COMPACTLY SUPPORTED BOUNDED FRAMES ON LIE GROUPS

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Abstract. Let $G = NH$ be a Lie group where N, H are closed connected subgroups of G, and N is an exponential solvable Lie group which is normal in G. Suppose furthermore that N admits a unitary character χ_λ corresponding to a linear functional λ of its Lie algebra. We assume that the map $h \mapsto Ad\left(h^{-1}\right)^* \lambda$ defines an immersion at the identity of H. Fixing a Haar measure on H, we consider the unitary representation π of G obtained by inducing χ_λ. This representation which is realized as acting in $L^2(H, d\mu_H)$ is generally not irreducible, and we do not assume that it satisfies any integrability condition. One of our main results establishes the existence of a countable set $\Gamma \subset G$ and a function $\mathbf{f} \in L^2(H, d\mu_H)$ which is compactly supported and bounded such that $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is a frame. Additionally, we prove that $\mathbf{f}$ can be constructed to be continuous. In fact, $\mathbf{f}$ can be taken to be as smooth as desired. Our findings extend the work started in [28] to the more general case where H is any connected Lie group. We also solve a problem left open in [28]. Precisely, we prove that in the case where H is an exponential solvable group, there exist a continuous (or smooth) function $\mathbf{f}$ and a countable set Γ such that $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is a Parseval frame. Since the concept of well-localized frames is central to time-frequency analysis, wavelet, shearlet and generalized shearlet theories, our results are relevant to these topics and our approach leads to new constructions which bear potential for applications. Moreover, our work sets itself apart from other discretization schemes in many ways. (1) We give an explicit construction of Hilbert frames and Parseval frames generated by bounded and compactly supported windows. (2) We provide a systematic method that can be exploited to compute frame bounds for our constructions. (3) We make no assumption on the irreducibility or integrability of the representations of interest.

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1. Introduction and overview of the work

Let H be a Lie group endowed with a fixed left Haar measure $d\mu_H$. A frame [3] for the Hilbert space $\mathfrak{H} = L^2(H, d\mu_H)$ is a collection of vectors $\{\mathbf{f}_k : k \in I\}$ such that there exist positive constants $A \leq B$ where

$$A \|\mathbf{g}\|_{\mathfrak{H}}^2 \leq \sum_{k\in I} \left|\langle \mathbf{g}, \mathbf{f}_k\rangle_{\mathfrak{H}}\right|^2 \leq B \|\mathbf{g}\|_{\mathfrak{H}}^2 \tag{1.1}$$

for all $\mathbf{g} \in \mathfrak{H}$. Note that the inequality (1.1) is a relaxation of the Parseval formula known for orthonormal bases. The constants A, B are called the frame bounds. The largest possible lower frame bound is called the optimal lower bound and the smallest possible upper-bound is the optimal upper frame bound. In the case where $A = B = 1$, we say that $\{\mathbf{f}_k : k \in I\}$ is a Parseval frame. Every frame $\{\mathbf{f}_k : k \in I\}$ for a Hilbert space $\mathfrak{H}$ gives rise to a basis-like expansion of the type $\mathbf{g} = \sum_{k\in I} a_k(\mathbf{g}) \mathbf{f}_k$ for every vector $\mathbf{g}$ in $\mathfrak{H}$. Moreover, the linear functionals a_k can be selected to be bounded on $\mathfrak{H}$. Consequently, for every $\mathbf{g}$ in $\mathfrak{H}$, there exist vectors $\{\mathbf{h}_k : k \in I\}$ in $\mathfrak{H}$ such that

$$\mathbf{g} = \sum_{k\in I} \langle \mathbf{g}, \mathbf{h}_k\rangle \mathbf{f}_k.$$

There is indeed a canonical choice for the system $\{\mathbf{h}_k : k \in I\}$ for which the expansion above is unconditional. Precisely, for a frame $\{\mathbf{f}_k : k \in I\}$, the associated frame operator

$$S : \mathbf{g} \mapsto S(\mathbf{g}) = \sum_{k\in I} \langle \mathbf{g}, \mathbf{f}_k\rangle_{\mathfrak{H}} \mathbf{f}_k \tag{1.2}$$

is bounded, self-adjoint, positive and invertible on $\mathfrak{H}$ and every vector $\mathbf{g}$ admits an expansion of the form $\mathbf{g} = \sum_{k\in I} \langle \mathbf{g}, S^{-1}\mathbf{f}_k\rangle_{\mathfrak{H}} \mathbf{f}_k$. The frame operator and its inverse are topological isomorphisms of $\mathfrak{H}$ and the series $\sum_{k\in I} \langle \mathbf{g}, S^{-1}\mathbf{f}_k\rangle_{\mathfrak{H}} \mathbf{f}_k$ converges unconditionally in the norm of $\mathfrak{H}$ for each $\mathbf{g}$ in $\mathfrak{H}$. In fact, the sequence $\{S^{-1}\mathbf{f}_k : k \in I\}$ is called the canonical dual frame of $\{\mathbf{f}_k : k \in I\}$. Moreover, each sequence $\left(\langle \mathbf{g}, S^{-1}\mathbf{f}_k\rangle_{\mathfrak{H}}\right)_{k\in I}$ is square-summable. However, the coefficients appearing in the expansion $\sum_{k\in I} \langle \mathbf{g}, S^{-1}\mathbf{f}_k\rangle_{\mathfrak{H}} \mathbf{f}_k$ need not be unique. In the case where $\{\mathbf{f}_k : k \in I\}$ is a Parseval frame, we have a much simpler reconstruction procedure.

The frame operator coincides with the identity map and as a result,

$$\mathbf{g} = \sum_{k \in I} \langle \mathbf{g}, \mathbf{f}_k \rangle_{\mathfrak{H}} \mathbf{f}_k \text{ for all } \mathbf{g} \in \mathfrak{H}.$$

It is well-known that the Schrödinger representations of the 3-dimensional Heisenberg Lie group provide the group-theoretic foundation for time-frequency analysis [19, 4, 21]. The 3-dimensional Heisenberg group is often realized as a semi-direct product $\mathbb{R}^2 \rtimes \mathbb{R}$ equipped with the following non-commutative operation

$$(v,t)(w,s) = \left(v + \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} w, t+s \right).$$

Three elementary classes of operators occur in the Schrödinger representations: scalar multiplications, modulation operators (phase translations) and translations. Although the role played by the operators acting by scalar multiplication is negligible, the combined actions of modulations and translations can be discretized to construct frames and orthonormal bases for $L^2(\mathbb{R})$. Gabor analysis offers a rigorous study of the various ways by which one can select a window function and a countable subset of the Heisenberg group to construct frames. Similarly, the ax+b group plays a significant role in classical wavelet analysis. Like the Heisenberg group, the ax+b group is also a solvable Lie group. However, unlike the Heisenberg group, it is not a nilpotent group. Very often, this affine group is realized as a semi-direct product $\mathbb{R} \rtimes \mathbb{R}$ endowed with the multiplication

$$(x,a)(y,b) = (x + e^a y, a+b).$$

Up to unitary equivalence, the ax+b group has two infinite-dimensional irreducible representations. These representations may be realized as acting in $L^2(\mathbb{R})$ as follows

$$\pi_{\pm}(x,a)\mathbf{f}(t) = e^{\pm 2\pi i e^{-t} x} \mathbf{f}(t-a).$$

Wavelet analysis provides means by which we can discretize the representations $\pi_{\pm}$ (or their direct sum) to construct nice frames (we will say more about this later.) A common feature of the examples of the Heisenberg group and the ax+b group worth noting is the following: the irreducible representations giving rise to wavelets and Gabor frames are all monomial (induced by characters.) Precisely, they are obtained by inducing a unitary character of a normal closed subgroup. It turns out that for a surprisingly large class of solvable Lie groups, similar constructions are possible [28, 20]. One may then ask the following: to which extent can we generalize the results in [28, 20] in a unified manner? The main purpose of this work is to address this question.

Let H be a connected Lie group endowed with a fixed left Haar measure $d\mu_H$. Let us suppose that π is a strongly continuous unitary representation of a Lie group G acting in the Hilbert space $\mathfrak{H} = L^2(H, d\mu_H)$. We investigate conditions under which, it is possible to find a discrete subset Γ of G and a function $\mathbf{f}$ such that the following holds. $\mathbf{f}$ is bounded and compactly supported on H and the system $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is either a frame or a Parseval frame or an orthonormal basis. We are especially interested in finding conditions under which the window function $\mathbf{f}$ is either continuous or smooth. In a nutshell, we seek to construct 'nice' frames, and when such a frame exists, we shall call it an H-localized π-frame. A practical motivation for the interest in the so called nice frames can be described as follows. In situations where the generator of $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is a smooth and compactly supported function, the Gramian of the corresponding system can be shown to have fast off-diagonal decay [1]. This property can be quite useful in computing the inverse of the frame operator as well as the dual frames corresponding to the system $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ (see for instance Proposition 20.)

Suppose L is the left regular representation of H acting in $\mathfrak{H}$, and H is not discrete. If $\mathbf{f}$ is compactly supported and bounded then for any countable set $\Gamma \subset H$, it is known that the system $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ can never be a frame for $\mathfrak{H}$ [15]. This observation excludes the left regular representation as a viable choice for the construction of nice frames on H.

In the situation where π is an irreducible representation which is also integrable, the coorbit theory [10, 17, 13] developed by Feichtinger and Gröchenig has proved to be a powerful discretization scheme for the construction of Hilbert space frames and atoms for other Banach spaces satisfying certain regular conditions.

Olafsson and Christensen recently introduced a generalization of coorbit theory in which, the integrability and irreducibility conditions of coorbit theory were removed [1, 2]. Their theory hinges on the existence of a cyclic vector satisfying a reproducing formula equation together with other technical assumptions. Other generalizations of coorbit theory can be found in [6, 12, 30].

Our work sets itself apart from other discretization schemes in several ways.

- We make no assumption on the irreducibility or integrability of the representations of interest.

[1]We thank the referee for pointing this fact to us

- We are mainly interested in the explicit construction of Hilbert space frames, Parseval frames and orthonormal bases generated by bounded and compactly supported windows.
- We are also interested in providing a systematic method that can be exploited to compute (optimal) frame bounds for our constructions.

Let G be a Lie group such that $G = NH$, H, N are connected closed subgroups, and N is simply connected and normal in G. Assume furthermore that π is a unitary representation of G [11] obtained by inducing a unitary character of N and is realized as acting in the Hilbert space $L^2(H, d\mu_H)$. As a starting point, we consider a unitary character χ of N associated to a linear functional λ of the Lie algebra of N. The conjugation action of H on N gives rise to a smooth action $\star$ of H on the linear functionals of the Lie algebra of N . This action plays a decisive role in our work. We prove that if the map $h \mapsto h \star \lambda$ defines an immersion at the identity of the group H (see Theorem 1) then H-localized π-frames can be constructed quite explicitly. These types of frames naturally arise in the context of shearlet, time-frequency analyses, and wavelet theory [23, 24, 25, 18, 14, 13, 16] where frames generated by functions satisfying certain regular conditions (such as fast decay and various degrees of smoothness) are often sought after. Therefore, the present project, which was initiated in [28] provides a unified discretization scheme relevant to these topics. Furthermore, our main results (see Theorem 1, Theorem 2 and Theorem 3) extend the work of [28] to a significantly broader class of representations (see Proposition 26 for an example). While the central results of [28] only focus on cases where the conormal subgroup H of G is completely solvable, we allow H to be any connected Lie group here (see for example, Proposition 26.)

1.1. **Assumptions.** Before we present our main results, let us first formally introduce some assumptions. Let H be a connected Lie group with Lie algebra $\mathfrak{h}$. Let us also assume that there exists a Lie algebra $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ such that $\mathfrak{n}$ is an exponential solvable ideal of $\mathfrak{g}$ satisfying $\dim(\mathfrak{n}) = n \geq \dim(\mathfrak{h}) = r$. Put

$$\mathfrak{n} = \sum_{k=1}^{n} \mathbb{R}X_k,\ \mathfrak{h} = \sum_{k=1}^{r} \mathbb{R}A_k \text{ and } N = \exp(\mathfrak{n}).$$

Furthermore, we assume that N and H are closed subgroups of G. Note that by assumption, the exponential map defines a global diffeomorphism between $\mathfrak{n}$ and N. As such, N is a simply connected normal (exponential) solvable subgroup of G.

Let

$$\lambda \in \mathfrak{n}^* = \sum_{k=1}^{n} \mathbb{R} X_k^*$$

be a linear functional of $\mathfrak{n}$ where $\{X_1^*, X_2^*, \cdots, X_n^*\}$ is a basis for the dual vector space $\mathfrak{n}^*$ satisfying $\left\langle X_j^*, X_k \right\rangle = \delta_{j,k}$ for $1 \leq j, k \leq n$. We also assume that the derived ideal

$$[\mathfrak{n}, \mathfrak{n}] = \mathbb{R}\text{-span}\,\{[X, Y] : X, Y \in \mathfrak{n}\}$$

is contained in the kernel of λ. Consequently, the linear functional λ determines a unitary character of the normal subgroup of N defined by

$$\chi_\lambda (\exp X) = e^{2\pi i \langle \lambda, X \rangle}$$

for $\exp(X) \in N$. Next, we consider the induced representation

$$\pi = \mathrm{ind}_N^{NH} (\chi_\lambda) \tag{1.3}$$

realized as acting on the Hilbert space $\mathfrak{H} = L^2 (H, d\mu_H)$ as follows. Given $s \in G$, we define the endomorphism $Ad(s)$ as the differential of the smooth map $y \mapsto sys^{-1}$ at the neutral element of G. Next, for $\mathbf{f} \in \mathfrak{H}$, $x \in N$, and $z \in H$, we have

$$[\pi (x)] \mathbf{f} (h) = e^{2\pi i \left\langle Ad\left(h^{-1}\right)^* \lambda, \log(x) \right\rangle} \mathbf{f} (h) \text{ and } [\pi (z)] \mathbf{f} (h) = \mathbf{f} \left(z^{-1} h\right). \tag{1.4}$$

The reader who is interested in learning about the fundamental principles of induced representations is invited to peruse Chapter 6 in the textbook of Folland [11].

