \phi\left(\exp(-W)\prod_{k=1}^{d}\exp(x_kX_k)\right).$$

Next, we observe that the map $\beta : \mathbb{R}^d \times \exp(\mathfrak{z}\oplus\mathfrak{b}) \to N$

$$((x_1, x_2, \cdots, x_d), \exp X) \mapsto \exp(x_1X_1)\exp(x_2X_2)\cdots\exp(x_dX_d)\exp X$$

is a diffeomorphism. Based on the properties of the Hilbert space (9), for $X \in \mathfrak{z}\oplus\mathfrak{b}$, and $\phi \in \mathbf{H}_\lambda$ if

$$n = \exp(x_1X_1)\exp(x_2X_2)\cdots\exp(x_dX_d)\exp X$$

then $\phi(n) = \phi(\exp(x_1X_1)\exp(x_2X_2)\cdots\exp(x_dX_d))\,e^{-2\pi i\lambda(X)}$. Thus, we may naturally identify the Hilbert completion of $\mathbf{H}_\lambda$ with

$$L^2\left(\prod_{k=1}^{d}\exp(\mathbb{R}X_k)\right) \cong L^2(\mathbb{R}^d).$$

Now, we will compute the action of π_λ. Letting $Y_j \in \mathfrak{b}$, then

$$\pi_\lambda(\exp(l_jY_j))\,\phi\left(\prod_{k=1}^{d}\exp(x_kX_k)\right) = \phi\left(\exp(-l_jY_j)\prod_{k=1}^{d}\exp(x_kX_k)\right).$$

Next, let $\mathbf{x} = \prod_{k=1}^{d}\exp(x_kX_k)$.

$$\begin{aligned}\exp(-l_jY_j)\,\mathbf{x} &= \mathbf{x}\left(\mathbf{x}^{-1}\exp(-l_jY_j)\,\mathbf{x}\right)\\ &= \mathbf{x}\exp\left(-l_jY_j + \sum_{k=1}^{d}x_kl_j[X_k, Y_j]\right)\\ &= \mathbf{x}\exp(-l_jY_j)\exp\left(\sum_{k=1}^{d}x_kl_j[X_k, Y_j]\right).\end{aligned}$$

Thus, $\pi_\lambda(\exp l_j Y_j)\phi(x) = e^{-2\pi i\left(\sum_{k=1}^{d} x_k l_j \lambda[X_k,Y_j]\right)}\phi(x)$ and

$$\pi_\lambda(\exp(-m_1X_1))\phi\left((\exp(x_1X_1)\prod_{k=2}^{d}\exp(x_kX_k)\right) = \phi\left(\exp((x_1-m_1)X_1)\prod_{k=2}^{d}\exp(x_kX_k)\right).$$

Also, for $j>1$, since

$$\exp(-x_rX_r)\exp(-m_jX_j)\exp(x_rX_r) = \exp(-m_jX_j + x_rm_j[X_r,X_j])$$

then $\exp(-m_jX_j)\exp(x_rX_r) = \exp(x_rX_r)\exp(-m_jX_j + x_rm_j[X_r,X_j])$ and

$$\begin{aligned}\exp(-m_jX_j)\,\mathbf{x} = {} & \exp(x_1X_1)\exp(x_2X_2)\cdots \\ & \exp((x_j-m_j)X_j)\cdots\exp(x_dX_d) \\ & \times\exp\left(\sum_{r=1}^{j-1} m_jx_r[X_r,X_j]\right).\end{aligned}$$

Thus,

$$\begin{aligned}\pi_\lambda(\exp(-m_jX_j))\phi(\mathbf{x}) = {} & e^{-2\pi i\lambda\left(\sum_{r=1}^{j-1} m_jx_r[X_r,X_j]\right)}\times \\ & \phi(\exp(x_1X_1)\exp(x_2X_2)\cdots\exp((x_j-m_j)X_j) \\ & \cdots\exp(x_dX_d)).\end{aligned}$$

Finally, $\pi_\lambda(\exp s_jZ_j)\phi(x) = e^{2\pi i\lambda(\exp s_jZ_j)}\phi(x)$. In conclusion, identifying $\mathbb{R}^d$ with $N/\exp(\mathfrak{z}\oplus\mathfrak{b})$,

$$\pi_\lambda(\exp l_jY_j)\phi(x) = e^{-2\pi i\lambda\left(\sum_{k=1}^{d} x_kl_j[X_k,Y_j]\right)}\phi(x)$$

For $j=1$, $\pi_\lambda(\exp(m_1X_1))\phi(x) = \phi(x_1-m_1,x_2,\cdots,x_j,\cdots x_d)$. For $j>1$, we obtain

$$\pi_\lambda(\exp(m_jX_j))\phi(x) = e^{-2\pi i\lambda\left(\sum_{r=1}^{j-1} m_jx_r[X_r,X_j]\right)}\phi(x_1,x_2,\cdots,x_j-m_j,\cdots x_d)$$

and finally, $\pi_\lambda(\exp s_jZ_j)\phi(x) = e^{2\pi i\lambda(s_jZ_j)}\phi(x)$. Thus, the proposition is proved by putting the elements $\exp m_kX_k$ in the appropriate order. □

Example 10. *Let N be a nilpotent Lie group with Lie algebra spanned by Z, Y_2, Y_1, X_2, X_1 with non-trivial Lie brackets $[X_1,X_2]=[X_1,Y_1]=[X_2,Y_2]=Z$. The unitary dual of N is parametrized by*

$$\Sigma = \{\lambda\in\mathfrak{n}^* : \lambda(Z)\neq 0, \lambda(Y_2)=\lambda(Y_1)=\lambda(X_2)=\lambda(X_1)=0\}.$$

With some straightforward computations, we obtain that

$$\pi_\lambda(z_2Z_2)\pi_\lambda(z_1Z_1)\pi_\lambda(l_2Y_2)\pi_\lambda(l_1Y_1)\pi_\lambda(k_2X_2)\pi_\lambda(k_1X_1)v(x_1,x_2)$$

is equal to

$$e^{2\pi z_2 i\lambda}e^{2\pi z_1 i\lambda}e^{-2\pi i x_2 l_2\lambda}e^{-2\pi i x_1 l_1\lambda}e^{-2\pi i x_1 k_2\lambda}v(x_1-k_1,x_2-k_2)$$

where $v\in L^2(\mathbb{R}^2)$.

