

# COMPUTING VERGNE POLARIZING SUBALGEBRAS

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**ABSTRACT.** According to Kirillov's theory, the construction of a unitary irreducible representation of a nilpotent Lie group requires a precise computation of some polarizing subalgebra subordinated to a linear functional in the linear dual of the corresponding Lie algebra. This important step is generally challenging from a computational viewpoint. In this paper, we provide an algorithmic approach to the construction of the well-known Vergne polarizing subalgebras [1]. The algorithms presented in this paper are specifically designed so that they can be implemented in Computer Algebra Systems. We also show there are instances where Vergne's construction could be refined for the sake of efficiency. Finally, we adapt our refined procedure to free nilpotent finite-dimensional Lie algebras of step-two to obtain simple and precise descriptions of Vergne polarizing algebras corresponding to all linear functionals in a dense open subset of the linear dual of the corresponding Lie algebra. Several explicit examples are given throughout the paper. Also, a program written for Mathematica is presented at the end of the paper.

## 1. INTRODUCTION

The Lie brackets of nilpotent Lie algebras have some natural combinatorial structures which make this class of groups very appealing; both from theoretical and computational aspects. The main purpose of this paper is to reconcile in a sense some of the theoretical nature of harmonic analysis on nilpotent Lie groups with some of the computations involved in the process of constructing unitary irreducible representations. It is worth noticing that Pedersen has written several programs in REDUCE to compute polarizing subalgebras and canonical coordinates for all nilpotent Lie algebras of dimensions less than seven. In fact, Michel Duflo sent us copies of a companion manuscript to [4] which contains outputs of his programs. Since Pedersen's programs were written a while ago, it is appropriate that we reintroduce some of his ideas through some more current technology. Let  $\mathfrak{g}$  be a real nilpotent Lie algebra. We denote by  $G$  the connected simply connected Lie group corresponding to  $\mathfrak{g}$  such that  $G = \exp(\mathfrak{g})$ . In this case,  $G$  is called a nilpotent Lie group. The problem of classification of the irreducible unitary representations of  $G$  is well understood [1]. Let  $\mathfrak{p}$  be a subalgebra of  $\mathfrak{g}$  and let  $\ell \in \mathfrak{g}^*$ . We say that  $\mathfrak{p} = \mathfrak{p}(\ell)$  is subordinated to  $\ell$  if  $\ell[\mathfrak{p}(\ell), \mathfrak{p}(\ell)] = 0$ . As a result, the formula  $\chi_\ell(\exp X) = \exp(2\pi i \ell(X))$  defines a unitary character on  $\exp(\mathfrak{p}(\ell))$ . The induced representation  $\text{Ind}_{\exp \mathfrak{p}(\ell)}^G(\chi_\ell)$  is an irreducible representation acting in the Hilbert space  $L^2\left(\frac{G}{\exp \mathfrak{p}(\ell)}\right)$  if and only if  $\mathfrak{p}(\ell)$  is a **maximal isotropic subspace** for the skew-symmetric bilinear form  $B_\ell$  defined by  $B_\ell(X, Y) = \ell[X, Y]$ . Such algebra is called a **polarizing subalgebra** [1, 2] for the linear functional  $\ell$  and, it is well-known that the condition that  $\mathfrak{p}(\ell)$  is a maximal isotropic subalgebra is equivalent to

$$(1) \quad [\mathfrak{p}(\ell), \mathfrak{p}(\ell)] \subset \ker(\ell)$$

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(2)  $\dim \mathfrak{p}(\ell) = n - d$  where  $\dim(G \cdot \ell) = 2d$ .

The notation  $\cdot$  is the coadjoint action of  $G$  on  $\mathfrak{g}^*$  which is defined as follows:

$$\exp X \cdot \ell(Y) = \ell(Ad_{\exp(-X)}Y) = \ell(\exp(ad_{-X})Y).$$

One of the complications of the representation theory of nilpotent Lie groups is that the algebra  $\mathfrak{p}(\ell)$  is not generally unique, and there is no canonical way of constructing it. However, there exist well-known procedures among which Vergne's construction ([1] Theorem 1.3.5) is probably the most known. The work presented in this paper is two-fold. In the second section of this paper, we revisit Vergne's construction and we show that under specific conditions, the description of Vergne is refinable. In the third section, we adapt our results to the class of finite-dimensional free nilpotent Lie algebras of step-two. We are then able to derive very simple descriptions of Vergne polarizing subalgebras for all linear functionals in a dense open subset of  $\mathfrak{g}^*$ . At the end of the paper, we also provide a program written in Mathematica to compute the polarizing subalgebra of any given nilpotent Lie algebra.