On one hand, we remark that given $x \in N$, the operator $\pi(x)$ acts in a way which resembles a modulation action. We call the action induced by such an operator a generalized modulation action. On the other hand, the operators $\pi(z)$ (where $z \in H$) act on $\mathfrak{H}$ by left translations and the combined action of the normal and conormal parts of the group gives an action resembling that of time-frequency translations [3]. Perhaps, it is also worth highlighting that all irreducible representations of groups connected to Gabor theory, wavelet and shearlet analyses naturally arise in this fashion (see [28, Section 1.4]).

We refer the reader who might be intrigued by the size of the class of groups and representations satisfying the properties described above to Proposition 26. It is proved in Proposition 26 that for any given connected matrix group H, there exists a matrix group $G = NH$ and a unitary representation π satisfying all conditions listed above.

1.2. **Main theorems.** The main results of our investigation are summarized in the following theorems.

Theorem 1. *Let N, H and π be as described in (1.4). Assume that the map $h \mapsto Ad(h^{-1})^*\lambda$ is an immersion at the identity of H. Then there exists a relatively compact neighborhood $\mathcal{O}$ of the identity in H such that if $\mathbf{f}$ is a continuous function supported on $\mathcal{O}$, then there exists a countable set $\Gamma \subset G$ such that $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is a frame for $L^2(H, d\mu_H)$. Moreover, $\mathbf{f}$ can be chosen to be in $C^k(H)$ for arbitrary natural number k (and even infinitely differentiable.)*

Theorem 2. *Let N, H and π be as described in (1.4). Assume that the map $h \mapsto Ad(h^{-1})^*\lambda$ is an immersion at the identity of H. If there exists a function defined on a relatively compact neighborhood of the identity in H with support satisfying some additional technical conditions [2], then it is possible to discretize π to construct Parseval frames and/or orthonormal bases for the Hilbert space $L^2(H, d\mu_H)$.*

We remark that in Theorem 1, the set Γ depends on a choice of $\mathbf{f}$ and $\mathcal{O}$. Additionally, the following establishes the existence of a Parseval frame generated by a continuous and compactly supported function when H is exponential and solvable. This does close a significant gap in our previous work [28].

Theorem 3. *Let N, H and π be as described in (1.4). If the map $h \mapsto Ad(h^{-1})^*\lambda$ is an immersion at the identity of H and if H is an exponential solvable Lie group then there exist a relatively compact neighborhood $\mathcal{O}$ of the identity in H, a continuous function $\mathbf{f}$ supported on $\mathcal{O}$ and a countable subset Γ of G such that $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is a Parseval frame.*

Note that in situations where there exists a unit vector satisfying the conditions described in Theorem 3, $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is an orthonormal basis for $L^2(H, d\mu_H)$.

A few additional observations are in order.

Remark 4.

(1) *In a systematic fashion, every frame $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ for the Hilbert space $L^2(H, d\mu_H)$ gives rise to new frames generated by a unitary representation of G as follows. Let α be a Lie group automorphism of G. According to [5, Lemma 2.1.3], there exists a unitary map*

$$Q : L^2(H, d\mu_H) \to L^2\left(NH/\alpha^{-1}(N), d\mu_{NH/\alpha^{-1}(N)}\right) \tag{1.5}$$

[2]See Proposition 21

such that

$$Q\left[\operatorname{Ind}_N^{NH}\left(\chi_\lambda\right)\left(\alpha\left(x\right)\right)\right]\mathbf{f}=\left[\operatorname{Ind}_{\alpha^{-1}(N)}^{NH}\left(\chi_\lambda\circ\alpha\right)\left(x\right)\right]Q\mathbf{f}$$

for all $x\in G$ and for all $\mathbf{f}\in L^2\left(H,d\mu_H\right)$. An explicit construction of a unitary map Q intertwining the representations above is described in the proof of [5, Lemma 2.1.3]. In fact, if π is an irreducible representation of G, according to Schur's lemma, the unitary operator described in the proof of [5, Lemma 2.1.3] is unique up to multiplication by a constant. In the definition of Q, (see (1.5)) the measure $d\mu_{NH/\alpha^{-1}(N)}$ is a quasi G-invariant measure on a cross-section for $NH/\alpha^{-1}\left(N\right)$ [11, Section 6.1]. Letting

$$\pi_{\lambda^{(\alpha)}}=\operatorname{Ind}_{\alpha^{-1}(N)}^{NH}\left(\chi_\lambda\circ\alpha\right)\ \text{ and } \pi_{\lambda_{(\alpha)}}=\operatorname{Ind}_N^{NH}\left(\chi_\lambda\circ\alpha\right),$$

the following is immediate. (a) $\left\{\pi\left(\gamma\right)\mathbf{f}:\gamma\in\Gamma\right\}$ is a frame if and only if

$$\left\{\pi_{\lambda^{(\alpha)}}\left(\gamma\right)Q\mathbf{f}:\gamma\in\alpha^{-1}\left(\Gamma\right)\right\}$$

is a frame for $L^2\left(NH/\alpha^{-1}\left(N\right),d\mu_{NH/\alpha^{-1}(N)}\right)$. (b) If α is an automorphism of G fixing the normal subgroup N then $\left\{\pi\left(\gamma\right)\mathbf{f}:\gamma\in\Gamma\right\}$ is a frame for $L^2\left(H,d\mu_H\right)$ if and only if the collection

$$\left\{\pi_{\lambda_{(\alpha)}}\left(\gamma\right)Q\mathbf{f}:\gamma\in\alpha^{-1}\left(\Gamma\right)\right\}$$

is a frame for $L^2\left(H,d\mu_H\right)$.

(2) *Suppose that $\left\{\pi\left(\gamma\right)\mathbf{f}:\gamma\in\Gamma\right\}$ is a frame for the Hilbert space $L^2\left(H,d\mu_H\right)$. If Q is a bounded and invertible linear operator commuting with all operators in the set $\left\{\pi\left(\gamma\right):\gamma\in\Gamma\right\}$ then clearly, $\left\{\pi\left(\gamma\right)Q\mathbf{f}:\gamma\in\Gamma\right\}$ is also a frame for $L^2\left(H,d\mu_H\right)$.*

(3) *The mapping $h\mapsto Ad\left(h^{-1}\right)^*$ is a linear representation of H acting on the dual of the Lie algebra of N. Moreover, the stabilizer subgroup*

$$H_\lambda=\left\{h\in H:Ad\left(h^{-1}\right)^*\lambda=\lambda\right\}$$

is a closed subgroup of H and $h\mapsto Ad(h^{-1})^*\lambda$ is an immersion at the identity of H if and only if H_λ is discrete (see Proposition 8.)

Intuitively, our strategy in proving our main results is two-fold. First, we look for a set M inside N containing a discrete set Γ_N such that the corresponding operators applied to a suitable function supported on a neighborhood of the identity in H generate what we shall call a 'local frame'. The construction of local frames follows from the fact that H is locally Euclidean and that under suitable change of variables, the action of N gives rise to some local Fourier series. It is important to note that M is generally a proper sub-manifold of N. In fact, the discretization of M for the construction of a local frame is not apparent and does require careful work. Secondly, via π, H acts on itself by left translations. For a suitable countable subset $\Gamma_H \subset H$, the operators $\pi(\gamma), \gamma \in \Gamma_H$ give a covering of H allowing us to construct nice frames for $L^2(H)$ from our so-called local frames.

In Section 2, we shall present a number of intermediate facts leading to the proof of our main results (Theorem 1 and Theorem 3). The third section of the work contains explicit examples showing how our scheme can be exploited for the construction of frames, Parseval frames and orthonormal bases.

2. Intermediate results and proofs of Theorem 1 and Theorem 3

2.1. **Preliminaries.** Let us start by fixing our notation.

- Let T be a linear operator acting on a vector. T^* stands for the adjoint of the linear operator T and the transpose of a matrix A is denoted A^T.
- Given a vector space V, the max-norm of an arbitrary vector $v \in V$ is $\|v\|_{\max} = \max_k |v_k|$.
- Given a Lebesgue measurable subset E of $\mathbb{R}^d$, $|E|$ stands for the Lebesgue measure of E and the indicator function of a set A is denoted 1_A.
- Given a finite set J, the number of elements in J is denoted $\sharp(J)$.
- The trace of a matrix M is denoted $\mathrm{Tr}\,(M)$.
- The zero square matrix of order m is often denoted 0_m.
- The identity endomorphism acting on a vector space V is often denoted id.

Define

$$\beta_\lambda : H \to \beta_\lambda(H) \subseteq \mathfrak{n}^* \tag{2.1}$$

such that

$$\beta_\lambda(h) = Ad\left(h^{-1}\right)^* \lambda. \tag{2.2}$$

We make the following additional assumption: The smooth function $h \mapsto Ad\left(h^{-1}\right)^* \lambda$ is an immersion at the identity of H. This implies that $n = \dim N \geq r = \dim H$. We make no assumption on the irreducibility of π, nor shall we assume that π is integrable or even square-integrable (see Example 28). In fact, as soon as the structure constants of the Lie algebra $\mathfrak{g}$ are determined, the immersion condition imposed on the function $h \mapsto Ad\,(h^{-1})^* \lambda$ is easily verified (see Proposition 5.) To this end, it suffices to show that the differential of the map $h \mapsto Ad\,(h^{-1})^* \lambda$ has a trivial null-space at the identity of H. Letting $r = \dim(H)$, there is a smooth chart

$$(\mathcal{O}, \varphi) \tag{2.3}$$

such that $\mathcal{O} \subset H$ is an open set around the identity of H, and

$$\varphi : \mathcal{O} \to \varphi(\mathcal{O}) \subseteq \mathfrak{h}$$

is a diffeomorphism taking the identity in H to the zero element in its Lie algebra such that $\beta_\lambda \circ \varphi^{-1}$ has a constant rank equal to r. Since the exponential map defines a local diffeomorphism between an open set containing the zero vector in $\mathfrak{h}$ and a suitable open set around the neutral element in H, for a sufficiently small neighborhood of the identity, the function φ can be taken to be the log function. Moving forward, the Jacobian of

$$h \mapsto Ad\left(h^{-1}\right)^* \lambda$$

in local coordinates takes the form

$$D = \mathrm{Jac}_{\beta_\lambda \circ \varphi^{-1}}(0) = \left[\frac{\partial \left(\beta_\lambda \circ \varphi^{-1}\right)_j (A)}{\partial A_k}(0) \right]_{1 \leq j \leq n, 1 \leq k \leq r} \tag{2.4}$$

and the following is immediate.

Proposition 5. *$h \mapsto Ad\,(h^{-1})^* \lambda$ is an immersion at the identity of H if and only if $\det\left(D^T D\right)$ is nonzero.*

Proof. By definition, $h \mapsto Ad\,(h^{-1})^* \lambda$ is an immersion at the identity of H if and only if the matrix D has trivial kernel. Next, the observation that D has trivial kernel if and only if its gramian matrix $D^T D$ is invertible gives the desired result. □

Proposition 6. *Let $\omega \in \mathfrak{n}^*$. Fixing $\{X_1, X_2, \cdots, X_n\}$ as a basis for $\mathfrak{n}$, and $\{A_1, A_2, \cdots, A_r\}$ as a basis for $\mathfrak{h}$, then*

$$D_\omega = \begin{bmatrix} \langle \omega, [X_1, A_1] \rangle & \cdots & \langle \omega, [X_1, A_r] \rangle \\ \vdots & \ddots & \vdots \\ \langle \omega, [X_n, A_1] \rangle & \cdots & \langle \omega, [X_n, A_r] \rangle \end{bmatrix}$$

is a matrix representation of $\mathrm{Jac}_{\beta_\omega \circ \varphi^{-1}}(0)$ *with respect to the fixed bases.*

Proof. Given $t \in \mathbb{R}$, $A \in \mathfrak{h}$ and $X \in \mathfrak{n}$, we have

$$\left\langle \lim_{t\to 0} \frac{[Ad(\exp(-tA))]^* \omega - \omega}{t}, X \right\rangle = \lim_{t\to 0} \frac{\langle [Ad(\exp(-tA))]^* \omega - \omega, X\rangle}{t}.$$

Next, since

$$\langle \omega, Ad(\exp(-tA))X\rangle = \left\langle \omega, \sum_{k=0}^{\infty} ad(-tA)^k X \right\rangle$$

the following holds true

$$\begin{aligned}\left\langle \lim_{t\to 0} \frac{[Ad(\exp(-tA))]^* \omega - \omega}{t}, X \right\rangle &= \lim_{t\to 0} \frac{\langle \omega, Ad(\exp(-tA))X\rangle}{t} - \lim_{t\to 0}\frac{\langle \omega, X\rangle}{t} \\ &= \lim_{t\to 0} \frac{\left\langle \omega, \sum_{k=1}^{\infty} ad(-tA)^k X\right\rangle}{t} \\ &= (*)\end{aligned}$$

Observing that

$$\sum_{k=1}^{\infty} ad(-tA)^k X = ad(-tA)X + \sum_{k=2}^{\infty} ad(-tA)^k X,$$

we obtain

$$\begin{aligned}(*) &= \left(\lim_{t\to 0} \frac{\langle \omega, ad(-tA)X\rangle}{t}\right) + \underbrace{\left(\lim_{t\to 0} \frac{\left\langle \omega, \sum_{k=2}^{\infty} (-t)^k ad(A)X\right\rangle}{t}\right)}_{=0} \\ &= -\langle \omega, [A, X]\rangle = \langle \omega, [X, A]\rangle\end{aligned}$$

and this gives the desired result. □

Proposition 7. *The set of linear functionals* ω *for which* β_ω *is a local immersion at the identity in* H *is a Zariski open subset of* $\mathfrak{n}^*$.