2.2. **Proof of Theorem 2.** Let Λ be a full rank lattice in $\mathbb{R}^{2d}$ and $v \in L^2\left(\mathbb{R}^d\right)$. We recall that the family of functions in $L^2\left(\mathbb{R}^d\right)$: $\mathcal{G}(v,\Lambda) = \left\{e^{2\pi i\langle k,x\rangle} v(x-n) : (n,k) \in \Lambda\right\}$ is called a **Gabor system**. We are now ready to prove Theorem 2. We will show that if $\lambda \in \Sigma$, and $v \in L^2\left(\mathbb{R}^d\right)$, then $\pi_\lambda(\Gamma_1) v = \mathcal{G}\left(v, B(\lambda)\mathbb{Z}^{2d}\right)$ where $B(\lambda)$ is a square matrix of order $2d$ described as follows.

$$B(\lambda) = \begin{bmatrix} 1_{d,d} & 0_{d,d} \\ -X(\lambda) & -S(\lambda) \end{bmatrix}, \; S(\lambda) = \begin{bmatrix} \lambda[X_1,Y_1] & \cdots & \lambda[X_1,Y_d] \\ \vdots & \ddots & \vdots \\ \lambda[X_d,Y_1] & \cdots & \lambda[X_d,Y_d] \end{bmatrix}, \tag{10}$$

and $X(\lambda)$ is a matrix with entries in the dual of the vector space $[\mathfrak{a},\mathfrak{a}]$ given by

$$X(\lambda) = \begin{bmatrix} 0 & \lambda[X_1,X_2] & \lambda[X_1,X_3] & \cdots & \lambda[X_1,X_d] \\ \vdots & 0 & \lambda[X_2,X_3] & \cdots & \lambda[X_2,X_d] \\ & & \ddots & \cdots & \vdots \\ \vdots & & & 0 & \lambda[X_{d-1},X_d] \\ 0 & \cdots & & \cdots & 0 \end{bmatrix}$$

Proof of Theorem 2. Regarding $B(\lambda)$ as a linear operator acting on $\mathbb{R}^{2d}$ which we identify with

$$\mathbb{R}\text{-span } \{X_1, X_2 \cdots X_d, Y_1, Y_2, \cdots, Y_d\},$$

we obtain

$$B(\lambda)\begin{bmatrix} m_1 \\ \vdots \\ m_d \\ l_1 \\ \vdots \\ l_d \end{bmatrix} = \begin{bmatrix} m_1 \\ \vdots \\ m_d \\ -\sum_{k=1}^d l_k \lambda[X_1,Y_k] - \sum_{k=2}^d m_k \lambda[X_1,X_k] \\ \vdots \\ -\sum_{k=1}^d l_k \lambda[X_{d-1},Y_k] - m_{k-1}\lambda[X_{d-1},X_d] \\ \sum_{k=1}^d l_k \lambda[X_d,Y_k] \end{bmatrix}.$$

Appealing to Proposition 9, we compute

$$\begin{aligned} &\pi_\lambda(\exp(l_d Y_d)\cdots\exp(l_1 Y_1)\exp(m_d X_d)\cdots\exp(m_1 X_1))\, v(x) \\ &= e^{-2\pi i\left(\sum_{j=1}^d \sum_{k=1}^d x_k l_j \lambda[X_k,Y_j] + \sum_{j=2}^d \sum_{r=1}^{j-1} m_j x_r [X_j,X_r]\right)} \times v(x_1 - m_1, \cdots, x_d - m_d). \end{aligned}$$

Factoring all the terms multiplying $x_1, x_2, \cdots$, and x_d in

$$-\sum_{j=1}^d \sum_{k=1}^d x_k l_j \lambda[X_k,Y_j] - \sum_{j=2}^d \sum_{r=1}^{j-1} m_j x_r [X_j,X_r],$$

we obtain that $\pi_\lambda(\Gamma_1) v = \mathcal{G}\left(v, B(\lambda)\mathbb{Z}^{2d}\right)$ and

$$\det B(\lambda) = -\det S(\lambda)\det(1_{d,d}) - \det X(\lambda) \times \det(0_{d,d}) = -\det S(\lambda).$$

$\square$

Example 11. *Let N be a simply connected, connected nilpotent Lie group with Lie algebra spanned by the ordered basis $\{Z_3, Z_2, Z_1, Y_2, Y_1, X_2, X_1\}$ with the following non-trivial Lie brackets*

$$[X_1, X_2] = Z_3, [X_1, Y_1] = Z_1, [X_1, Y_2] = Z_2, [X_2, Y_1] = Z_2, [X_2, Y_2] = Z_1.$$

Given $v \in L^2(\mathbb{R}^4)$, we have $\pi_\lambda(\Gamma_1) v = \mathcal{G}(v, B(\lambda)\mathbb{Z}^4)$ and

$$B(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\lambda(Z_3) & -\lambda(Z_1) & -\lambda(Z_2) \\ 0 & 0 & -\lambda(Z_2) & -\lambda(Z_1) \end{bmatrix}.$$