## 2. POLARIZING ALGEBRAS

Let  $Z_1, Z_2, \dots, Z_n$  be a fixed strong Malcev basis [1] for the Lie algebra  $\mathfrak{g}$  passing through the center of  $\mathfrak{g}$ . Let

$$(0) \subseteq \mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \dots \subseteq \mathfrak{g}_n$$

be a chain of ideals in  $\mathfrak{g}$  such that  $\dim \mathfrak{g}_j = j$ . Given  $\ell \in \mathfrak{g}^*$ , let  $\ell_j = \ell|_{\mathfrak{g}_j}$  where  $\ell_j$  is the restriction of the linear functional  $\ell$  to the vector space  $\mathfrak{g}_j$ . Then, the polarizing subalgebra subordinated to the linear functional  $\ell$  is described as follows

$$\mathfrak{p}(\ell) = \sum_{j=1}^n \mathfrak{r}(\ell_j)$$

where  $\mathfrak{r}(\ell_j)$  is the radical of the skew-symmetric bilinear form  $B_{\ell_j}$ . By definition

$$(2.1) \quad \mathfrak{r}(\ell_j) = \{Y \in \mathfrak{g}_j : B_{\ell_j}(X, Y) = 0 \text{ for all } X \in \mathfrak{g}_j\}.$$

A direct consequence of (2.1) is that

$$(2.2) \quad \mathfrak{p}(\ell) = \sum_{j=1}^n \text{nullspace}(M(\ell_j))$$

where  $M(\ell_j)$  is a skew-symmetric matrix of order  $j$  given by

$$M(\ell_j) = \begin{bmatrix} 0 & \ell_j([Z_1, Z_2]) & \cdots & \ell_j([Z_1, Z_j]) \\ \ell_j([Z_2, Z_1]) & 0 & \cdots & \ell_j([Z_2, Z_j]) \\ \vdots & \vdots & \ddots & \vdots \\ \ell_j([Z_{j-1}, Z_1]) & \ell_j([Z_{j-1}, Z_2]) & \cdots & \ell_j([Z_{j-1}, Z_j]) \\ \ell_j([Z_j, Z_1]) & \ell_j([Z_j, Z_2]) & \cdots & 0 \end{bmatrix}.$$

It is worth mentioning that for Lie algebras of arbitrary large dimensions, it is a very difficult and tedious task to compute Formula 2.1 by hands. Therefore, there is a need to provide methods for the construction of Vergne polarizing subalgebras that can be implemented in Computer Algebra Systems and on other computational platforms. From a computational point of view, Formula

2.2 is slightly more appealing because of the use of linear algebra terms. Moreover, we should point out that the null-space of  $M(\ell_j)$  is obtained with respect to the fixed strong Malcev basis  $Z_1, Z_2, \dots, Z_j$  for the Lie algebra  $\mathfrak{g}$ .

Let us now provide an algorithm for the construction of a Vergne polarizing subalgebra subordinated to an arbitrary linear functional  $\ell \in \mathfrak{g}^*$ .

**Algorithm 1.** *Let*

$$\ell \in \mathfrak{g}^*.$$

*First, we set*

$$(2.3) \quad \mathbf{M}(\ell) = \begin{bmatrix} 0 & \ell[Z_1, Z_2] & \cdots & \ell[Z_1, Z_{n-1}] & \ell[Z_1, Z_n] \\ -\ell[Z_1, Z_2] & 0 & \cdots & \ell[Z_2, Z_{n-1}] & \ell[Z_2, Z_n] \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\ell[Z_1, Z_{n-1}] & -\ell[Z_2, Z_{n-1}] & \cdots & 0 & \ell[Z_{n-1}, Z_n] \\ -\ell[Z_1, Z_n] & -\ell[Z_2, Z_n] & \cdots & -\ell[Z_{n-1}, Z_n] & 0 \end{bmatrix}$$