Proof. Let $\mathcal{Q} = \{\omega \in \mathfrak{n}^* : \beta_\omega \text{ is an immersion at the identity in } H\}$. To prove the stated result, it suffices to establish that $\mathcal{Q}$ is the complement of a zero set of a polynomial defined on $\mathfrak{n}^*$. Put

$$D(\omega)=\left[\begin{array}{ccc}\langle\omega,[X_1,A_1]\rangle & \cdots & \langle\omega,[X_1,A_r]\rangle\\ \vdots & \ddots & \vdots\\ \langle\omega,[X_n,A_1]\rangle & \cdots & \langle\omega,[X_n,A_r]\rangle\end{array}\right] \text{ and } p(\omega)=D(\omega)D(\omega)^T.$$

Clearly p is a polynomial defined on $\mathfrak{n}^*$. Appealing to Proposition 5 and Proposition 6,

$$\mathcal{Q}=\left\{\omega\in\mathfrak{n}^*:\det\left(D(\omega)D(\omega)^T\right)\neq 0\right\}. \tag{2.5}$$

□

Since n is greater than or equal to r and D has an invertible submatrix of order r then there is at least a subset J of $\{1,2,\cdots,n\}$ such that $\sharp(J)=r=\dim H$ and the submatrix of D obtained by retaining all rows of D corresponding to the elements of J is an invertible square matrix of order r. By the Inverse Function Theorem (see Theorem 5.11, [26]), there is a linear projection

$$P=P_J:\mathfrak{n}\to\mathfrak{n} \tag{2.6}$$

of rank r (depending on the set J) given by

$$P(X_k)=\begin{cases}X_k \text{ if } k\in J\\ 0 \text{ if } k\notin J\end{cases} \tag{2.7}$$

such that

$$\Theta_{J,\lambda}=\Theta_\lambda=P^*\beta_\lambda\varphi^{-1}:\varphi(\mathcal{O})\to\Theta_\lambda(\varphi(\mathcal{O}))$$

is a **local analytic diffeomorphism** at the zero element in $\mathfrak{h}$.

From here on, we fix a set J as described above. Generally, the set $\mathcal{O}$ needs not be relatively compact. However, it is always possible to select a sufficiently small connected open and relatively compact subset $\mathcal{O}_\circ\subset\mathcal{O}$ around the identity in H such that

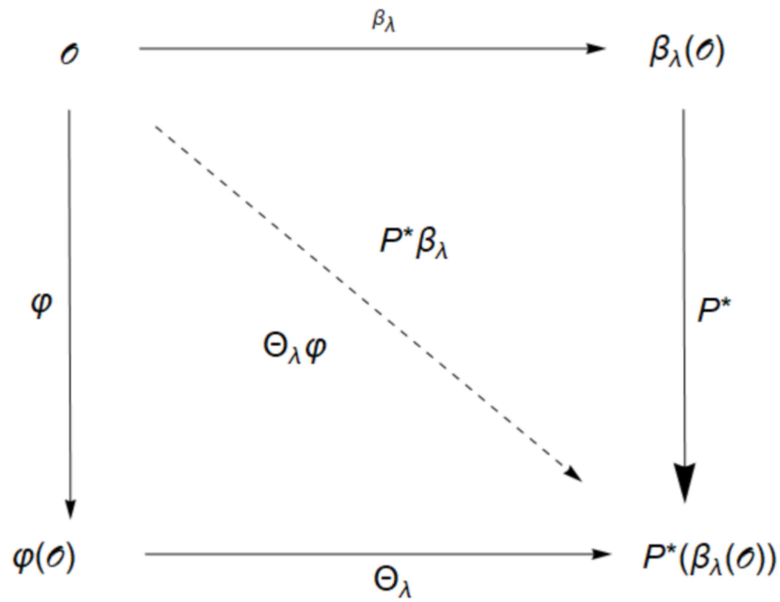
(a) The restriction

$$\Theta_\lambda|_{\varphi(\mathcal{O}_\circ)}:\varphi(\mathcal{O}_\circ)\to\Theta_\lambda(\varphi(\mathcal{O}_\circ))$$

of Θ_λ to $\varphi(\mathcal{O}_\circ)$ is a diffeomorphism.

(b) $\Theta_\lambda(\varphi(\mathcal{O}_\circ))$ is relatively compact in $P^*(\mathfrak{n}^*)$.

To avoid cluster of notation, for the remainder of this work, we simply replace $\mathcal{O}_\circ$ with $\mathcal{O}$. The following diagram summarizes the connection between the maps $\beta_\lambda, P^*, \varphi$ and $\Theta_\lambda|_{\mathcal{O}}$ defined above



(2.8)

A diagram summarizing the connection between the maps $\beta_\lambda, P^*, \varphi$ and $\Theta_\lambda|_{\mathcal{O}}$.

Proposition 8. *The smooth function $h \mapsto Ad\left(h^{-1}\right)^* \lambda$ is an immersion at the identity of H if and only if $H_\lambda = \left\{h \in H : Ad\left(h^{-1}\right)^* \lambda = \lambda\right\}$ is a discrete subgroup of H.*

Proof. The isotropy group H_λ is a closed subgroup of H and the map $F : H/H_\lambda \to \beta_\lambda(H)$ defined by $F(hH_\lambda) = Ad\left(h^{-1}\right)^* \lambda$ is an equivariant diffeomorphism (Theorem 7.19, [26].) As such, $\dim(H/H_\lambda) = \dim \beta_\lambda(H)$. Let us suppose that $h \mapsto Ad\left(h^{-1}\right)^* \lambda$ is an immersion at the identity of H with rank r. Then $r = \dim(H/H_\lambda) = \dim \beta_\lambda(H) = \dim H$. This implies that H_λ is a discrete subgroup of H. Suppose on the other hand that H_λ is a discrete subgroup of H. Then there exists an open set O_e around the identity of H such that $H_\lambda \cap O_e$ is trivial. Restricting $h \mapsto Ad\left(h^{-1}\right)^* \lambda$ to such an open set, it is then clear that $h \mapsto Ad\left(h^{-1}\right)^* \lambda$ must be an immersion at the identity of H. □

The purpose of the results presented below is to set the stage for the proofs of our main theorems. Our primary objective is to prove that given a continuous function $\mathbf{f}$ supported on $\mathcal{O}$, it is possible to find a countable set $\Gamma \subset G$ such that $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is a frame for $L^2(H, d\mu_H)$. To this end, we will need to show that there exist positive constants A, B such that

$$A\left\Vert \mathbf{g}\right\Vert ^{2}\leq\sum_{\gamma\in\Gamma}\left\vert \left\langle \mathbf{g},\pi\left(\gamma\right)\mathbf{f}\right\rangle \right\vert ^{2}\leq B\left\Vert \mathbf{g}\right\Vert ^{2}$$

for all $\mathbf{g}\in L^{2}\left(H,d\mu_{H}\right).$ We will be mainly interested in countable sets of the type $\Gamma=\Gamma_{H}\Gamma_{N}$ where Γ_{H} and Γ_{N} are some discrete subsets of H and $\exp\left(P\mathfrak{n}\right)\subseteq N$ respectively. Since we are committed to providing a unified discretization scheme for the class of representations described above, the following observation is in order.

Remark 9. *Generally, it is important to choose our sampling set to be of the form $\Gamma_{H}\Gamma_{N}$ as opposed to $\Gamma_{N}\Gamma_{H}$ (which may at first appear to be a more natural ordering). To see this, let us consider the case of the ax+b Lie group which we realize as follows. Let*

$$G=NH=\left\{ \left[\begin{array}{cc} e^{t} & x\\ 0 & 1 \end{array}\right]:t,x\in\mathbb{R}\right\}$$

be the ax+b group with Lie algebra $\mathfrak{g}$ spanned by

$$X=\left[\begin{array}{cc} 0 & 1\\ 0 & 0 \end{array}\right],A=\left[\begin{array}{cc} 1 & 0\\ 0 & 0 \end{array}\right].$$

The only non-trivial Lie bracket of $\mathfrak{g}$ is given by $[A,X]=X$. Next, let $\lambda=X_{1}^{\ast}$. We consider the induced representation $\pi_{\lambda}=\mathrm{ind}_{N}^{NH}\left(\chi_{\lambda}\right)$ which we realize as acting in the Hilbert space $L^{2}\left(\mathbb{R}\right)=L^{2}\left(H\right)$ as follows

$$\pi_{\lambda}\left(\exp xX\right)\mathbf{f}\left(t\right)=e^{2\pi ie^{-t}x}\cdot\mathbf{f}\left(t\right)\text{ and }\pi_{\lambda}\left(\exp aA\right)\mathbf{f}\left(t\right)=\mathbf{f}\left(t-a\right).$$

For some fixed positive real number ϵ, let $I=(-\epsilon,\epsilon)$ be an interval in $\mathbb{R}$. It is shown in [28] that there exist a positive and a continuous function $\mathbf{f}\in L^{2}\left(\mathbb{R}\right)$ supported on I and some positive real numbers α,δ such that

$$\left\{ \pi_{\lambda}\left(\exp\kappa A\right)\pi_{\lambda}\left(\exp jX\right)\mathbf{f}:j\in\alpha\mathbb{Z},\kappa\in\delta\mathbb{Z}\right\}$$

is a frame for $L^{2}\left(\mathbb{R}\right).$ Next, we consider the following system

$$\left\{ \pi_{\lambda}\left(\exp jX\right)\pi_{\lambda}\left(\exp\kappa A\right)\mathbf{f}:j\in\alpha\mathbb{Z},\kappa\in\delta\mathbb{Z}\right\}$$

obtained by changing the order of the original sampling set. For $\mathbf{h}\in L^{2}\left(\mathbb{R}\right),$ we have

$$\sum_{j\in\alpha\mathbb{Z}}\sum_{\kappa\in\delta\mathbb{Z}}\left\vert \left\langle \mathbf{h},\pi_{\lambda}\left(\exp jX\right)\pi_{\lambda}\left(\exp\kappa A\right)\mathbf{f}\right\rangle \right\vert ^{2}=\sum_{j\in\alpha\mathbb{Z}}\sum_{\kappa\in\delta\mathbb{Z}}\left\vert \int_{\kappa+I}\mathbf{h}\left(t\right)e^{-2\pi ie^{-t}j}\cdot\overline{\mathbf{f}\left(t-\kappa\right)}dt\right\vert ^{2}.$$

The change of variable $\xi = e^{-t}$ gives

$$\begin{aligned}&\sum_{j\in\alpha\mathbb{Z}}\sum_{\kappa\in\delta\mathbb{Z}}\left|\langle \mathbf{h},\pi_\lambda(\exp jX)\,\pi_\lambda(\exp\kappa A)\,\mathbf{f}\rangle\right|^2\\ &=\sum_{j\in\alpha\mathbb{Z}}\sum_{\kappa\in\delta\mathbb{Z}}\left|\int_{e^{\kappa+I}}\frac{\mathbf{h}\left(\log\left(\xi^{-1}\right)\right)\cdot\overline{\mathbf{f}\left(\log\left(\xi^{-1}\right)-\kappa\right)}}{\xi}e^{-2\pi i\xi j}d\xi\right|^2.\end{aligned}$$

Moreover, suppose that $\mathbf{h}$ *is a nonzero function which is compactly supported and additionally satisfies the following conditions: (1)*

$$\begin{aligned}&\sum_{j\in\alpha\mathbb{Z}}\sum_{\kappa\in\delta\mathbb{Z}}\left|\int_{e^{\kappa+I}}\frac{\mathbf{h}\left(\log\left(\xi^{-1}\right)\right)\cdot\overline{\mathbf{f}\left(\log\left(\xi^{-1}\right)-\kappa\right)}}{\xi}e^{-2\pi i\xi j}d\xi\right|^2\\ &=\sum_{j\in\alpha\mathbb{Z}}\left|\int_{e^{\kappa_0+I}}\frac{\mathbf{h}\left(\log\left(\xi^{-1}\right)\right)\cdot\overline{\mathbf{f}\left(\log\left(\xi^{-1}\right)-\kappa_0\right)}}{\xi}e^{-2\pi i\xi j}d\xi\right|^2\end{aligned}$$

for some fixed element $\kappa_0\in\delta\mathbb{Z}$*. (2) The Lebesgue measure of the set* e^{κ_0+I} *is strictly larger than* α^{-1}*. Note that the second condition is due to the expansive property of the exponential function* $x\mapsto e^x$ *on the positive real line. Consequently, the collection of trigonometric exponentials* $\left\{e^{-2\pi i\xi j}: j\in\alpha\mathbb{Z}\right\}$ *cannot be total in* $L^2\left(e^{\kappa_0+I}, d\xi\right)$*. It is then possible to choose a nonzero function* $\mathbf{h}\in L^2(\mathbb{R})$ *such that*

$$\int_{e^{\kappa_0+I}}\frac{\mathbf{h}\left(\log\left(\xi^{-1}\right)\right)\cdot\overline{\mathbf{f}\left(\log\left(\xi^{-1}\right)-\kappa_0\right)}}{\xi}e^{-2\pi i\xi j}d\xi=0$$

for every integer j*. Fixing such a function* $\mathbf{h}$*, we obtain*

$$\sum_{j\in\mathbb{Z}}\sum_{\kappa\in\mathbb{Z}}\left|\langle \mathbf{h},\pi_\lambda(\exp jX)\,\pi_\lambda(\exp\kappa A)\,\mathbf{f}\rangle\right|^2=0.$$

Since $\mathbf{h}$ *is not the zero function, we conclude that* $\{\pi_\lambda(\exp jX)\,\pi_\lambda(\exp\kappa A)\,\mathbf{f} : j\in\alpha\mathbb{Z},\kappa\in\delta\mathbb{Z}\}$ *is not a frame for* $L^2(\mathbb{R})$*.*

Given $\mathbf{g}$ and $\mathbf{f}$ in $L^2(H, d\mu_H)$, we formally define

$$\mathrm{s}\left(\Gamma_H,\Gamma_N,\mathbf{g},\mathbf{f}\right)=\sum_{\ell\in\Gamma_H}\sum_{\exp PX\in\Gamma_N}\left|\left\langle \mathbf{g},\pi\left(\ell^{-1}\right)\pi\left(\exp\left(PX\right)\right)\mathbf{f}\right\rangle_{\mathfrak{H}}\right|^2. \tag{2.9}$$