2.3. **Plancherel Theory.** Now, we recall well-known facts about the Plancherel theory for the class of groups considered in this paper. Assume that N is endowed with its canonical Haar measure which is the Lebesgue measure in our situation. For $\lambda = (\lambda_1, \cdots, \lambda_{n-2d}, 0, \cdots, 0) \in \Sigma$, (see (8))

$$d\mu(\lambda) = |\det B(\lambda)| \, d\lambda = |\det S(\lambda)| \, d\lambda \tag{11}$$

is the Plancherel measure (see chapter 4 in [1]), and the matrix $B(\lambda)$ is defined in (10). We have

$$\mathcal{F} : L^2(N) \to \int_\Sigma^\oplus L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \, d\mu(\lambda)$$

where the Fourier transform is defined on $L^2(N) \cap L^1(N)$ by

$$\mathcal{F}(f)(\lambda) = \int_\Sigma f(n) \pi_\lambda(n) \, dn$$

and the Plancherel transform $\mathcal{P}$ is the **extension** of the Fourier transform to $L^2(N)$ inducing the equality

$$\|f\|_{L^2(N)}^2 = \int_\Sigma \|\mathcal{P}(f)(\lambda)\|_{\mathcal{HS}}^2 \, d\mu(\lambda).$$

In fact, $\|\cdot\|_{\mathcal{HS}}$ denotes the Hilbert-Schmidt norm on $L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$. We recall that the inner product of two rank-one operators in $L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$ is given by $\langle u \otimes v, w \otimes y \rangle_{\mathcal{HS}} = \langle u, w \rangle_{L^2(\mathbb{R}^d)} \langle v, y \rangle_{L^2(\mathbb{R}^d)}$. We also have that $L \cong \mathcal{P} L \mathcal{P}^{-1} = \int_\Sigma^\oplus \pi_\lambda \otimes \mathbf{1}_{L^2(\mathbb{R}^d)} d\mu(\lambda)$, and $\mathbf{1}_{L^2(\mathbb{R}^d)}$ is the identity operator on $L^2(\mathbb{R}^d)$. Finally, for $\lambda \in \Sigma$, it is well-known that

$$\mathcal{P}(L(x)\phi)(\lambda) = \pi_\lambda(x) \circ (\mathcal{P}\phi)(\lambda). \tag{12}$$

3. Reconstruction of Band-limited Vectors and Proof of Theorem 3

3.1. **Properties of Band-limited Hilbert Subspaces.** We will start this section by introducing a natural concept of band-limitation on the class of groups considered in this paper. The following set will be of special interest, and we will be mainly interested in multiplicity-free subspaces. Let

$$\mathbf{E} = \{\lambda \in \Sigma : |\det S(\lambda)| \leq 1\}$$

and let $\mathbf{m}$ be the Lebesgue measure on $\mathfrak{z}^*$. We remark that depending on the structure constants of the Lie algebra, the set $\mathbf{E}$ is either bounded or unbounded. Furthermore, we need the following lemma to hold.

Lemma 12. $\mathbf{E}$ *is a set of positive Lebesgue measure*

Proof. Since $\det S(\lambda)$ is a homogeneous polynomial, there exists $a > 0$ such that

$$\{\lambda \in \mathfrak{z}^* : |\lambda_k| < a \text{ and } |\det S(\lambda)| \neq 0\} \subset \mathbf{E}.$$

It is easy to see that

$$\{\lambda \in \mathfrak{z}^* : |\lambda_k| < a \text{ and } |\det S(\lambda)| \neq 0\} = \{\lambda \in \mathfrak{z}^* : |\lambda_k| < a\} - \{\lambda \in \mathfrak{z}^* : \det S(\lambda) = 0\}.$$

However $\{\lambda \in \mathfrak{z}^* : \det S(\lambda) = 0\}$ is a set of $\mathbf{m}$-measure zero. As a result,

$$\mathbf{m}(\{\lambda \in \mathfrak{z}^* : |\lambda_k| < a \text{ and } |\det S(\lambda)| \neq 0\}) = \mathbf{m}(\{\lambda \in \mathfrak{z}^* : |\lambda_k| < a\}).$$

Thus, $\mathbf{m}(\{\lambda \in \mathfrak{z}^* : |\lambda_k| < a \text{ and } |\det S(\lambda)| \neq 0\}) > 0$ and it follows that $\mathbf{E}$ is a set of positive Lebesgue measure. □

Definition 13. *Let* $\mathbf{A} \subset \Sigma$ *be a measurable bounded set. We say a function* $f \in L^2(N)$ *is* $\mathbf{A}$-***band-limited*** *if its Plancherel transform is supported on* $\mathbf{A}$. *Fix* $\mathbf{u} = \{\mathbf{u}_\lambda : \lambda \in \Sigma\}$ *a measurable field of unit vectors in* $L^2(\mathbb{R}^d)$ *which is parametrized by* Σ. *The Hilbert space*

$$\mathbf{H}_{\mathbf{u}} = \mathcal{P}^{-1}\left(\int_\Sigma^\oplus L^2(\mathbb{R}^d) \otimes \mathbf{u}_\lambda \, d\mu(\lambda)\right)$$

is a multiplicity-free subspace of $L^2(N)$. *For any measurable subset of* $\mathbf{A}$ *of* Σ, *we define the Hilbert space*

$$\mathbf{H}_{\mathbf{u},\mathbf{A}} = \mathcal{P}^{-1}\left(\int_{\mathbf{A}}^\oplus L^2(\mathbb{R}^d) \otimes \mathbf{u}_\lambda \, d\mu(\lambda)\right). \tag{13}$$