*such that  $\mathbf{M}(\ell)$  is a singular skew-symmetric matrix of order  $n$ . Next, we define a submatrix of  $\mathbf{M}(\ell)$  of order  $j$  as follows:*

$$\mathbf{M}_j(\ell) = \begin{bmatrix} 0 & \ell[Z_1, Z_2] & \cdots & \ell[Z_1, Z_{j-1}] & \ell[Z_1, Z_j] \\ -\ell[Z_1, Z_2] & 0 & \cdots & \ell[Z_2, Z_{j-1}] & \ell[Z_2, Z_j] \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\ell[Z_1, Z_{j-1}] & -\ell[Z_2, Z_{j-1}] & \cdots & 0 & \ell[Z_{j-1}, Z_j] \\ -\ell[Z_1, Z_j] & -\ell[Z_2, Z_j] & \cdots & -\ell[Z_{j-1}, Z_j] & 0 \end{bmatrix}.$$

*Finally*

$$\mathfrak{p}(\ell) = \sum_{k=1}^n \text{nullspace}(\mathbf{M}_k(\ell)).$$

**Remark 2.** *Although, it is fairly easy to implement Algorithm 1, it is not generally the most efficient way to compute the algebra  $\mathfrak{p}(\ell)$ . In the remainder of this section, we will provide a refined version of Algorithm 1.*

Let  $\mathfrak{z}(\mathfrak{g})$  be the central ideal for the Lie algebra  $\mathfrak{g}$ .

**Lemma 3.** *If  $j \leq \dim \mathfrak{z}(\mathfrak{g}) + 1$  then  $\text{nullspace}(M(\ell_j)) = \mathbb{R}Z_1 + \cdots + \mathbb{R}Z_j$ .*

*Proof.* This lemma follows from the fact that if we assume that  $j \leq \dim \mathfrak{z}(\mathfrak{g}) + 1$  then the matrix  $M(\ell_j)$  is simply a zero matrix of order  $j$  which we consider as a linear operator acting on the vector space  $\mathfrak{g}_j = \mathbb{R}Z_1 + \cdots + \mathbb{R}Z_j$ . Thus its null-space is equal to the whole vector space  $\mathfrak{g}_j$ .  $\square$

Now, let us assume that

$$j > \dim \mathfrak{z}(\mathfrak{g}) + 1.$$

Then the matrix  $M(\ell_j)$  is equal to

$$(2.4) \quad \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \ell_j [Z_{\dim \mathfrak{z}(\mathfrak{g})+1}, Z_{\dim \mathfrak{z}(\mathfrak{g})+2}] & \cdots & \ell_j [Z_{\dim \mathfrak{z}(\mathfrak{g})+1}, Z_j] \\ 0 & \cdots & 0 & -\ell_j [Z_{\dim \mathfrak{z}(\mathfrak{g})+1}, Z_{\dim \mathfrak{z}(\mathfrak{g})+2}] & 0 & \cdots & \ell_j [Z_{\dim \mathfrak{z}(\mathfrak{g})+2}, Z_j] \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -\ell_j [Z_{\dim \mathfrak{z}(\mathfrak{g})+1}, Z_j] & -\ell_j [Z_{\dim \mathfrak{z}(\mathfrak{g})+2}, Z_j] & \cdots & 0 \end{bmatrix}.$$

We will consider the following submatrix of  $M(\ell_j)$  which we denote by

$$M_0(\ell_j) = \begin{bmatrix} 0 & \ell_j [Z_{\dim \mathfrak{z}(\mathfrak{g})+1}, Z_{\dim \mathfrak{z}(\mathfrak{g})+2}] & \cdots & \ell_j [Z_{\dim \mathfrak{z}(\mathfrak{g})+1}, Z_j] \\ -\ell_j [Z_{\dim \mathfrak{z}(\mathfrak{g})+1}, Z_{\dim \mathfrak{z}(\mathfrak{g})+2}] & 0 & \cdots & \ell_j [Z_{\dim \mathfrak{z}(\mathfrak{g})+2}, Z_j] \\ \vdots & \vdots & \ddots & \vdots \\ -\ell_j [Z_{\dim \mathfrak{z}(\mathfrak{g})+1}, Z_j] & -\ell_j [Z_{\dim \mathfrak{z}(\mathfrak{g})+2}, Z_j] & \cdots & 0 \end{bmatrix}$$