At this point, it is not clear for which quadruple $(\Gamma_H,\Gamma_N,\mathbf{g},\mathbf{f})$ should the series defined by $\mathrm{s}(\Gamma_H,\Gamma_N,\mathbf{g},\mathbf{f})$ be convergent (or divergent). Suppose that $\mathbf{f}$ is continuous and supported on $\mathcal{O}$ and for the time being, we shall overlook all convergence issues related to the definition of $\mathrm{s}(\Gamma_H,\Gamma_N,\mathbf{g},\mathbf{f})$. Since

$$[\pi(\exp(X))]\,\mathbf{f}(h)=e^{2\pi i\left\langle Ad\left(h^{-1}\right)^{*}\lambda,X\right\rangle}\mathbf{f}(h)\ \text{ and }\ [\pi(\ell)]\,\mathbf{f}(h)=\mathbf{f}\left(\ell^{-1}h\right),$$

it is clear that in a formal sense, we have

$$I_{\mathbf{g},\mathbf{f},\ell,X}=\left\langle \mathbf{g},\pi\left(\ell^{-1}\right)\pi(\exp(PX))\,\mathbf{f}\right\rangle_{\mathfrak{H}}=\int_{\mathcal{O}}\mathbf{g}\left(\ell^{-1}h\right)e^{-2\pi i\left\langle Ad\left(h^{-1}\right)^{*}\lambda,PX\right\rangle}\overline{\mathbf{f}(h)}\,d\mu_H(h).$$

Recall that $\beta_\lambda : H\to\beta_\lambda(H)\subseteq\mathfrak{n}^*$ is defined as follows: $\beta_\lambda(h)=Ad\left(h^{-1}\right)^{*}\lambda$. Next, for each $h\in\mathcal{O}$, there exists a unique element A in $\varphi(\mathcal{O})$ (which is a subset of the Lie algebra of H) such that $h=\varphi^{-1}(A)$ and

$$I_{\mathbf{g},\mathbf{f},\ell,X}=\int_{\varphi(\mathcal{O})}\left(\mathbf{g}\left(\ell^{-1}\varphi^{-1}(A)\right)e^{-2\pi i\left\langle\beta_\lambda\left(\varphi^{-1}(A)\right),PX\right\rangle}\overline{\mathbf{f}\left(\varphi^{-1}(A)\right)}\right)d\mu_H\left(\varphi^{-1}(A)\right).$$

Since P is a linear projection, we obtain

$$\left\langle\beta_\lambda\left(\varphi^{-1}(A)\right),PX\right\rangle=\left\langle P^*\beta_\lambda\left(\varphi^{-1}(A)\right),PX\right\rangle;$$

and consequently,

$$I_{\mathbf{g},\mathbf{f},\ell,X}=\int_{\varphi(\mathcal{O})}\left(\mathbf{g}\left(\ell^{-1}\varphi^{-1}(A)\right)\overline{\mathbf{f}\left(\varphi^{-1}(A)\right)}\right)e^{-2\pi i\left\langle P^*\beta_\lambda\left(\varphi^{-1}(A)\right),PX\right\rangle}d\mu_H\left(\varphi^{-1}(A)\right).$$

In local coordinates, the action of $P^*\beta_\lambda$ is given by the diffeomorphism

$$\Theta_\lambda=P^*\beta_\lambda\varphi^{-1}:\varphi(\mathcal{O})\to\Theta_\lambda(\varphi(\mathcal{O}))$$

and for every $A\in\varphi(\mathcal{O})$, there exists a unique element ξ in $\Theta_\lambda(\mathcal{O})\subset\mathfrak{p}^*$ such that, given $A=\Theta_\lambda^{-1}(\xi)$, and

$$F_{\mathbf{g},\mathbf{f}}(\xi)=\mathbf{g}\left(\ell^{-1}\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right)\overline{\mathbf{f}\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right)}$$

we have

$$I_{\mathbf{g},\mathbf{f},\ell,X}=\int_{\Theta_\lambda(\varphi(\mathcal{O}))}F_{\mathbf{g},\mathbf{f}}(\xi)\,e^{-2\pi i\langle\xi,PX\rangle}d\mu_H\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right).$$

Note that $d\mu_H\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right)$ is a measure on $\Theta_\lambda(\varphi(\mathcal{O}))$. In fact, $\Theta_\lambda(\varphi(\mathcal{O}))$ is merely the pushforward of the Haar measure on H under the map $\Theta_\lambda\circ\varphi$.

Let $d\xi$ be the Lebesgue measure on $P^*(\mathfrak{n}^*)$. To simplify the integral $I_{\mathbf{g},\mathbf{f},\ell,X}$, we will need to compute the Radon-Nikodym derivative

$$\frac{d\mu_H\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right)}{d\xi},$$

denoted by $\mathbf{W}_\lambda : \Theta_\lambda(\varphi(\mathcal{O})) \to (0,\infty)$ where

$$\mathbf{W}_\lambda(\xi) = \frac{d\mu_H\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right)}{d\xi}. \tag{2.10}$$

In light of the discussion above, the following is immediate.

Lemma 10. *Given discrete sets* $\Gamma_H \subset H$, $\Gamma_N \subset \exp(P\mathfrak{n})$, $\mathbf{g}$ *and* $\mathbf{f}$ *in* $L^2(H, d\mu_H)$ *such that* $\mathbf{f}$ *is a continuous function compactly supported in* $\mathcal{O}$,

$$\begin{aligned} &\mathrm{s}(\Gamma_H, \Gamma_N, \mathbf{g}, \mathbf{f}) \\ &= \sum_{\ell \in \Gamma_H} \sum_{\exp(PX) \in \Gamma_N} \left| \int_{\Theta_\lambda(\varphi(\mathcal{O}))} \left[\mathbf{g}\left(\ell^{-1}\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right) \overline{\mathbf{f}\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right)} \mathbf{W}_\lambda(\xi) \right] e^{-2\pi i \langle \xi, PX \rangle} d\xi \right|^2. \end{aligned}$$

Lemma 11. $\mathbf{W}_\lambda$ *is a strictly positive function defined on* $\Theta_\lambda(\varphi(\mathcal{O}))$. *More precisely,*

$$\mathbf{W}_\lambda(\xi) = \left| \prod_{k=1}^{r} v_k(\xi) \right| \cdot \left| \det \left[\frac{\partial \left[\Theta_\lambda^{-1}(\xi)\right]_j}{\partial \xi_k} \right]_{1 \le j,k \le n} \right|$$

where for $k \in \{1, 2, \cdots, r\}$,

$$v_k(\xi) = \begin{cases} \dfrac{e^{w_k(\xi)} - 1}{w_k(\xi)} \text{ if } w_k(\xi) \neq 0 \\ 1 \text{ if } w_k(\xi) = 0 \end{cases}$$

and $(w_k(\xi))_{k=1}^r$ *is a sequence of eigenvalues for the endomorphism* $-ad_{\mathfrak{h}}\left(\Theta_\lambda^{-1}(\xi)\right)$.

Proof. Referring back to the diagram given in Figure 2.8, we proceed as follows. First, let $\mathbf{f}$ be a positive function compactly supported on $\mathcal{O}$. Given $h \in \mathcal{O}$, since there exists a unique $A \in \varphi(\mathcal{O}) \subset \mathfrak{h}$ such that $h = \varphi^{-1}(A)$, we have

$$\int_{\mathcal{O}} \mathbf{f}(h)\, d\mu_H(h) = \int_{\varphi(\mathcal{O})} \mathbf{f}\left(\varphi^{-1}(A)\right) \cdot d\mu_H\left(\varphi^{-1}(A)\right).$$

Next,

$$\int_{\mathcal{O}} \mathbf{f}(h)\, d\mu_H(h) = \int_{\Theta_\lambda(\varphi(\mathcal{O}))} \mathbf{f}\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right) d\mu_H\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right).$$

Taking φ to be the logarithmic map (we assume here that $\mathcal{O}$ is sufficiently small), we obtain

$$\int_{\mathcal{O}} \mathbf{f}(h)\, d\mu_H(h) = \int_{\Theta_\lambda(\varphi(\mathcal{O}))} \mathbf{f}\left(\exp\left(\Theta_\lambda^{-1}(\xi)\right)\right) d\mu_H\left(\exp\left(\Theta_\lambda^{-1}(\xi)\right)\right).$$

Chain Rule together with [9, Proposition 5.5.6] gives the following. Defining

$$A(\xi) = \frac{\mathrm{id} - \exp\left(-ad_{\mathfrak{h}}\left(\Theta_{\lambda}^{-1}(\xi)\right)\right)}{ad_{\mathfrak{h}}\left(\Theta_{\lambda}^{-1}(\xi)\right)};$$

up to multiplication by a positive constant, we have

$$\mathbf{W}_{\lambda}(\xi) = \left| \det(A(\xi)) \cdot \det \left[\frac{\partial \left[\Theta_{\lambda}^{-1}(\xi)\right]_j}{\partial \xi_k} \right]_{1 \leq j,k \leq n} \right|. \tag{2.11}$$

Note that the function $\xi \mapsto \det A(\xi)$ is nonzero on $\Theta_{\lambda}(\varphi(\mathcal{O}))$ since each linear operator

$$A(\xi) = \sum_{k=0}^{\infty} \frac{\left(-ad_{\mathfrak{h}}\left(\Theta_{\lambda}^{-1}(\xi)\right)\right)^k}{(k+1)!} \tag{2.12}$$

is invertible [8, Page 25]. Precisely, there is an ordering $(v_1(\xi), \cdots, v_r(\xi))$ for the eigenvalues of the linear operator $\frac{\mathrm{id} - \exp\left(-ad_{\mathfrak{h}}\left(\Theta_{\lambda}^{-1}(\xi)\right)\right)}{ad_{\mathfrak{h}}\left(\Theta_{\lambda}^{-1}(\xi)\right)}$ as well as an ordering $(w_1(\xi), \cdots, w_r(\xi))$ for the eigenvalues of $-ad_{\mathfrak{h}}\left(\Theta_{\lambda}^{-1}(\xi)\right)$ such that

$$|\det(A(\xi))| = \left| \prod_{k=1}^{r} v_k(\xi) \right| \tag{2.13}$$

and for each index k,

$$v_k(\xi) = \begin{cases} \dfrac{e^{w_k(\xi)} - 1}{w_k(\xi)} \text{ if } w_k(\xi) \neq 0 \\ 1 \text{ if } w_k(\xi) = 0 \end{cases}.$$

As a result,

$$\mathbf{W}_{\lambda}(\xi) = \left| \prod_{k=1}^{r} v_k(\xi) \right| \cdot \left| \det \left(\left[\frac{\partial \left[\Theta_{\lambda}^{-1}(\xi)\right]_j}{\partial \xi_k} \right]_{1 \leq j,k \leq n} \right) \right| \tag{2.14}$$

and for $\xi \in \Theta_{\lambda}(\varphi(\mathcal{O}))$,

$$\mathbf{W}_{\lambda}(\xi) > 0.$$

□

Lemma 12. *For any continuous function* $\mathbf{s}$ *compactly supported on* $\mathcal{O}$*, the function*

$$h \mapsto \mathbf{s}(h) \sqrt{\mathbf{W}_{\lambda}(\Theta_{\lambda}(\varphi(h)))}$$

is also continuous and compactly supported on $\mathcal{O}$*.*

Proof. According to Lemma 11, $\mathbf{W}_{\lambda}$ is a strictly positive function defined on $\Theta_{\lambda}(\varphi(\mathcal{O}))$. Thus, the function $h \mapsto \sqrt{\mathbf{W}_{\lambda}(\Theta_{\lambda}(\varphi(h)))}$ is a positive continuous function on $\mathcal{O}$. Furthermore, for any continuous function $\mathbf{s}$ supported on $\mathcal{O}$, it is clear that

$$h \mapsto \mathbf{s}(h) \cdot \sqrt{\mathbf{W}_{\lambda}(\Theta_{\lambda}(\varphi(h)))}$$

is also continuous and supported on $\mathcal{O}$. □

Remark 13. *Although Lemma 11 is useful for the proofs of the main results, it is not practical for specific examples (see the examples given in Section 3.2). To compute* $\mathbf{W}_{\lambda}$ *it is clearly important to have at our disposal, an explicit formula for the Haar measure of* H*. In many cases, this can be achieved inductively as follows. Suppose that* $H = ST$ *where* S, T *are closed subgroups,* $S \cap T$ *is compact and* $S \times T \ni (s,t) \mapsto st \in H$ *is a homeomorphism. Then*

$$\int_{H} \mathbf{f}(h)\, d\mu_{H}(h) = \int_{S}\int_{T} \mathbf{f}(st) \frac{\det(Ad_{T}(t))}{\det(Ad_{S}(s))} d\mu_{T}(t)\, d\mu_{S}(s)$$

and $d\mu_{S}(s)\, d\mu_{T}(t)$ *are left Haar measure for* S *and* T *respectively [22, Proposition 5.26].*

In our quest to describe quadruples $(\Gamma_H, \Gamma_N, \mathbf{g}, \mathbf{f})$ for which the series $\mathrm{s}(\Gamma_H, \Gamma_N, \mathbf{g}, \mathbf{f})$ is convergent, we will need to also address the following question: Let $\mathbf{f}$ be a nonzero positive function which is compactly supported on H. Can we find a discrete set $\Lambda \subset H$ such that for all $x \in H$, there exist positive real numbers $A \leq B$ not depending on x such that $A \leq \sum_{\gamma \in \Lambda} \mathbf{f}(\gamma x) \leq B$? A positive answer to this question is given by Lemma 14; and its proof [3] requires the following concepts.

- A discrete set $\Gamma \subset H$ is U-separated if $\gamma U \cap \gamma' U$ is empty for $\gamma \neq \gamma'$ in Γ.
- A discrete set $\Gamma \subset H$ is called V-dense if $H = \Gamma V$.

Lemma 14. *Let* $\mathbf{f}$ *be a nonzero positive function which is compactly supported on* H*. Then there exists a discrete set* $\Gamma_H \subset H$ *such that the compound inequalities*

$$A \leq \sum_{\gamma \in \Gamma_H} \mathbf{f}(\gamma x) \leq B$$

hold for all $x \in H$ *with* $0 < A \leq B < \infty$ *independent of* x*.*

[3]Many thanks go to H. Führ for sharing with us a complete proof of Lemma 14.