Clearly $\mathbf{H}_{\mathbf{u},\mathbf{A}}$ is a Hilbert subspace of $L^2(N)$ which contains vectors whose Fourier transforms are rank-one operators and are supported on the set $\mathbf{A}$. Next, we recall the following standard facts in frame theory. A sequence $\{f_n : n \in \mathbb{Z}\}$ of elements in a Hilbert space $\mathbf{H}$ is called a **frame** [8, 7, 10, 9] if there are constant $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_n |\langle f, f_n\rangle|^2 \leq B\|f\|^2 \text{ for all } f \in \mathbf{H}.$$

The numbers A, B in the definition of a frame are called **lower** and **upper bounds** respectively. A frame is a **tight frame** if $A = B$ and a **normalized tight frame** or **Parseval frame** if $A = B = 1$.

Definition 14. *A set in a Hilbert space* $\mathbf{H}$ *is total if the closure of its linear span is equal to* $\mathbf{H}$.

We will need the following theorem known as the **Density Theorem for Lattices** ([10] Theorem 10).

Theorem 15. *(**Density Theorem**) Let* $v \in L^2(\mathbb{R}^d)$ *and let* $\Lambda = A\mathbb{Z}^{2d}$ *where* A *is an invertible matrix of order* $2d$. *Then the following holds*

(1) *If* $|\det(A)| > 1$ *then* $\mathcal{G}(v,\Lambda)$ *is not total in* $L^2(\mathbb{R}^d)$.
(2) *If* $\mathcal{G}(v,\Lambda)$ *is a frame for* $L^2(\mathbb{R}^d)$ *then* $0 < |\det(A)| \leq 1$.

In light of the theorem above, we have the following.

Proposition 16. *Let* $\mathbf{J}$ *be a measurable subset of* Σ. *If* $\mathbf{J}-\mathbf{E}$ *is a set of positive measure then it is not possible to find a function* $g \in \mathbf{H}_{\mathbf{u},\mathbf{J}}$ *such that* $L(\Gamma)g$ *is total in* $\mathbf{H}_{\mathbf{u},\mathbf{J}}$. *In other words, the representation* $(L, \mathbf{H}_{\mathbf{u},\mathbf{J}})$ *is not cyclic.*

Proof. To prove Part 1, if $\mathbf{J}-\mathbf{E}$ is a non-null set, by the Density Theorem for lattices (see 15) $\pi_\lambda(\Gamma)\circ\mathcal{P}g(\lambda) = \pi_\lambda(\Gamma)u_\lambda \otimes \mathbf{u}_\lambda$ cannot be total in $L^2(\mathbb{R}^d)\otimes\mathbf{u}_\lambda$ for all $\lambda \in \mathbf{J}-\mathbf{E}$ since $\pi_\lambda(\Gamma)u_\lambda = \mathcal{G}(u_\lambda, B(\lambda)\mathbb{Z}^{2d})$ cannot be total in $L^2(\mathbb{R}^d)$ for any $\lambda \in \mathbf{J}-\mathbf{E}$. Thus $\overline{\mathcal{P}(\text{span }(L(\Gamma)g))}$ is contained but not equal to $\int_{\mathbf{J}}^{\oplus} L^2(\mathbb{R}^d)\otimes \mathbf{u}_\lambda d\mu(\lambda)$. □

Let $\mathbf{K}$ be a measurable fundamental domain for $\mathbb{Z}^{n-2d}\cap\mathfrak{z}^*$ such that $\mathbf{m}(\mathbf{K}\cap\mathbf{E})$ is positive. Clearly such set always exists. In fact, we define $\mathbf{K}$ to be the unit cube around the zero linear functional in $\mathfrak{z}^*$ as follows:

$$\mathbf{K} = \left\{\lambda \in \mathfrak{z}^* : \lambda(Z_k) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \text{ for } 1 \leq k \leq n-2d\right\}.$$

Put $\mathbf{I} = \mathbf{E}\cap\mathbf{K}$.

Definition 17. *We say a set* $\mathcal{T}$ *is a* ***tiling set*** *for a lattice* $\mathcal{L}$ *if and only*

(1) $\bigcup_{l\in\mathcal{L}}(\mathcal{T}+l) = \mathbb{R}^d$ *a.e.*
(2) $(\mathcal{T}+l)\cap(\mathcal{T}+l')$ *has Lebesgue measure zero for any* $l \neq l'$ *in* $\mathcal{L}$.

Definition 18. *We say that* $\mathcal{T}$ *is a* ***packing set*** *for a lattice* $\mathcal{L}$ *if and only if* $(\mathcal{T}+l)\cap(\mathcal{T}+l')$ *has Lebesgue measure zero for any* $l \neq l'$ *in* $\mathcal{L}$.

Let M be a matrix of order d. We define the norm of M as follows.

$$\|M\|_\infty = \sup\left\{Mx : x\in\mathbb{R}^d, \|x\|_{\max} = 1\right\} \text{ where } \|x\|_{\max} = \max_{1\leq k\leq d}|x_k|.$$

Now, put

$$\mathbf{Q} = \left\{\lambda \in \Sigma\text{: } \left\|S(\lambda)^{Tr}\right\|_\infty < 1\right\}.$$

It is clear that $\mathbf{Q}$ is a set of positive measure.