where  $j = \dim \mathfrak{z}(\mathfrak{g}) + s_j$ ,

$$(2.5) \quad M(\ell_j) = \begin{bmatrix} 0 & 0 \\ 0 & M_0(\ell_j) \end{bmatrix}$$

and  $s_j > 1$ . We observe that if  $j_1 \leq j_2$  then  $M_0(\ell_{j_1})$  is a submatrix of  $M_0(\ell_{j_2})$ . It is fairly easy to check that the above implies that if  $j = \dim(\mathfrak{z}(\mathfrak{g})) + s_j$  then

$$\text{nullspace}(M(\ell_{\dim(\mathfrak{z}(\mathfrak{g})) + s_j})) = \mathfrak{z}(\mathfrak{g}) + \text{nullspace}(M_0(\ell_{\dim(\mathfrak{z}(\mathfrak{g})) + s_j})).$$

Now, let

$$\mathbf{I}(\ell) = \mathbf{I} = \{1 < s_j \leq n - \dim(\mathfrak{z}(\mathfrak{g})) : \text{rank}(M_0(\ell_{\dim(\mathfrak{z}(\mathfrak{g})) + s_j})) = s_j\} \subset \{2, 3, \dots, \dim(\mathfrak{g})\}$$

Clearly, it is possible for the set  $\mathbf{I}$  to be empty. However, in the case where  $\mathbf{I}$  is nonempty, it makes sense to attempt to refine Formula 2.2.

**Lemma 4.** *If  $s_j \in \mathbf{I}$  then*

$$\text{nullspace}(M(\ell_j)) = \text{nullspace}(M(\ell_{\dim \mathfrak{z}(\mathfrak{g}) + s_j})) = \mathfrak{z}(\mathfrak{g}).$$

*Proof.* If  $s_j \in \mathbf{I}$  then  $M_0(\ell_{\dim \mathfrak{z}(\mathfrak{g}) + s_j})$  is a skew-symmetric matrix of even full-rank. Thus, its null-space is the trivial vector space. Therefore,

$$\text{nullspace}(M(\ell_j)) = \mathfrak{z}(\mathfrak{g}).$$

□

Appealing to the above lemma, and to Formula 2.2, the following is immediate.

**Theorem 5.** *Let  $\ell$  be a linear functional in  $\mathfrak{g}^*$ . A Vergne polarizing subalgebra subordinated to the linear functional  $\ell$  is*

$$\mathfrak{p}(\ell) = \mathfrak{z}(\mathfrak{g}) + \mathbb{R}Z_{\dim \mathfrak{z}(\mathfrak{g})+1} + \sum_{s_j \notin \mathbf{I}(\ell)} \text{nullspace}(M_0(\ell_{\dim \mathfrak{z}(\mathfrak{g}) + s_j}))$$

### 3. VERGNE POLARIZING SUBALGEBRAS FOR FREE NILPOTENT LIE ALGEBRAS

In this section, we will adapt the algorithms provided in the previous section to a class of nilpotent Lie algebras known as free nilpotent Lie algebras of step-two. Applying the refined algorithms described in Theorem 5, we are able to provide very simple descriptions of Vergne polarizing algebras subordinated to a family of linear functionals in a Zariski open (dense) subset of the linear dual of the corresponding Lie algebra. Let  $\mathfrak{g}$  be the free nilpotent Lie algebra of step-two on  $m$  generators ( $m > 1$ ). If  $\{Z_1, \dots, Z_m\}$  is the generating set for  $\mathfrak{g}$  then

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) + \mathbb{R}\text{-span } \{Z_1, \dots, Z_m\}$$

such that  $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\text{-span } \{Z_{ik} : 1 \leq i \leq m \text{ and } i < k \leq m\}$ . The non-trivial Lie brackets of this Lie algebra are described as follows.

$$[Z_i, Z_j] = Z_{ij} \text{ for } (1 \leq i \leq m \text{ and } i < j \leq m).$$