Proof. Fixing a set $\Gamma \subset H$ which is U-separated, for some open set U around the neutral element of H, we select an open symmetric set V such that $V^2 \subset U$. Next, let

$$\mathrm{supp}\,(f) = \bigcup_{i=1}^{k} x_i V \text{ where } x_i \in \Gamma.$$

Then each set $x_i V$ contains no more than one element from Γ. Suppose otherwise. Then given $\kappa, \gamma \in \Gamma$ in $x_i V$ one has that $\kappa^{-1}\gamma \in V^2 \subset U$ and this contradicts the fact that Γ is U-separated. Thus,

$$\sum_{\gamma\in\Gamma} \mathbf{f}\,(\gamma x) \leq k \cdot (\sup\{f\,(x) : x \in H\}) < \infty.$$

Next, let W be an open set contained in H such that $\mathbf{f}\,(x) > \epsilon$ for all $x \in W$ and for some positive real number $\epsilon > 0$. Without loss of generality, we may assume that W is an open set around the neutral element in H. Let us select a symmetric open set V around the identity in H such that $V^2 \subset W$. Next, observe that every V-dense subset Γ must intersect W. To see this, let $y \in V$ and $\gamma \in \Gamma$ such that $y \in \gamma V$. Then clearly $\gamma \in yV^{-1} \subset V^2 \subset W$ and

$$\sum_{\gamma\in\Gamma} \mathbf{f}\,(\gamma x) \geq \epsilon > 0.$$

To complete the proof, it is enough to select a V-dense, U-separated set with sufficiently small V. To this end, pick Z contained in V such that Z is symmetric, Z^2 is contained in V and pick a Z-separated subset which is maximal with respect to inclusion. Then such a set is V-dense and Z-separated. □

Remark 15. *In the case where H is completely solvable, an explicit construction of the set Γ_H in Lemma 14 can be found in [28, Lemma 29].*

Let

$$\mathbf{w}\,(h) = \mathbf{W}_\lambda\,(\Theta_\lambda\,(\varphi\,(h)))\,. \tag{2.15}$$

Remark 16. *Given a continuous (or a smooth) function $\mathbf{f}$ which is supported inside $\mathcal{O}$, since the product $\mathbf{f}\sqrt{\mathbf{w}}$ is continuous and supported inside $\mathcal{O}$, according to Lemma 14, there exists a discrete subset Γ_H of H such that given*

(2.16)

$$m_{H,\mathbf{f}} = \mathrm{essinf}_{h\in H}\left(\sum_{\ell\in\Gamma_H}\left|\mathbf{f}\,(\ell h)\cdot\sqrt{\mathbf{w}\,(\ell h)}\right|^2\right) \text{ and } M_{H,\mathbf{f}} = \mathrm{esssup}_{h\in H}\left(\sum_{\ell\in\Gamma_H}\left|\mathbf{f}\,(\ell h)\cdot\sqrt{\mathbf{w}\,(\ell h)}\right|^2\right),$$

we have

$$0 < m_{H,\mathbf{f}} \leq M_{H,\mathbf{f}} < \infty.$$

We recall that by assumption $\Theta_\lambda(\varphi(\mathcal{O}))$ is relatively compact in $P^*(\mathfrak{n}^*)$ and $\Theta_\lambda(\varphi(\mathcal{O}))$ has positive Lebesgue measure in

$$\mathfrak{p}^* = P^*(\mathfrak{n}^*) \simeq \mathbb{R}^r. \tag{2.17}$$

Given $C \subset \mathfrak{p}^*$, and Γ_N contained in $\exp(\mathfrak{p})$, we define

$$\mathcal{E}(C, \Gamma_N) = \left\{ \frac{e^{2\pi i \langle \xi, Y \rangle}}{|C|^{1/2}} : Y \in \log(\Gamma_N) \right\}. \tag{2.18}$$

Put

$$\mathfrak{p} = P_J(\mathfrak{n}) = P(\mathfrak{n}). \tag{2.19}$$

Lemma 17. *There exists a set C which is a compact subset of $\mathfrak{p}^*$ such that $\Theta_\lambda(\varphi(\mathcal{O})) \subseteq C$ and Γ_N is a discrete subset of $\exp(\mathfrak{p})$ such that the trigonometric system $\mathcal{E}(C, \Gamma_N)$ is an orthonormal basis for $L^2(C, d\xi)$.*

Proof. Define a positive real number ς such that

$$\varsigma = \operatorname{esssup}_{h \in H} \left\{ \|\Theta_\lambda(\varphi(h))\|_{\max} : h \in \mathcal{O} \right\}.$$

Next, let

$$J = \{j_1 < \cdots < j_r\} = \{1, 2, \cdots, n\}.$$

Put $C = \sum_{k=1}^r [-\varsigma, \varsigma) X_{j_k}^* \subset \mathfrak{p}^*$ and define

$$\Gamma_N = \exp\left(\sum_{k=1}^r \frac{1}{2\varsigma} \mathbb{Z} X_{j_k} \right) \subset N.$$

Then

$$\left\{ C + k : k \in \sum_{k=1}^r 2\varsigma \mathbb{Z} X_{j_k}^* \right\}$$

tiles $\mathfrak{p}^*$. Moreover, the Lebesgue measure of C is equal $(2\varsigma)^r$ and the system $\mathcal{E}(C, \Gamma_N)$ is an orthonormal basis for $L^2(C, d\xi)$. □

To simplify our presentation, we list below some important conditions which we shall refer to by their corresponding numbers throughout this work.

A collection of technical conditions

(**C1**) Γ_H is a discrete subset of H such that

$$0 < m_{H,\mathbf{f}} = \operatorname{essinf}_{h\in H}\left(\sum_{\ell\in\Gamma_H}\left|\mathbf{f}\left(\ell h\right)\sqrt{\mathbf{w}\left(\ell h\right)}\right|^2\right)$$
$$\leq M_{H,\mathbf{f}} = \operatorname{esssup}_{h\in H}\left(\sum_{\ell\in\Gamma_H}\left|\mathbf{f}\left(\ell h\right)\sqrt{\mathbf{w}\left(\ell h\right)}\right|^2\right) < \infty$$

as defined in (2.16).

(**C2**) C is a compact subset of $\mathfrak{p}^*$ such that

$$\Theta_\lambda\left(\varphi\left(\mathcal{O}\right)\right) \subseteq C$$

and Γ_N is a discrete subset of $\exp\left(\mathfrak{p}\right)$ such that the trigonometric system

$$\mathcal{E}\left(C,\Gamma_N\right) = \left\{\frac{e^{2\pi i\langle\xi,Y\rangle}}{\left|C\right|^{1/2}} : Y \in \log\left(\Gamma_N\right)\right\}$$

is an orthonormal basis for $L^2\left(C, d\xi\right)$ as defined (2.18).

(**C3**) $m_{H,\mathbf{f}} = M_{H,\mathbf{f}}$ (see (2.16))

(**C4**) $m_{H,\mathbf{f}} = M_{H,\mathbf{f}} = \left|C\right|^{-1}$ (see (2.16) and (2.18))

(**C5**) $\int_H \left|\mathbf{f}\left(h\right)\right|^2 d\mu_H\left(h\right) = 1$

(**C6**) There is a discrete subset Γ_H of H such that $\left\{\ell^{-1}\mathcal{O} : \ell \in \Gamma_H\right\}$ is a tiling of H.

(**C7**) $\mathbf{f}$ is a vector in $L^2\left(\mathcal{O}, d\mu_H\right)$ such that

$$\mathbf{f}\left(\varphi^{-1}\left(\Theta_\lambda^{-1}\left(\xi\right)\right)\right) = \mathbf{W}_\lambda\left(\xi\right)^{-1/2} \cdot 1_{\Theta_\lambda\left(\varphi\left(\mathcal{O}\right)\right)}\left(\xi\right)$$

and

$$\mathbf{W}_\lambda\left(\xi\right) = \frac{d\mu_H\left(\varphi^{-1}\left(\Theta_\lambda^{-1}\left(\xi\right)\right)\right)}{d\xi}.$$

(**C8**) $\mathbf{f}$ is a vector in $L^2\left(\mathcal{O}, d\mu_H\right)$ such that

$$\frac{\left\|\mathbf{f}\right\|_{\mathfrak{H}}}{\sqrt{\left|C\right|}} = 1.$$

For $\Gamma \subset G$, we define

$$\mathcal{S}\left(\mathbf{f},\Gamma\right)=\left\{\pi\left(x\right)\mathbf{f}:x\in\Gamma\right\}. \tag{2.20}$$

Lemma 18. *Let* $\mathbf{f}$ *be a continuous (or a smooth) function which is compactly supported on* $\mathcal{O}$. *Next, let* Γ_H *and* Γ_N *be discrete sets satisfying the conditions described in (**C1**), and (**C2**). Then* $\mathcal{S}\left(\mathbf{f},\Gamma_H^{-1}\Gamma_N\right)$ *is a frame for* $\mathfrak{H}$ *with frame bounds*

$$0<m_{H,\mathbf{f}}\cdot|C|\leq M_{H,\mathbf{f}}\cdot|C|<\infty.$$

Additionally, the following holds true.

(1) *If (**C3**) holds then* $\mathcal{S}\left(\mathbf{f},\Gamma_H^{-1}\Gamma_N\right)$ *is a tight frame for* $\mathfrak{H}$.
(2) *If (**C4**) and (**C5**) hold then* $\mathcal{S}\left(\mathbf{f},\Gamma_H^{-1}\Gamma_N\right)$ *is an orthonormal basis for* $\mathfrak{H}$.

Proof. Let $\mathbf{g}$ be a continuous function which is compactly supported in H. Put

$$\mathrm{s}\left(\Gamma_H,\Gamma_N,\mathbf{g},\mathbf{f}\right)=\sum_{\ell\in\Gamma_H}\sum_{\exp(PX)\in\Gamma_N}\left|\left\langle\mathbf{g},\pi\left(\ell^{-1}\right)\pi\left(\exp\left(PX\right)\right)\mathbf{f}\right\rangle_{\mathfrak{H}}\right|^2.$$

According to Lemma 10,

$$\begin{aligned}&\mathrm{s}\left(\Gamma_H,\Gamma_N,\mathbf{g},\mathbf{f}\right)\\&=\sum_{\ell\in\Gamma_H}\sum_{\exp(PX)\in\Gamma_N}\left|\int_{\Theta_\lambda(\varphi(\mathcal{O}))}\left[\mathbf{g}\left(\ell^{-1}\varphi^{-1}\left(\Theta_\lambda^{-1}\left(\xi\right)\right)\right)\overline{\mathbf{f}\left(\varphi^{-1}\left(\Theta_\lambda^{-1}\left(\xi\right)\right)\right)}\mathbf{W}_\lambda\left(\xi\right)\right]e^{-2\pi i\langle\xi,PX\rangle}d\xi\right|^2.\end{aligned}$$

To simplify notation, we let

$$u\left(\xi\right)=\varphi^{-1}\left(\Theta_\lambda^{-1}\left(\xi\right)\right)\in H.$$

Then

$$\mathrm{s}\left(\Gamma_H,\Gamma_N,\mathbf{g},\mathbf{f}\right)=\sum_{\ell\in\Gamma_H}\sum_{\exp(PX)\in\Gamma_N}\left|\int_{\Theta_\lambda(\varphi(\mathcal{O}))}\mathbf{g}\left(\ell^{-1}u\left(\xi\right)\right)\overline{\mathbf{f}\left(u\left(\xi\right)\right)}\mathbf{W}_\lambda\left(\xi\right)e^{-2\pi i\langle\xi,PX\rangle}d\xi\right|^2.$$

Letting $\mathfrak{F}_C$ be the Fourier series defined on $L^2(C)$, it is clear that

$$\begin{aligned}&\int_{\Theta_\lambda(\varphi(\mathcal{O}))}\mathbf{g}\left(\ell^{-1}u\left(\xi\right)\right)\overline{\mathbf{f}\left(u\left(\xi\right)\right)}\mathbf{W}_\lambda\left(\xi\right)e^{-2\pi i\langle\xi,PX\rangle}d\xi\\&=\mathfrak{F}_C\left(\xi\mapsto\mathbf{g}\left(\ell^{-1}u\left(\xi\right)\right)\overline{\mathbf{f}\left(u\left(\xi\right)\right)}\mathbf{W}_\lambda\left(\xi\right)1_{\Theta_\lambda(\varphi(\mathcal{O}))}\left(\xi\right)\right)\left(PX\right).\end{aligned}$$

Appealing to Plancherel's theorem,

$$\mathrm{s}\left(\Gamma_H,\Gamma_N,\mathbf{g},\mathbf{f}\right)=|C|\cdot\sum_{\ell\in\Gamma_H}\int_{\Theta_\lambda(\varphi(\mathcal{O}))}\left|\mathbf{g}\left(\ell^{-1}u\left(\xi\right)\right)\overline{\mathbf{f}\left(u\left(\xi\right)\right)}\mathbf{W}_\lambda\left(\xi\right)^{1/2}\right|^2\mathbf{W}_\lambda\left(\xi\right)d\xi.$$

Setting

$$h=u\left(\xi\right)=\varphi^{-1}\left(\Theta_\lambda^{-1}\left(\xi\right)\right),$$

we obtain

$$\mathrm{s}\left(\Gamma_H,\Gamma_N,\mathbf{g},\mathbf{f}\right)=\sum_{\ell\in\Gamma_H}\int_{\ell^{-1}\mathcal{O}}\left|\mathbf{g}\left(h\right)\right|^2\left|\mathbf{f}\left(\ell h\right)\mathbf{w}\left(\ell h\right)^{1/2}\right|^2d\mu_H\left(h\right)\tag{2.21}$$

$$=\sum_{\ell\in\Gamma_H}\int_{H}\left|\mathbf{g}\left(h\right)\right|^2\left|\mathbf{f}\left(\ell h\right)\mathbf{w}\left(\ell h\right)^{1/2}\right|^2d\mu_H\left(h\right).\tag{2.22}$$

Using Tonelli's Theorem to interchange the sum and integral in (2.22) yields

$$\mathrm{s}\left(\Gamma_H,\Gamma_N,\mathbf{g},\mathbf{f}\right)=|C|\cdot\int_H\left|\mathbf{g}\left(h\right)\right|^2\sum_{\ell\in\Gamma_H}\left|\mathbf{f}\left(\ell h\right)\mathbf{w}\left(\ell h\right)^{1/2}\right|^2d\mu_H\left(h\right).$$

In summary,

$$\mathrm{s}\left(\Gamma_H,\Gamma_N,\mathbf{g},\mathbf{f}\right)\leq|C|\cdot M_{H,\mathbf{f}}\int_H\left|\mathbf{g}\left(h\right)\right|^2d\mu_H\left(h\right)=|C|\,M_{H,\mathbf{f}}\left\|\mathbf{g}\right\|_{\mathfrak{H}}^2$$

and

$$\mathrm{s}\left(\Gamma_H,\Gamma_N,\mathbf{g},\mathbf{f}\right)\geq|C|\cdot m_{H,\mathbf{f}}\int_H\left|\mathbf{g}\left(h\right)\right|^2d\mu_H\left(h\right)=|C|\,m_{H,\mathbf{f}}\left\|\mathbf{g}\right\|_{\mathfrak{H}}^2.$$

Since the set of all continuous and compactly supported functions is dense in $\mathfrak{H}$, it follows that the collection

$$\left\{\pi\left(\ell^{-1}\kappa\right)\mathbf{f}:(\ell,\kappa)\in\Gamma_H\times\Gamma_N\right\}$$

is a frame for the Hilbert space $\mathfrak{H}$ with frame bounds $|C|\cdot m_{H,\mathbf{f}}\leq|C|\cdot M_{H,\mathbf{f}}$. This proves the first part of the result.