Lemma 19. *For any* $\lambda \in \mathbf{Q}$, *then* $\left[-\frac{1}{2},\frac{1}{2}\right]^d$ *is a packing set for* $S(\lambda)^{-Tr}\mathbb{Z}^d$ *and a tiling set for* $\mathbb{Z}^d$.

Proof. Clearly $\left[-\frac{1}{2},\frac{1}{2}\right)^d$ is a tiling set for $\mathbb{Z}^d$. To show that the lemma holds, it suffices to show that $\left[-\frac{1}{2},\frac{1}{2}\right)^d$ is a packing set for $S(\lambda)^{-Tr}\mathbb{Z}^d$. Let us suppose that there exist $\kappa_1,\kappa_2 \in S(\lambda)^{-Tr}\mathbb{Z}^d$ and $\sigma_1,\sigma_2 \in \left[-\frac{1}{2},\frac{1}{2}\right]^d$, $\sigma_1\neq\sigma_2$ such that $\sigma_1+\kappa_1 = \sigma_2+\kappa_2$. Then there exist $j_2, j_1 \in \mathbb{Z}^d$ such that $\sigma_1 - \sigma_2 = \left(S(\lambda)^{Tr}\right)^{-1}(j_2-j_1)$. So $S(\lambda)^{Tr}(\sigma_1-\sigma_2) = j_2 - j_1$ and $\left\|S(\lambda)^{Tr}(\sigma_1-\sigma_2)\right\|_{\max} = \|j_2-j_1\|_{\max}$. Since $j_2 - j_1 \neq 0$ then

$$\left\|S(\lambda)^{Tr}(\sigma_1-\sigma_2)\right\|_{\max} \geq 1$$

and $\left\|S(\lambda)^{Tr}(\sigma_1-\sigma_2)\right\|_{\max} \leq \left\|S(\lambda)^{Tr}\right\|_{\infty} < 1$. Thus

$$1 > \left\|S(\lambda)^{Tr}(\sigma_1-\sigma_2)\right\|_{\max} \geq 1.$$

That would be a contradiction. □

From now on, we will assume that $\mathbf{I}$ is replaced with $\mathbf{Q}\cap\mathbf{I}$.

Example 20. *Let N be a nilpotent Lie group with Lie algebra $\mathfrak{n}$ spanned by*

$$\{Z_2, Z_1, Y_2, Y_1, X_2, X_1\}$$

with the following non-trivial Lie brackets.

$$[X_1,X_2]=Z_2,[X_1,Y_1]=Z_1,[X_2,Y_1]=Z_2,[X_3,Y_1]=-Z_1,[X_1,Y_2]\ =Z_2,[X_2,Y_2]=-Z_1$$
$$[X_3,Y_2]=Z_1,[X_1,Y_3]=-Z_1,[X_2,Y_3]=Z_1,[X_3,Y_3]=Z_2.$$

Let $\lambda\in\mathfrak{n}^*$, we write $\lambda=(\lambda_1,\lambda_2,\lambda_3,\cdots,\lambda_n)$ where $\lambda_k=\lambda(Z_k)$. Then

$$\mathbf{I}=\left\{\begin{array}{c}(\lambda_1,\lambda_2,0,\cdots,0)\in\mathfrak{z}^*:-3\lambda_1^2\lambda_2-\lambda_2^3\neq 0,|3\lambda_1^2\lambda_2+\lambda_2^3|\leq 1,2|\lambda_1|+|\lambda_2|<1\\ -1/2\leq\lambda_1,\lambda_2\leq 1/2\end{array}\right\}.$$

Next, we define the unitary operator $\mathcal{U}:L^2(\mathbb{R}^d)\longrightarrow L^2(\mathbb{R}^d)$ such that

$$\mathcal{U}f(t)=e^{-2\pi i\langle t,X(\lambda)t\rangle}f(t)$$

Lemma 21. *For every linear functional $\lambda\in\mathbf{I}$,*

$$\mathcal{G}\left(|\det S(\lambda)|^{1/2}\,\mathcal{U}\chi_{[-1/2,1/2]^d},B(\lambda)\mathbb{Z}^{2d}\right)$$

is a Parseval frame in $L^2(\mathbb{R}^d)$.

Proof. Given $v\in L^2(\mathbb{R}^d)$, we write $M_lv(t)=e^{2\pi i\langle k,t\rangle}v(t)$ and $T_kv(t)=v(t-k)$. Thus,

$$\mathcal{G}\left(v,B(\lambda)\mathbb{Z}^{2d}\right)=\left\{M_{-S(\lambda)l-X(\lambda)k}T_kv:(k,l)\in\mathbb{Z}^{2d}\right\}.$$

Next, we will see that $\mathcal{U}M_{-S(\lambda)l}T_k\mathcal{U}^{-1}v(t)=M_{-S(\lambda)l-X(\lambda)k}T_kv(t)$. Indeed,

$$\begin{aligned}\mathcal{U}M_{-S(\lambda)l}T_k\mathcal{U}^{-1}v(t)&=e^{-2\pi i\langle t,X(\lambda)t\rangle}M_{-S(\lambda)l}T_k\mathcal{U}^{-1}v(t)\\&=e^{-2\pi i\langle t,X(\lambda)t\rangle}e^{-2\pi i\langle S(\lambda)l,t\rangle}\mathcal{U}^{-1}v(t-k)\\&=e^{-2\pi i\langle t,X(\lambda)t\rangle}e^{-2\pi i\langle S(\lambda)l,t\rangle}e^{2\pi i\langle t,X(\lambda)(t-k)\rangle}v(t-k)\\&=e^{-2\pi i\langle t,X(\lambda)t\rangle}e^{-2\pi i\langle S(\lambda)l,t\rangle}e^{2\pi i\langle t,X(\lambda)t\rangle}e^{-2\pi i\langle t,X(\lambda)k\rangle}v(t-k)\\&=e^{-2\pi i\langle S(\lambda)l,t\rangle}e^{-2\pi i\langle t,X(\lambda)k\rangle}v(t-k)\\&=M_{-S(\lambda)l-X(\lambda)k}T_kv(t).\end{aligned}$$