It is then easy to see that

$$\dim(\mathfrak{z}(\mathfrak{g})) = \frac{m(m-1)}{2}.$$

In these settings, we define recursively the matrix

$$M_0(\ell_j) = \begin{bmatrix} 0 & \ell_j[Z_1, Z_2] & \cdots & \ell_j[Z_1, Z_{j-1}] & \ell_j[Z_1, Z_j] \\ \ell_j[Z_2, Z_1] & 0 & \cdots & \ell_j[Z_2, Z_{j-1}] & \ell_j[Z_2, Z_j] \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \ell_j[Z_{j-1}, Z_1] & \ell_j[Z_{j-1}, Z_2] & \cdots & 0 & \ell_j[Z_{j-1}, Z_j] \\ \ell_j[Z_j, Z_1] & \ell_j[Z_j, Z_2] & \cdots & \ell_j[Z_j, Z_{j-1}] & 0 \end{bmatrix}$$

such that

$$M_0(\ell_j) = \begin{bmatrix} M_0(\ell_{j-1}) & v(\ell_j) \\ w(\ell_j) & 0 \end{bmatrix}$$

where

$$v(\ell_j) = \begin{bmatrix} \ell_j[Z_1, Z_j] \\ \ell_j[Z_2, Z_j] \\ \vdots \\ \ell_j[Z_{j-1}, Z_j] \end{bmatrix} \text{ and } w(\ell_j) = [\ell_j[Z_j, Z_1] \quad \ell_j[Z_j, Z_2] \quad \cdots \quad \ell_j[Z_j, Z_{j-1}]].$$

From now on, we will assume that  $\ell \in \mathfrak{g}^*$  and

$$(3.1) \quad \ell \in \{f \in \mathfrak{g}^* : \det M_0(f_{j-1}) \neq 0 \text{ for odd } j > 1\}.$$

Let  $k$  be a natural number smaller than the dimension of the Lie algebra  $\mathfrak{g}$ . Furthermore, for each  $k$  we define the embedding map  $\mu_k : \mathbb{R}^k \rightarrow \mathfrak{g}$  such that  $\mu_k$  is a map which sends the column vector  $[z_1, \dots, z_k]^t$  to the vector  $z_1 Z_1 + \cdots + z_k Z_k \in \mathfrak{g}$  and  $[z_1, \dots, z_k]^t$  is the transpose of  $[z_1, \dots, z_k]$ .

**Lemma 6.** *If  $j > 1$  is odd then the null-space of the matrix  $M_0(\ell_j)$  is equal to*

$$\mathbb{R} \left( Z_j - \mu_{j-1} \left( \begin{bmatrix} 0 & \ell_j[Z_1, Z_2] & \cdots & \ell_j[Z_1, Z_{j-1}] \\ \ell_j[Z_2, Z_1] & 0 & \cdots & \ell_j[Z_2, Z_{j-1}] \\ \vdots & \vdots & \ddots & \vdots \\ \ell_j[Z_{j-1}, Z_1] & \ell_j[Z_{j-1}, Z_2] & \cdots & 0 \end{bmatrix}^{-1} \begin{bmatrix} \ell_j[Z_1, Z_j] \\ \ell_j[Z_2, Z_j] \\ \vdots \\ \ell_j[Z_{j-1}, Z_j] \end{bmatrix} \right) \right).$$

If  $j$  is even then

$$\text{nullspace}(M_0(\ell_j)) = (0).$$

*Proof.* For any linear functional  $\ell$  satisfying (3.1)), we will show that the set  $\mathbf{I}(\ell)$  only contains even indices and the matrix  $M_0(\ell_{j-1})$ . First of all, let us suppose that  $j$  is odd. Then  $M_0(\ell_{j-1})$  is a skew-symmetric matrix of even **full-rank** (see (3.1)). So

$$\begin{bmatrix} M_0(\ell_{j-1})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M_0(\ell_{j-1}) & v(\ell_j) \\ w(\ell_j) & 0 \end{bmatrix} = \begin{bmatrix} I & M_0(\ell_{j-1})^{-1} v(\ell_j) \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} I & M_0(\ell_{j-1})^{-1} v(\ell_j) \\ 0 & 0 \end{bmatrix}$$