For Part (1), assuming additionally that $m_{H,\mathbf{f}}=M_{H,\mathbf{f}}$, the lower and upper frame bound coincide and consequently, $\mathcal{S}\left(\mathbf{f},\Gamma_H^{-1}\Gamma_N\right)$ is a tight frame for $\mathfrak{H}$.

Regarding Part (2), under the assumption that

$$|C|\cdot m_{H,\mathbf{f}}=|C|\cdot M_{H,\mathbf{f}}=1$$

and

$$\int_H\left|\mathbf{f}\left(h\right)\right|^2d\mu_H\left(h\right)=1,$$

we obtain $\mathcal{S}\left(\mathbf{f},\Gamma_H^{-1}\Gamma_N\right)$ is a unit norm Parseval frame and therefore, it must be necessarily be an orthonormal basis. □

Proposition 19. *Let* $\mathbf{f}$ *be a continuous (or a smooth) function which is compactly supported on* $\mathcal{O}$*. Next, let* Γ_N *be a discrete set defined as described in (**C2**). Given a discrete set* Λ *of* H*, if the system* $\mathcal{S}\left(\mathbf{f},\Lambda^{-1}\Gamma_N\right)$ *is a frame for* $\mathfrak{H}$ *with frame bounds* A, B *then for almost every* $h \in H$*, we have*

$$A\left|C\right|^{-1} \leq \sum_{\ell\in\Lambda}\left|\mathbf{f}\left(\ell h\right)\sqrt{\mathbf{w}\left(\ell h\right)}\right|^2 \leq B\left|C\right|^{-1}.$$

In particular, $\mathbf{f}$ *must be bounded.*

Proof. The proof goes as follows. Put

$$\mathrm{s}\left(\Lambda,\Gamma_N,\mathbf{g},\mathbf{f}\right) = \sum_{\ell\in\Lambda}\sum_{\exp(PX)\in\Gamma_N}\left|\left\langle \mathbf{g},\pi\left(\ell^{-1}\right)\pi\left(\exp\left(PX\right)\right)\mathbf{f}\right\rangle_{\mathfrak{H}}\right|^2.$$

By assumption, given $\mathbf{g} \in \mathfrak{H}$,

$$\begin{aligned}\mathrm{s}\left(\Lambda,\Gamma_N,\mathbf{g},\mathbf{f}\right) &= \int_H \left|\mathbf{g}\left(h\right)\right|^2\left(\left|C\right|\sum_{\ell\in\Lambda}\left|\mathbf{f}\left(\ell h\right)\mathbf{w}\left(\ell h\right)^{1/2}\right|^2\right)d\mu_H\left(h\right)\\ &\geq \int_H A\left|\mathbf{g}\left(h\right)\right|^2 d\mu_H\left(h\right).\end{aligned}$$

Consequently,

$$\int_H \left|\mathbf{g}\left(h\right)\right|^2\left(\left|C\right|\sum_{\ell\in\Lambda}\left|\mathbf{f}\left(\ell h\right)\mathbf{w}\left(\ell h\right)^{1/2}\right|^2 - A\right)d\mu_H\left(h\right) \geq 0.$$

Suppose that

$$\left|C\right|\sum_{\ell\in\Gamma_H}\left|\mathbf{f}\left(\ell h\right)\mathbf{w}\left(\ell h\right)^{1/2}\right|^2 < A$$

on some subset $E \subseteq H$ of positive and finite Haar measure. Letting $\mathbf{g}$ be the indicator function of the set E,

$$\begin{aligned}\mathrm{s}\left(\Lambda,\Gamma_N,\mathbf{g},\mathbf{f}\right) &= \int_E \left|C\right|\cdot\sum_{\ell\in\Lambda}\left|\mathbf{f}\left(\ell h\right)\mathbf{w}\left(\ell h\right)^{1/2}\right|^2 d\mu_H\left(h\right)\\ &< A\cdot\int_E d\mu_H\left(h\right)\\ &= A\left\|\mathbf{g}\right\|^2.\end{aligned}$$

Thus, $A\left\|\mathbf{g}\right\|^2 > \mathrm{s}\left(\Lambda,\Gamma_N,\mathbf{g},\mathbf{f}\right)$. This contradicts our assumption. Similarly, suppose that

$$|C|\cdot\sum_{\ell\in\Lambda}\left|\mathbf{f}\left(\ell h\right)\mathbf{w}\left(\ell h\right)^{1/2}\right|^2 > B$$

on some subset F of positive and finite Haar measure. Letting $\mathbf{g}$ be the indicator function of the set F,

$$\sum_{\ell\in\Lambda}\sum_{\exp(PX)\in\Gamma_N}\left|\left\langle\mathbf{g},\pi\left(\ell^{-1}\right)\pi\left(\exp\left(PX\right)\right)\mathbf{f}\right\rangle_{\mathfrak{H}}\right|^2 > B\left\|\mathbf{g}\right\|^2$$

and this contradicts that $\mathcal{S}\left(\mathbf{f},\Lambda^{-1}\Gamma_N\right)$ is a frame for $\mathfrak{H}$ with upper frame bound B. □

Proposition 20. [4] *Let* $\mathbf{f}$ *be a continuous (or a smooth) function which is compactly supported on* $\mathcal{O}$. *Next, let* Γ_H *and* Γ_N *be the discrete sets as described in* ***(C1)***, *and* ***(C2)***. *Put*

$$\mathbf{F}\left(x\right)=\sum_{\ell\in\Gamma_H}\left|\mathbf{f}\left(\ell x\right)\mathbf{w}\left(\ell x\right)^{1/2}\right|^2.$$

Then the frame operator S *corresponding to* $\mathcal{S}\left(\mathbf{f},\Gamma_H^{-1}\Gamma_N\right)$ *and its inverse are given by* $S\mathbf{h}=\left(|C|\cdot\mathbf{F}\right)\mathbf{h}$ *and* $S^{-1}\mathbf{h}=\left(|C|\cdot\mathbf{F}\right)^{-1}\mathbf{h}$ *respectively for all* $\mathbf{h}\in\mathfrak{H}$.

Proof. Let us assume that

$$\mathcal{S}\left(\mathbf{f},\Gamma_H^{-1}\Gamma_N\right)=\left\{\pi\left(\ell^{-1}\kappa\right)\mathbf{f}:\left(\ell,\kappa\right)\in\Gamma_H\times\Gamma_N\right\}$$

is a frame for $\mathfrak{H}$ with frame operator $S:\mathfrak{H}\to\mathfrak{H}$ given by

$$S\mathbf{h}=\sum_{(\ell,\kappa)\in\Gamma_H\times\Gamma_N}\left\langle\mathbf{h},\pi\left(\ell^{-1}\kappa\right)\mathbf{f}\right\rangle_{\mathfrak{H}}\pi\left(\ell^{-1}\kappa\right)\mathbf{f}\tag{2.23}$$

for $\mathbf{h}\in\mathfrak{H}$. Assuming that $\mathbf{h}$ has compact support, the inner product of $S\mathbf{h}$ with $\mathbf{h}$ is

$$\begin{aligned}\left\langle S\mathbf{h},\mathbf{h}\right\rangle_{\mathfrak{H}}&=\sum_{(\ell,\kappa)\in\Gamma_H\times\Gamma_N}\left|\left\langle\mathbf{h},\pi\left(\ell^{-1}\kappa\right)\right\rangle_{\mathfrak{H}}\right|^2\\&=|C|\cdot\int_H\left|\mathbf{h}\left(x\right)\right|^2\left(\sum_{\ell\in\Gamma_H}\left|\mathbf{f}\left(\ell x\right)\mathbf{w}\left(\ell x\right)^{1/2}\right|^2\right)d\mu_H\left(x\right).\end{aligned}$$

[4]We thank the reviewer for making suggestions that lead us to the statement and proof of Proposition 20

Since the set of all continuous functions which are compactly supported is dense in $\mathfrak{H}$, and since S is continuous, it follows that for all $\mathbf{h} \in \mathfrak{H}$, we have

$$\begin{aligned}\langle S\mathbf{h}, \mathbf{h}\rangle_{\mathfrak{H}} &= |C| \cdot \int_H |\mathbf{h}(x)|^2 \mathbf{F}(x)\, d\mu_H(x) \\ &= \int_H (|C| \cdot \mathbf{h}(x)\mathbf{F}(x))\overline{\mathbf{h}(x)} d\mu_H(x) \\ &= \langle |C| \cdot \mathbf{F}\mathbf{h}, \mathbf{h}\rangle_{\mathfrak{H}}.\end{aligned}$$

Thus, the action of the operator S on a vector $\mathbf{h}$ in the Hilbert space $\mathfrak{H}$ is given by $S\mathbf{h} = |C| \cdot \mathbf{F}\mathbf{h}$. Similarly, the inverse of S also acts by multiplication by $\frac{1}{|C|\cdot\mathbf{F}}$ as follows $S^{-1}\mathbf{h} = \frac{1}{|C|\cdot\mathbf{F}}\mathbf{h}$. □

The following theorem offers concrete conditions for the existence of Parseval frames and orthonormal bases of the type

$$\mathcal{S}\left(\mathbf{s} \cdot 1_A, \Gamma_H^{-1}\Gamma_N\right)$$

for some measurable function $\mathbf{s}$ defined on $\mathcal{O}$ and a $d\mu_H$-measurable subset A of $\mathcal{O}$.

Proposition 21.

(1) *Let $\mathbf{f}$ be a vector in $L^2(\mathcal{O}, d\mu_H)$ satisfying **(C7)**. If additionally **(C2)** and **(C6)** hold then $\mathcal{S}\left(|C|^{-1/2}\mathbf{f}, \Gamma_H^{-1}\Gamma_N\right)$ is a Parseval frame for $\mathfrak{H}$.*

(2) *Let $\mathbf{f}$ be a vector in $L^2(\mathcal{O}, d\mu_H)$ satisfying **(C7)** and **(C8)**. If additionally, **(C2)** and **(C6)** hold then $\mathcal{S}\left(|C|^{-1/2}\mathbf{f}, \Gamma_H^{-1}\Gamma_N\right)$ is an orthonormal basis for $\mathfrak{H}$.*

Proof. First, we observe that if $\mathbf{f}$ is a function defined on $\mathcal{O}$ such that

$$\mathbf{f}\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right)\mathbf{W}_\lambda(\xi)^{1/2} = 1_{\Theta_\lambda(\varphi(\mathcal{O}))}(\xi),$$

then $\mathbf{f} \in L^2(\mathcal{O}, d\mu_H)$. Indeed,

$$\begin{aligned}\int_{\mathcal{O}} |\mathbf{f}(h)|^2 d\mu_H(h) &= \int_{\varphi(\mathcal{O})} \left|\mathbf{f}\left(\varphi^{-1}(A)\right)\right|^2 d\mu_H\left(\varphi^{-1}(A)\right) \\ \text{(see (3.2))} &= \int_{\Theta_\lambda(\varphi(\mathcal{O}))} \left|\mathbf{f}\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right)\right|^2 \mathbf{W}_\lambda(\xi)\, d\xi\end{aligned}$$

and

$$\begin{aligned}\int_{\mathcal{O}} |\mathbf{f}(h)|^2 d\mu_H(h) &= \int_{\Theta_\lambda(\varphi(\mathcal{O}))} \left|\mathbf{W}_\lambda(\xi)^{1/2}\mathbf{f}\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right)\right|^2 d\xi \\ &= |\Theta_\lambda(\varphi(\mathcal{O}))| < \infty.\end{aligned}$$

Given $X \in \mathfrak{p}$, and letting $\mathbf{g}$ be a continuous function defined on $\mathcal{O}$, we obtain

$$\begin{aligned}
I &= \langle \mathbf{g}, \pi(\exp X)\mathbf{f}\rangle_{\mathfrak{H}} \\
&= \int_{\mathcal{O}} \mathbf{g}(h)\, e^{2\pi i\left\langle Ad\left(h^{-1}\right)^{*}\lambda, X\right\rangle}\overline{\mathbf{f}(h)}d\mu_H(h) \\
\left(\beta_\lambda\left(\varphi^{-1}(A)\right)=Ad\left(h^{-1}\right)^{*}\lambda\right) &= \int_{\varphi(\mathcal{O})} \mathbf{g}\left(\varphi^{-1}(A)\right) e^{2\pi i\left\langle \beta_\lambda\left(\varphi^{-1}(A)\right), X\right\rangle}\overline{\mathbf{f}\left(\varphi^{-1}(A)\right)}d\mu_H\left(\varphi^{-1}(A)\right) \\
\text{Since } X=P^{*}X\in\mathfrak{p} &= \int_{\varphi(\mathcal{O})} \mathbf{g}\left(\varphi^{-1}(A)\right) e^{2\pi i\left\langle P^{*}\beta_\lambda\varphi^{-1}(A), X\right\rangle}\overline{\mathbf{f}\left(\varphi^{-1}(A)\right)}d\mu_H\left(\varphi^{-1}(A)\right) \\
\Theta_\lambda(A)=P^{*}\beta_\lambda\varphi^{-1}(A) &= \int_{\varphi(\mathcal{O})} \mathbf{g}\left(\varphi^{-1}(A)\right) e^{2\pi i\langle \Theta_\lambda(A), X\rangle}\overline{\mathbf{f}\left(\varphi^{-1}(A)\right)}d\mu_H\left(\varphi^{-1}(A)\right).
\end{aligned}$$