Put $w=\mathcal{U}^{-1}v$. Then $\mathcal{U}M_{-S(\lambda)l}T_k\mathcal{U}^{-1}v=\mathcal{U}M_{-S(\lambda)l}T_kw$. Now, let $w=|\det S(\lambda)|^{1/2}\chi_{E(\lambda)}$ such that $E(\lambda)$ is a tiling set for $\mathbb{Z}^d$ and a packing set for $\left(S(\lambda)^{Tr}\right)^{-1}\mathbb{Z}^d$. Then $\mathcal{G}\left(w,\mathbb{Z}^d\times S(\lambda)\mathbb{Z}^d\right)$ is a Parseval frame in $L^2(\mathbb{R}^d)$ (see Proposition 3.1 [16]) and

$\mathcal{G}\left(\left|\det S\left(\lambda\right)\right|^{1/2}\mathcal{U}\chi_{E(\lambda)}, B\left(\lambda\right)\mathbb{Z}^{2d}\right)$ is a Parseval frame in $L^2\left(\mathbb{R}^d\right)$. The proof of the lemma is completed by replacing $E\left(\lambda\right)$ with $\left[-1/2,1/2\right]^d$. □

Proposition 22. *Let $\mathbf{H}_{\mathbf{u},\mathbf{I}}$ be a Hilbert space consisting of $\mathbf{I}$-band-limited functions. There exists a function f in $\mathbf{H}_{\mathbf{u},\mathbf{I}}$ such that $L\left(\Gamma\right)f$ is a Parseval frame in $\mathbf{H}_{\mathbf{u},\mathbf{I}}$ and $\|f\|^2_{\mathbf{H}_{\mathbf{u},\mathbf{I}}} = \mu\left(\mathbf{I}\right)$.*

Proof. Define $f \in \mathbf{H}_{\mathbf{u},\mathbf{I}}$ such that $\mathcal{P}f\left(\lambda\right) = \left|\det B\left(\lambda\right)\right|^{-1/2}\phi\left(\lambda\right)\otimes\mathbf{u}_\lambda$ such that

$$\phi(\lambda) = \left|\det S\left(\lambda\right)\right|^{1/2}\mathcal{U}\chi_{[-1/2,1/2]^d}.$$

We recall that $\mathcal{P}\left(L\left(\gamma\right)f\right)\left(\lambda\right) = \pi_\lambda\left(\gamma\right)\circ\mathcal{P}f\left(\lambda\right) = \left|\det B\left(\lambda\right)\right|^{-1/2}\pi_\lambda\left(\gamma\right)\phi\left(\lambda\right)\otimes\mathbf{u}_\lambda$. Let g be any function in $\mathbf{H}_{\mathbf{u},\mathbf{I}}$ such that $\mathcal{P}g\left(\lambda\right) = u_\lambda\otimes\mathbf{u}_\lambda$. Next, we write $\gamma\in\Gamma$ such that $\gamma = k\eta$, where k is in the center of the Lie group N and η is in $\Gamma_1 = \exp\mathbb{Z}Y_d\cdots\exp\mathbb{Z}Y_1\exp\mathbb{Z}X_d\cdots\exp\mathbb{Z}X_1$.

$$\begin{aligned}\sum_{\gamma\in\Gamma}\left|\left\langle g, L\left(\gamma\right)f\right\rangle_{\mathbf{H}_{\mathbf{u},\mathbf{I}}}\right|^2 &= \sum_{\gamma\in\Gamma}\left|\int_{\mathbf{I}}\left\langle u_\lambda\otimes\mathbf{u}_\lambda, \left|\det B\left(\lambda\right)\right|\pi_\lambda\left(\gamma\right)\left(\left|\det B\left(\lambda\right)\right|^{-1/2}\phi\left(\lambda\right)\right)\otimes\mathbf{u}_\lambda\right\rangle_{\mathcal{HS}}d\lambda\right|^2\\
&= \sum_{\gamma\in\Gamma}\left|\int_{\mathbf{I}}\left\langle u_\lambda\otimes\mathbf{u}_\lambda, \pi_\lambda\left(\gamma\right)\left(\left|\det B\left(\lambda\right)\right|^{1/2}\phi\left(\lambda\right)\right)\otimes\mathbf{u}_\lambda\right\rangle_{\mathcal{HS}}d\lambda\right|^2\\
&= \sum_{\gamma\in\Gamma}\left|\int_{\mathbf{I}}\left\langle u_\lambda\otimes\mathbf{u}_\lambda, \pi_\lambda\left(\gamma\right)\left(\left|\det B\left(\lambda\right)\right|^{1/2}\phi\left(\lambda\right)\right)\otimes\mathbf{u}_\lambda\right\rangle_{\mathcal{HS}}d\lambda\right|^2\\
&= \sum_{\gamma\in\Gamma}\left|\int_{\mathbf{I}}\left\langle u_\lambda, \pi_\lambda\left(\gamma\right)\left|\det B\left(\lambda\right)\right|^{1/2}\phi\left(\lambda\right)\right\rangle_{L^2\left(\mathbb{R}^d\right)}\left\langle\mathbf{u}_\lambda,\mathbf{u}_\lambda\right\rangle_{L^2\left(\mathbb{R}^d\right)}d\lambda\right|^2\\
&= \sum_{\gamma\in\Gamma}\left|\int_{\mathbf{I}}\left\langle u_\lambda, \left|\det B\left(\lambda\right)\right|^{1/2}\pi_\lambda\left(\gamma\right)\phi\left(\lambda\right)\right\rangle_{L^2\left(\mathbb{R}^d\right)}d\lambda\right|^2\\
&= \sum_{\eta\in\Gamma_1}\sum_{k\in\mathbb{Z}^{n-2d}}\left|\int_{\mathbf{I}}e^{-2\pi i\langle k,\lambda\rangle}\left\langle u_\lambda, \left|\det B\left(\lambda\right)\right|^{1/2}\pi_\lambda\left(\eta\right)\phi\left(\lambda\right)\right\rangle_{L^2\left(\mathbb{R}^d\right)}d\lambda\right|^2. \qquad (14)\end{aligned}$$