is just the row-reduced form of the matrix

$$\begin{bmatrix} M_0(\ell_{j-1}) & v(\ell_j) \\ w(\ell_j) & 0 \end{bmatrix}.$$

Therefore,

$$\text{nullspace}(M_0(\ell_j)) = \mathbb{R}(Z_j - \mu_{j-1}(M_0(\ell_{j-1})^{-1} v(\ell_j)))$$

and the above is precisely equal to

$$\mathbb{R} \left( Z_j - \mu_{j-1} \left( \begin{bmatrix} 0 & \ell_j[Z_1, Z_2] & \cdots & \ell_j[Z_1, Z_{j-1}] \\ \ell_j[Z_2, Z_1] & 0 & \cdots & \ell_j[Z_2, Z_{j-1}] \\ \vdots & \vdots & \ddots & \vdots \\ \ell_j[Z_{j-1}, Z_1] & \ell_j[Z_{j-1}, Z_2] & \cdots & 0 \end{bmatrix}^{-1} \begin{bmatrix} \ell_j[Z_1, Z_j] \\ \ell_j[Z_2, Z_j] \\ \vdots \\ \ell_j[Z_{j-1}, Z_j] \end{bmatrix} \right) \right).$$

Secondly, by assumption, if  $j$  is even then  $M_0(\ell_j)$  is a skew-symmetric matrix of even full-rank. Therefore the null-space of this matrix is trivial.  $\square$

**Proposition 7.** *If  $m$  is odd ( $m = 2s + 1$  for some natural number  $s$ ) then*

$$\mathfrak{p}(\ell) = \mathfrak{z}(\mathfrak{g}) + \mathbb{R}Z_1 + \left( \sum_{k=1}^s \mathbb{R}(Z_{2k+1} - \mu_{2k}(M_0(\ell_{2k})^{-1} v(\ell_{2k+1})) \right).$$

*If  $m$  is even ( $m = 2s$  for some natural number  $s$ ) then*

$$\mathfrak{p}(\ell) = \mathfrak{z}(\mathfrak{g}) + \mathbb{R}Z_1 + \left( \sum_{k=1}^{s-1} \mathbb{R}(Z_{2k+1} - \mu_{2k}(M_0(\ell_{2k})^{-1} v(\ell_{2k+1})) \right).$$

The above proposition is a direct application of Lemma 6 and Theorem 5.

## A MATHEMATICA PROGRAM

Let  $\mathfrak{g}$  be a real nilpotent Lie algebra of dimension  $n$  with a fixed strong Malcev basis:  $Z_1, Z_2, \dots, Z_n$ . We will present a program written in Mathematica which can be used to compute the Vergne polarizing subalgebra with respect to a linear functional  $\ell \in \mathfrak{g}^*$ . The only argument for this program is the matrix

$$\mathbf{M}(\ell)$$

```
Polarization[M0_] := Module[{M = M0, X, i, j, s, k}, X = Table[Table[M[[i, j]], {i, 1, s}, {j, 1, s}], {s, 1, Length[M]}];
DeleteDuplicates[Table[Table[z[k] Z[k], {k, 1,
Length[Total[NullSpace[X[[k]]]]}], {k, 1, Length[X]}]]]
```

FIGURE 1. A program written in Mathematica

The output of this program is a spanning set for a polarizing subalgebra subordinated to  $\ell$ . Here are some actual Mathematica outputs

$$\text{Polarization} \left[ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda[1, 2] & \lambda[1, 3] & \lambda[1, 4] & \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda[1, 2] & 0 & \lambda[2, 3] & \lambda[2, 4] \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda[1, 3] & -\lambda[2, 3] & 0 & \lambda[3, 4] \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda[1, 4] & -\lambda[2, 4] & -\lambda[3, 4] & 0 \end{pmatrix} \right] // \text{MatrixForm}$$

FIGURE 2. Some Mathematica Outputs

## ACKNOWLEDGMENT

We thank Michel Duflo for bringing to our attention that Niels Pedersen has already written programs in REDUCE to compute polarizing subalgebras for all nilpotent Lie algebras of dimensions less than seven, and we also thank him for supplying reference [4].

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