Next, the change of variable

$$\xi = \Theta_\lambda(A) \Leftrightarrow A = \Theta_\lambda^{-1}(\xi)$$

yields

$$I = \int_{\Theta_\lambda(\varphi(\mathcal{O}))} e^{2\pi i\langle \xi, X\rangle}\left(\mathbf{g}\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right)\overline{\mathbf{f}\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right)}\mathbf{W}_\lambda(\xi)\right)d\xi.$$

Let C be a compact subset of $\mathfrak{n}^*$ containing $\Theta_\lambda(\varphi(\mathcal{O}))$ such that

$$\left\{\frac{e^{2\pi i\langle \xi, X\rangle}\cdot 1_{\Theta_\lambda(\varphi(\mathcal{O}))}(\xi)}{|C|^{1/2}} : \exp(X) \in \Gamma_N\right\} \tag{2.24}$$

is a Parseval frame for $L^2(\Theta_\lambda(\varphi(\mathcal{O})))$. Then

$$\begin{aligned}
I_1 &= \sum_{\exp X\in\Gamma_N}\left|\langle \mathbf{g}, \pi(\exp X)\mathbf{f}\rangle_{\mathfrak{H}}\right|^2 \\
&= \sum_{\exp X\in\Gamma_N}\left|\int_{\Theta_\lambda(\varphi(\mathcal{O}))}\frac{e^{2\pi i\langle \xi, X\rangle}}{|C|^{1/2}}\mathbf{g}\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right)|C|^{1/2}\,\mathbf{W}_\lambda(\xi)^{1/2}d\xi\right|^2.
\end{aligned}$$

Letting

$$u(\xi) = \varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right),$$

we obtain

$$\begin{aligned}
I_1 &= \int_{\Theta_\lambda(\varphi(\mathcal{O}))}\left||C|^{1/2}\,\mathbf{g}(u(\xi))\,\mathbf{W}_\lambda(\xi)^{1/2}\right|^2 d\xi \\
&= \int_{\Theta_\lambda(\varphi(\mathcal{O}))}|C|\cdot|\mathbf{g}(u(\xi))|^2\,\mathbf{W}_\lambda(\xi)\,d\xi
\end{aligned}$$

$$= |C| \cdot \int_{\mathcal{O}} |\mathbf{g}(h)|^2 \, d\mu_H(h).$$

Assuming the existence of a discrete set $\Gamma_H \subset H$ such that $\{\ell^{-1}\mathcal{O} : \ell \in \Gamma_H\}$ is a tiling of H; since the operators $\pi(\ell), \ell \in \Gamma_H$ are unitary, it immediately follows that

$$\begin{aligned}
\sum_{\ell \in \Gamma_H} \sum_{\exp X \in \Gamma_N} \left|\left\langle \mathbf{g}, \pi\left(\ell^{-1}\right) \pi(\exp X) \mathbf{f} \right\rangle\right|^2 &= \sum_{\ell \in \Gamma_H} \sum_{\exp X \in \Gamma_N} \left|\langle \pi(\ell) \mathbf{g} \cdot 1_{\mathcal{O}}, 1_{\mathcal{O}} \cdot \pi(\exp X) \mathbf{f} \rangle\right|^2 \\
&= |C| \cdot \sum_{\ell \in \Gamma_H} \|\pi(\ell)(\mathbf{g} \cdot 1_{\mathcal{O}})\|^2 \\
&= |C| \cdot \sum_{\ell \in \Gamma_H} \int_{\mathcal{O}} \left|\mathbf{g}\left(\ell^{-1} h\right)\right|^2 d\mu_H(h) \\
&= |C| \cdot \sum_{\ell \in \Gamma_H} \int_{\ell^{-1}\mathcal{O}} |\mathbf{g}(h)|^2 \, d\mu_H(h) \\
&= |C| \cdot \int_H |\mathbf{g}(h)|^2 \, d\mu_H(h).
\end{aligned}$$

Therefore $\{\pi(\ell^{-1}\kappa)\mathbf{f} : \kappa \in \Gamma_N, \ell \in \Gamma_H\}$ is a tight frame for $L^2(H, d\mu_H(h))$ with frame bound $|C|$. As a result,

$$\left\{\pi\left(\ell^{-1}\kappa\right) |C|^{-1/2} \mathbf{f} : \kappa \in \Gamma_N, \ell \in \Gamma_H\right\}$$

is a Parseval frame.

If additionally,

$$|C|^{-1/2} \cdot \|\mathbf{f}\|_{\mathfrak{H}} = |C|^{-1/2} \cdot |\Theta_\lambda(\varphi(\mathcal{O}))|^{1/2} = 1 \tag{2.25}$$

then

$$\begin{aligned}
\left\| |C|^{-1/2} \mathbf{f} \right\|_{\mathfrak{H}} &= \left(|C|^{-1} \int_{\mathcal{O}} |\mathbf{f}(x)|^2 \, d\mu_H(x) \right)^{1/2} \\
&= \left(|C|^{-1} \int_{\Theta_\lambda(\varphi(\mathcal{O}))} \left|\mathbf{f}\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right)\right|^2 d\mu_H\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right) \right)^{1/2} \\
&= \left(|C|^{-1} \int_{\Theta_\lambda(\varphi(\mathcal{O}))} \left|\mathbf{f}\left(\varphi^{-1}\left(\Theta_\lambda^{-1}(\xi)\right)\right) \mathbf{W}_\lambda(\xi)^{1/2}\right|^2 d\xi \right)^{1/2} \\
&= \left(|C|^{-1} \int_{\Theta_\lambda(\varphi(\mathcal{O}))} \frac{1_{\Theta_\lambda(\varphi(\mathcal{O}))}(\xi)}{(\mathbf{W}_\lambda(\xi))^{1/2}} (\mathbf{W}_\lambda(\xi))^{1/2} \, d\xi \right)^{1/2}
\end{aligned}$$

$$= \left(|C|^{-1}\int_{\Theta_\lambda(\varphi(\mathcal{O}))} d\xi\right)^{1/2}$$
$$\overset{\text{By (2.25)}}{=} |C|^{-1/2}\cdot|\Theta_\lambda(\varphi(\mathcal{O}))|^{1/2} = 1.$$

Therefore,

$$\left\{\pi\left(\ell^{-1}\kappa\right)\left(|C|^{-1/2}\,\mathbf{f}\right) : \kappa\in\Gamma_N, \ell\in\Gamma_H\right\}$$

is an orthonormal basis for $\mathfrak{H}$. □

Remark 22. *(The construction of orthonormal bases) For the special case where NH is a nilpotent Lie group, Proposition 21 Part 2 gives sufficient conditions for the existence of orthonormal bases generated by a function of the type $\mathbf{s}\cdot 1_A$. K. Gröchenig and D. Rottensteiner recently [20] proved the following. Let G be a graded Lie group (thus, nilpotent Lie group) with a one-dimensional center. Next, let π be a square-integrable irreducible representation of G realized as acting in $L^2\left(\mathbb{R}^d\right)$ which is square-integrable modulo its center. Then there exist a set $\Gamma\subset G$ and a compact set F such that*

$$\left\{|F|^{-1/2}\,\pi(\gamma)\,1_F : \gamma\in\Gamma\right\}$$

is an orthonormal basis. We would also like to remark that there exist unitary representations of nilpotent Lie groups which are not square-integrable modulo the center which we can still discretize to construct orthonormal bases for the corresponding Hilbert space. To see this, let G be a six-dimensional simply connected nilpotent Lie group with Lie algebra spanned by $X_{23}, X_{13}, X_{12}, X_3, X_2, X_1$ with corresponding dual vector space spanned by the dual basis

$$X_{23}^*, X_{13}^*, X_{12}^*, X_3^*, X_2^*, X_1^*$$

such that $\langle X_k^*, X_j\rangle = \delta_{kj}$. The non-trivial brackets of the Lie algebra are given by $[X_i, X_j] = X_{ij}$ for $i<j$. Thus, the elements X_{23}, X_{13}, X_{12} span the center of the Lie algebra of G and G is a step-two free nilpotent Lie group on three generators. Fix a linear functional

$$\lambda = \lambda_{23}X_{23}^* + \lambda_{13}X_{13}^* + \lambda_{12}X_{12}^* + \lambda_3X_3^* + \lambda_2X_2^* + \lambda_1X_1^*$$

such that λ_{23} is nonzero, and consider a subalgebra $\mathfrak{n}$ spanned by the vectors

$$X_{23}, X_{13}, X_{12}, \lambda_{12}X_3 - \lambda_{13}X_2 + \lambda_{23}X_1.$$

Then $\mathfrak{n}$ is a maximal algebra of $\mathfrak{g}$ such that $[\mathfrak{n},\mathfrak{n}]$ is contained in the kernel of the linear functional λ. $\mathfrak{n}$ is called a polarizing algebra subordinated to the linear functional λ [27].

Next, we consider the corresponding irreducible representation π_λ of G realized as acting in $L^2(\mathbb{R})$ as follows

$$\pi_\lambda(\exp tX) f(x) = \begin{cases} f(x-t) & \text{if } X = X_2 \\ e^{-2\pi i \lambda_{23} x t} f(x) & \text{if } X = X_3 \\ e^{2\pi i t \lambda_1} e^{2\pi i x t \lambda_{12}} e^{-2\pi i \frac{t^2 \lambda_{12}\lambda_{13}}{\lambda_{23}}} f\left(x - \frac{t\lambda_{12}}{\lambda_{23}}\right) & \text{if } X = X_1 \\ e^{2\pi i t \lambda_{ij}} f(x) & \text{if } X = X_{ij} \end{cases}. \tag{2.26}$$

Let

$$\mathfrak{r}_\lambda = \{X \in \mathfrak{g} : ad(X)^* \lambda = 0\}$$

be the so-called radical of λ. Then $\mathfrak{r}_\lambda$ contains the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ and since

$$\dim(\mathfrak{r}_\lambda) = 4 > 3 = \dim(\mathfrak{z}(\mathfrak{g})),$$

according to [5, 4.5.4 Corollary]), π_λ is not square-integrable modulo the center of G. Although, the action given by $\pi_\lambda(\exp X_1)$ is quite complicated, we do not need to take it into account for the construction of orthonormal bases. To this end, it suffices to select our discrete set to be

$$\Gamma = \exp\left(\lambda_{23}^{-1}\mathbb{Z}X_3\right)\exp(\mathbb{Z}X_2).$$

Given $k, j \in \mathbb{Z}$ and $\mathbf{f} \in L^2(\mathbb{R})$, we have

$$\pi_\lambda(\exp\left(\lambda_{23}^{-1}kX_3\right))\mathbf{f}(x) = e^{-2\pi i k x} f(x) \qquad \pi_\lambda(\exp(jX_2))\mathbf{f}(x) = \mathbf{f}(x-j).$$

and the collection $\left\{\pi_\lambda(\gamma) 1_{[0,1]} : \gamma \in \Gamma\right\}$ is an orthonormal basis for $L^2(\mathbb{R})$. Note that in this example, Theorem 1 does not even apply since G cannot be written as a semi-direct product of the type NH for some closed subgroups N, H satisfying our assumptions. We intend to generalize these constructions in a future investigation.

We are now in position to establish our main results.

2.2. **Proof of Theorem 1.** Appealing to Lemma 14 and Lemma 17, we see that the sufficient conditions stated of Lemma 18 hold for any function $\mathbf{f}$ which is continuous (or smooth) and supported on $\mathcal{O}$.

2.3. **Proof of Theorem 2.** Theorem 2 follows from Proposition 21.

2.4. **Proof of Theorem 3.** We will start this section by reviewing some fundamental concepts of exponential solvable Lie groups [31]. Let $\mathfrak{h}$ be a real solvable Lie algebra of dimension

r. We define the derived series of $\mathfrak{h}$ inductively as follows

$$D^1\mathfrak{h} = [\mathfrak{h},\mathfrak{h}]\,,\ D^k\mathfrak{h} = \left[D^{k-1}\mathfrak{h},D^{k-1}\mathfrak{h}\right].$$

Here $[\mathfrak{h},\mathfrak{h}]$ is the real span of $[X,Y]$ for $X,Y \in \mathfrak{h}$. Moreover, we say that $\mathfrak{h}$ is solvable if there exists a natural number m such that $\dim(D^m\mathfrak{h}) = 0$. The smallest such natural number m is called the length of $\mathfrak{h}$. For example, it is not hard to verify that the algebra of upper triangular matrices is a solvable Lie algebra.

Next, for a Lie algebra $\mathfrak{h}$, the descending central series is defined by the sequence

$$\mathfrak{h}^{(0)} = \mathfrak{h}, \mathfrak{h}^{(1)} = [\mathfrak{h},\mathfrak{h}]\,, \cdots, \mathfrak{h}^{(k)} = \left[\mathfrak{h}^{(k-1)},\mathfrak{h}\right]$$

and $\mathfrak{h}$ is called nilpotent if there exists a natural number m such that $\dim \mathfrak{h}^{(m)} = 0$. It is easy to verify that all nilpotent Lie algebras are solvable. However, the containment of nilpotent Lie algebras in the set of solvable algebras is strictly proper.

A connected Lie group H is said to be solvable if its Lie algebra is solvable. Keep in mind that even if H is simply connected and solvable, the exponential map may still fail to be bijective. For a solvable Lie group H with Lie algebra $\mathfrak{h}$, when the corresponding exponential map is a bijection, we say that such a group is **exponential** and its Lie algebra is also called exponential. Such a group is necessarily connected and simply connected as well.