Since $\left\{e^{-2\pi i\langle k,\lambda\rangle}\chi_{\mathbf{I}}\left(\lambda\right) : k\in\mathbb{Z}\right\}$ is a Parseval frame in $L^2\left(\mathbf{I}\right)$, letting

$$c_\eta\left(\lambda\right) = \left\langle u_\lambda, \left|\det B\left(\lambda\right)\right|^{1/2}\pi_\lambda\left(\eta\right)\phi\left(\lambda\right)\right\rangle_{L^2\left(\mathbb{R}^d\right)},$$

Equation (14) becomes

$$\sum_{\eta\in\Gamma_1}\sum_{k\in\mathbb{Z}^{n-2d}}\left|\int_{\mathbf{I}}e^{2\pi i\langle k,\lambda\rangle}c_\eta\left(\lambda\right)d\lambda\right|^2 = \sum_{\eta\in\Gamma_1}\sum_{k\in\mathbb{Z}^{n-2d}}\left|\widehat{c}_\eta\left(k\right)\right|^2 = \sum_{\eta\in\Gamma_1}\left\|c_\eta\right\|^2_{L^2(\mathbf{I})}.$$

Next,

$$\begin{aligned}\sum_{\gamma\in\Gamma}\left|\langle g, L(\gamma) f\rangle_{\mathbf{H}_{\mathbf{u},\mathbf{I}}}\right|^2 &= \sum_{\eta\in\Gamma_1}\int_{\mathbf{I}}\left|\left\langle u_\lambda, |\det B(\lambda)|^{1/2}\pi_\lambda(\eta)\phi(\lambda)\right\rangle_{L^2(\mathbb{R}^d)}\right|^2 d\lambda \\ &= \int_{\mathbf{I}}\sum_{\eta\in\Gamma_1}\left|\left\langle u_\lambda, |\det B(\lambda)|^{1/2}\pi_\lambda(\eta)\phi(\lambda)\right\rangle_{L^2(\mathbb{R}^d)}\right|^2 d\lambda \\ &= \int_{\mathbf{I}}\sum_{\eta\in\Gamma_1}\left|\langle u_\lambda, \pi_\lambda(\eta)\phi(\lambda)\rangle_{L^2(\mathbb{R}^d)}\right|^2 |\det B(\lambda)|\, d\lambda.\end{aligned}$$

Using the fact that $\mathcal{G}\left(\phi(\lambda), B(\lambda)\mathbb{Z}^{2d}\right)$ is a Parseval frame for almost every $\lambda\in\mathbf{I}$ (see Lemma 21), we obtain

$$\begin{aligned}\sum_{\eta\in\Gamma_1}\left|\langle u_\lambda, \pi_\lambda(\eta)\phi(\lambda)\rangle_{L^2(\mathbb{R}^d)}\right|^2 &= \|u_\lambda\|^2_{L^2(\mathbb{R}^d)} \\ &= \|u_\lambda\|^2_{L^2(\mathbb{R}^d)}\|\mathbf{u}_\lambda\|^2_{L^2(\mathbb{R}^d)} \\ &= \|u_\lambda\otimes\mathbf{u}_\lambda\|^2_{\mathcal{HS}}\end{aligned}$$

and

$$\sum_{\gamma\in\Gamma}\left|\langle g, L(\gamma) f\rangle_{\mathbf{H}_{\mathbf{u},\mathbf{I}}}\right|^2 = \int_{\mathbf{I}}\|\mathcal{P}g(\lambda)\|^2_{\mathcal{HS}}|\det B(\lambda)|\, d\lambda = \|g\|^2_{\mathbf{H}_{\mathbf{u},\mathbf{I}}}.$$

Now, to make sure that $f\in\mathbf{H}_{\mathbf{u},\mathbf{I}}$, we will show that its norm is finite. Clearly,

$$\begin{aligned}\|f\|^2_{\mathbf{H}_{\mathbf{u},\mathbf{I}}} &= \int_{\mathbf{I}}\|\mathcal{P}f(\lambda)\|^2_{\mathcal{HS}}|\det B(\lambda)|\, d\lambda \\ &= \int_{\mathbf{I}}\left\||\det B(\lambda)|^{-1/2}\phi(\lambda)\otimes\mathbf{u}_\lambda\right\|^2_{\mathcal{HS}}|\det B(\lambda)|\, d\lambda \\ &= \int_{\mathbf{I}}\|\phi(\lambda)\otimes\mathbf{u}_\lambda\|^2_{\mathcal{HS}}|\det B(\lambda)|^{-1}|\det B(\lambda)|\, d\lambda \\ &= \int_{\mathbf{I}}\|\phi(\lambda)\otimes\mathbf{u}_\lambda\|^2_{\mathcal{HS}}\, d\lambda \\ &= \int_{\mathbf{I}}\|\phi(\lambda)\|^2_{L^2(\mathbb{R}^d)}\, d\lambda.\end{aligned}$$

Since $\mathbf{I}$ is bounded then $\|f\|^2_{\mathbf{H}_{\mathbf{u},\mathbf{I}}}$ is clearly finite. In fact $\|\phi(\lambda)\|^2 = |\det S(\lambda)| = |\det B(\lambda)|$ and $\|f\|^2_{\mathbf{H}_{\mathbf{u},\mathbf{I}}} = \mu(\mathbf{I})$. □

3.2. **Proof of Theorem 3.** Finally, we are able to offer a proof of Theorem 3.

Proof of Theorem 3. Let f be a function in $\mathbf{H}_{\mathbf{u},\mathbf{I}}$ such that $\mathcal{P}f(\lambda) = \mathcal{U}\chi_{[-1/2,1/2]^d}\otimes\mathbf{u}_\lambda$. We recall the coefficient function $V_f:\mathbf{H}_{\mathbf{u},\mathbf{I}}\longrightarrow L^2(N)$ such that $V_fh(x) = \langle h, L(x) f\rangle = h * f^*$ where $*$ is the convolution operation and $f^*(x) = \overline{f(x^{-1})}$. We will first show that V_f is an isometry. In other words, f is an admissible vector.

$$\begin{aligned}\|V_f h\|^2 &= \int_{\mathbf{I}} \|\mathcal{P}h(\lambda) \circ \mathcal{P}(f^*)(\lambda)\|^2_{\mathcal{HS}} \; d\mu(\lambda) \\ &= \int_{\mathbf{I}} \|\mathcal{P}h(\lambda) \circ \mathcal{P}(f^*)(\lambda)\|^2_{\mathcal{HS}} \; d\mu(\lambda) \\ &= \int_{\mathbf{I}} \|\mathcal{P}h(\lambda)\|^2_{\mathcal{HS}} \;\; \|\mathcal{P}(f^*)(\lambda)\|^2_{\mathcal{HS}} \, d\mu(\lambda) \\ &= \int_{\mathbf{I}} \|\mathcal{P}h(\lambda)\|^2_{\mathcal{HS}} \; d\mu(\lambda) \\ &= \|h\|^2 .\end{aligned}$$

The third equality above is justified because, the operators involved are rank-one operators. Since f is an admissible vector for the representation $(L, \mathbf{H}_{\mathbf{u},\mathbf{I}})$ and since $L(\Gamma) f$ is a Parseval frame then according to Proposition 2.54. [4], $V_f(\mathbf{H}_{\mathbf{u},\mathbf{I}})$ is a sampling space with respect to Γ with sinc function $V_f(f)$. □

Example 23. *Let N a be Lie group with Lie algebra $\mathfrak{n}$ spanned by*

$$\{Z_3, Z_2, Z_1, Y_2, Y_1, X_2, X_1\}$$

with the following non-trivial Lie brackets.

$$[X_1, Y_1] = Z_1, [X_1, Y_2] = Z_2, [X_2, Y_1] = Z_2, [X_2, Y_2] = Z_3, [X_1, X_2] = Z_1 - Z_3.$$

Define a bounded subset $\mathbf{I}$ of $\mathfrak{z}^*$ given by

$$\mathbf{I} = \left\{ \begin{array}{c} \lambda \in \mathfrak{z}^* : \lambda(Z_1)\lambda(Z_3) - \lambda(Z_2)^2 \neq 0, \left|\lambda(Z_1)\lambda(Z_3) - \lambda(Z_2)^2\right| \leq 1 \\ \max\{|\lambda(Z_1) + \lambda(Z_2)|, |\lambda(Z_2) + \lambda(Z_3)|\} < 1 \\ (\lambda(Z_1), \lambda(Z_2), \lambda(Z_3)) \in [-1/2, 1/2]^3 \end{array} \right\}.$$

Put $f \in \mathbf{H}_{\mathbf{u},\mathbf{I}}$ such that $\widehat{f}(\lambda) = e^{-2\pi i(\lambda(Z_1)t_1t_2 - \lambda(Z_3)t_1t_2)} \chi_{[-1/2,1/2]^2}(t_1, t_2) \otimes \mathbf{u}_\lambda$ where $\{\mathbf{u}_\lambda : \lambda \in \mathbf{I}\}$ is a family of unit vectors in $L^2(\mathbb{R}^2)$. Then $V_f(\mathbf{H}_{\mathbf{u},\mathbf{I}})$ is a sampling space with respect to the discrete set

$$\begin{aligned}\Gamma = &\exp(\mathbb{Z}Z_3)\exp(\mathbb{Z}Z_2)\exp(\mathbb{Z}Z_1)\exp(\mathbb{Z}Y_2) \\ &\exp(\mathbb{Z}Y_1)\exp(\mathbb{Z}X_2)\exp(\mathbb{Z}X_1)\end{aligned}$$

with sinc function $s = V_f f$. Thus given any $h \in V_f(\mathbf{H}_{\mathbf{u},\mathbf{I}})$, h is determined by its sampled values $(h(\gamma))_{\gamma\in\Gamma}$ and $h(x) = \sum_{\gamma\in\Gamma} h(\gamma)\, s(\gamma^{-1}x)$.

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