The following lemmas will be instrumental

Lemma 23. *[31, Corollary 3.7.5.] Let $\mathfrak{h}$ be a solvable Lie algebra over the reals. Then we can find subalgebras $\mathfrak{h} = \mathfrak{h}_1, \mathfrak{h}_2, \cdots, \mathfrak{h}_r = \{0\}$ such that (1) Each $\mathfrak{h}_{k+1} \subseteq \mathfrak{h}_k$ and $\mathfrak{h}_{k+1}$ is an ideal of $\mathfrak{h}_k$ and (2) $\dim(\mathfrak{h}_k/\mathfrak{h}_{k+1}) = 1$ for $k \in \{1,\cdots,r\}$.*

Lemma 24. *[31, Theorem 3.18.11] Let H be a simply connected solvable Lie group with Lie algebra $\mathfrak{h}$. Suppose that $\{A_1,\cdots,A_r\}$ is a basis [5] for $\mathfrak{h}$ satisfying the following property: $\mathfrak{h}_{(j)} = \sum_{k=1}^{j} \mathbb{R}A_k$ is a subalgebra of $\mathfrak{h}$ and each $\mathfrak{h}_{(i)}$ is an ideal in $\mathfrak{h}_{(i+1)}$. Then the map*

$$(a_1,\cdots,a_r) \mapsto \exp(a_1A_1)\cdots\exp(a_rA_r)$$

is an analytic diffeomorphism of $\mathbb{R}^r$ onto H.

Moving forward, we will also need the following lemma. Let $\mathcal{E}(C,\Gamma_N)$ be as defined in (2.18).

[5]The existence of such a basis a direct consequence of Lemma 23

Lemma 25. *Let* $\mathbf{s}$ *be a continuous (or a smooth) function which is supported on* $\mathcal{O}$. *Suppose that* Γ_H *is a discrete subset of* H *such that*

$$\sum_{\ell\in\Gamma_H} |\mathbf{s}(\ell h)|^2 = 1$$

for almost every $h \in H$. *Let* C *be a compact subset of* $\mathfrak{p}^*$ *such that* $\Theta_\lambda(\varphi(\mathcal{O})) \subseteq C$ *and* Γ_N *is a discrete subset of* $\exp(\mathfrak{p})$ *such that the trigonometric system* $\mathcal{E}(C, \Gamma_N)$ *is an orthonormal basis for* $L^2(C, d\xi)$. *Then the system*

$$\left\{\left[\pi\left(\ell^{-1}\kappa\right)\right]|C|^{-1/2}\mathbf{s}\mathbf{w}^{-1/2} : (\ell,\kappa)\in\Gamma_H\times\Gamma_N\right\}$$

is a Parseval frame generated by the compactly supported and continuous function $\mathbf{s}\mathbf{w}^{-1/2}|C|^{-1/2}$.

Proof. Let us suppose that $\mathbf{s}$ is a continuous (or a smooth) function which is compactly supported on $\mathcal{O}$ such that $\sum_{\ell\in\Gamma_H}|\mathbf{s}(\ell h)|^2 = 1$ for all $h \in H$. Next, let $\mathbf{f} = \mathbf{s}\mathbf{w}^{-1/2}$. Then

$$\sum_{\kappa\in\Gamma_H}\left|\mathbf{f}(\kappa h)\sqrt{\mathbf{w}(\kappa h)}\right|^2 = \sum_{\kappa\in\Gamma_H}\left|\frac{\mathbf{s}(\kappa h)}{\sqrt{\mathbf{w}(\kappa h)}}\sqrt{\mathbf{w}(\kappa h)}\right|^2 = \sum_{\kappa\in\Gamma_H}|\mathbf{s}(kh)|^2 = 1.$$

Appealing to Lemma 18 Part (1), the stated result is immediate. □

Suppose that H is a connected exponential solvable Lie group. We will prove by induction on the dimension of H that there exists a continuous function $\mathbf{s}$ compactly supported in $\mathcal{O}$ such that

$$\sum_{\ell\in\Gamma_H}|\mathbf{s}(\ell h)|^2 = 1$$

for every element h in H.

For the base case, let us suppose that the dimension of H is equal to 1. Moreover, let us suppose that $\mathcal{O} = (-\epsilon^{-1}, \epsilon^{-1})$ for some positive real number ϵ. Put

$$\mathbf{s}(t) = \left(1_{[-1/2,1/2]} * 1_{[-1/2,1/2]}\right)^{\frac{1}{2}}(\epsilon t).$$

In the above, $*$ stands for the usual convolution product for functions defined on the real line given by

$$(f * g)(y) = \int_{\mathbb{R}} f(x)\, g(y-x)\, dx.$$

Since

$$\sum_{k\in\mathbb{Z}}\left(1_{[-1/2,1/2]} * 1_{[-1/2,1/2]}\right)(t+k) = 1 \text{ for all } t\in\mathbb{R},$$

it follows that

$$\sum_{k\in\mathbb{Z}}\left|\mathbf{s}\left(t+\epsilon^{-1}k\right)\right|^{2}=\sum_{k\in\mathbb{Z}}\left(1_{[-1/2,1/2]}*1_{[-1/2,1/2]}\right)(t\epsilon+k)=1$$

holds for every real number t. Note that the smoothness of $\mathbf{s}$ can be improved if $\mathbf{s}$ is replaced by a suitable spline-type function of higher order. To this end, let $\mathbf{b}_n$ be the function obtained by convolving the function $1_{[-1/2,1/2]}$ with itself n-many times. Next, let

$$\mathbf{s}_n(t)=(\mathbf{b}_n)^{\frac{1}{2}}\left(\frac{\epsilon n}{2}t\right).$$

Then

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left|\mathbf{s}_n\left(t+\frac{2}{\epsilon n}k\right)\right|^{2}&=\sum_{k\in\mathbb{Z}}\left|(\mathbf{b}_n)^{\frac{1}{2}}\left(\frac{\epsilon n}{2}\left(t+\frac{2}{\epsilon n}k\right)\right)\right|^{2}\\&=\sum_{k\in\mathbb{Z}}\mathbf{b}_n\left(k+\frac{nt\epsilon}{2}\right)\\&=1\end{aligned}$$

for every $t\in\mathbb{R}$ as desired.

Next, let us assume that our result holds true whenever, $\dim H\leq r$ for some natural number $r\geq 1$. Suppose next that $\dim(H)=r+1$. Since H is an exponential solvable Lie group, we may write $H=H_{\mathbf{n}}H_{\mathbf{c}}$ such that $H_{\mathbf{n}}$ is normal in H and $H_{\mathbf{c}}$ is a 1-parameter subgroup of H (by appealing to Lemma 23 and Lemma 24) which we shall identify with the set of real numbers. Without loss of generality, we may assume that $\mathcal{O}=\mathcal{O}_{\mathbf{n}}\mathcal{O}_{\mathbf{c}}$ such that $\mathcal{O}_{\mathbf{n}}\subset H_{\mathbf{n}}$ and $\mathcal{O}_{\mathbf{c}}\subset H_{\mathbf{c}}$ such that (inductive hypothesis) the following holds true. There exist $\Gamma_{\mathbf{n}}\subset H_{\mathbf{n}}$ and $\Gamma_{\mathbf{c}}\subset H_{\mathbf{c}}$ and continuous functions (or smooth functions) $\mathbf{s}_{\mathbf{n}}$ and $\mathbf{s}_{\mathbf{c}}$ such that

$$\sum_{\ell_{\mathbf{n}}\in\Gamma_{\mathbf{n}}}\left|\mathbf{s}_{\mathbf{n}}\left(\ell_{\mathbf{n}}h'\right)\right|^{2}=1$$

for almost every $h'\in H_{\mathbf{n}}$ and

$$\sum_{\ell_{\mathbf{c}}\in\Gamma_{\mathbf{c}}}\left|\mathbf{s}_{\mathbf{c}}\left(\ell_{\mathbf{c}}h''\right)\right|^{2}=1$$

for almost every $h''\in H_{\mathbf{c}}$. Next, we define a continuous function $\mathbf{s}$ on H as follows. Given $h\in H$, since we can uniquely factor h such that $h=h_{\mathbf{n}}h_{\mathbf{c}}$ with $h_{\mathbf{n}}\in H_{\mathbf{n}}$ and $h_{\mathbf{c}}\in H_{\mathbf{c}}$, we define

$$\mathbf{s}(h)=\mathbf{s}(h_{\mathbf{n}}h_{\mathbf{c}})=\mathbf{s}_{\mathbf{n}}(h_{\mathbf{n}})\cdot\mathbf{s}_{\mathbf{c}}(h_{\mathbf{c}})$$

for all h in H. Thus, for $h \in H$,

$$\sum_{\ell_{\mathbf{c}} \in \Gamma_{\mathbf{c}}} \sum_{\ell_{\mathbf{n}} \in \Gamma_{\mathbf{n}}} \left|\mathbf{s}\left(\ell_{\mathbf{n}} \ell_{\mathbf{c}} h\right)\right|^2 = \sum_{\ell_{\mathbf{c}} \in \Gamma_{\mathbf{c}}} \sum_{\ell_{\mathbf{n}} \in \Gamma_{\mathbf{n}}} \left|\mathbf{s}\left(\ell_{\mathbf{n}} \ell_{\mathbf{c}} h_{\mathbf{n}} h_{\mathbf{c}}\right)\right|^2.$$

Rewriting the identity in H as $e = \ell_{\mathbf{c}}^{-1} \ell_{\mathbf{c}}$, we obtain:

$$\sum_{\ell_{\mathbf{c}} \in \Gamma_{\mathbf{c}}} \sum_{\ell_{\mathbf{n}} \in \Gamma_{\mathbf{n}}} \left|\mathbf{s}\left(\ell_{\mathbf{n}} \ell_{\mathbf{c}} h\right)\right|^2 = \sum_{\ell_{\mathbf{c}} \in \Gamma_{\mathbf{c}}} \sum_{\ell_{\mathbf{n}} \in \Gamma_{\mathbf{n}}} \left|\mathbf{s}\left(\ell_{\mathbf{n}} \ell_{\mathbf{c}} h_{\mathbf{n}} \left(\ell_{\mathbf{c}}^{-1} \ell_{\mathbf{c}}\right) h_{\mathbf{c}}\right)\right|^2.$$

Next, appealing to the normality of $H_{\mathbf{n}}$ in H, it is clear that $\ell_{\mathbf{n}} \ell_{\mathbf{c}} h_{\mathbf{n}} \ell_{\mathbf{c}}^{-1} \in H_{\mathbf{n}}$. Moving forward,

$$\sum_{\ell_{\mathbf{c}} \in \Gamma_{\mathbf{c}}} \sum_{\ell_{\mathbf{n}} \in \Gamma_{\mathbf{n}}} \left|\mathbf{s}\left(\left(\ell_{\mathbf{n}} \ell_{\mathbf{c}} h_{\mathbf{n}} \ell_{\mathbf{c}}^{-1}\right)\left(\ell_{\mathbf{c}} h_{\mathbf{c}}\right)\right)\right|^2 = \sum_{\ell_{\mathbf{c}} \in \Gamma_{\mathbf{c}}} \sum_{\ell_{\mathbf{n}} \in \Gamma_{\mathbf{n}}} \left|\mathbf{s}_{\mathbf{n}}\left(\ell_{\mathbf{n}} \ell_{\mathbf{c}} h_{\mathbf{n}} \ell_{\mathbf{c}}^{-1}\right)\right|^2 \cdot \left|\mathbf{s}_{\mathbf{c}}\left(\ell_{\mathbf{c}} h_{\mathbf{c}}\right)\right|^2.$$

However, by assumption, for any fixed $\ell_{\mathbf{c}} \in \Gamma_{\mathbf{c}}$, since $\ell_{\mathbf{c}} h_{\mathbf{n}} \ell_{\mathbf{c}}^{-1} \in H_{\mathbf{n}}$, it is clearly the case that

$$\sum_{\ell_{\mathbf{n}} \in \Gamma_{\mathbf{n}}} \left|\mathbf{s}_{\mathbf{n}}\left(\ell_{\mathbf{n}} \ell_{\mathbf{c}} h_{\mathbf{n}} \ell_{\mathbf{c}}^{-1}\right)\right|^2 = 1.$$

Therefore,

$$\sum_{\ell_{\mathbf{c}} \in \Gamma_{\mathbf{c}}} \sum_{\ell_{\mathbf{n}} \in \Gamma_{\mathbf{n}}} \left|\mathbf{s}_{\mathbf{n}}\left(\ell_{\mathbf{n}} \ell_{\mathbf{c}} h_{\mathbf{n}} \ell_{\mathbf{c}}^{-1}\right)\right|^2 \cdot \left|\mathbf{s}_{\mathbf{c}}\left(\ell_{\mathbf{c}} h_{\mathbf{c}}\right)\right|^2 = \sum_{\ell_{\mathbf{c}} \in \Gamma_{\mathbf{c}}} \left|\mathbf{s}_{\mathbf{c}}\left(\ell_{\mathbf{c}} h_{\mathbf{c}}\right)\right|^2 = 1,$$

and we conclude that

$$\sum_{\ell_{\mathbf{c}} \in \Gamma_{\mathbf{c}}} \sum_{\ell_{\mathbf{n}} \in \Gamma_{\mathbf{n}}} \left|\mathbf{s}\left(\ell_{\mathbf{n}} \ell_{\mathbf{c}} h\right)\right|^2 = 1$$

for all h in H.

In light of Lemma 25, Theorem 3 is immediate.

3. Examples and applications

3.1. **A class of matrix groups satisfying the assumptions.** Proposition 26 gives us a procedure for constructing an extensive collection of matrix groups satisfying the conditions listed in Section 1.1. We thank Gestur Olafsson for showing us the class of linear groups described in Proposition 26. This class of groups is inspired by some earlier work of J. Wolf on the representation theory of maximal parabolic subgroups of classical Lie groups [32].

Proposition 26. *Let*

$$H = \left\{ \begin{bmatrix} h & 0_n \\ 0_n & \mathrm{Id}_n \end{bmatrix} : h \in K \right\} \text{ and } N = \left\{ \begin{bmatrix} \mathrm{Id}_n & x \\ 0_n & \mathrm{Id}_n \end{bmatrix} : x \in \mathfrak{gl}\left(n, \mathbb{R}\right) \right\}$$

where K is a closed and connected matrix subgroup of $GL(n,\mathbb{R})$. Put $G = NH$. Then there exists a linear functional $\lambda \in \mathfrak{n}^*$ such that the smooth map $h \mapsto \beta_\lambda(h) = Ad(h^{-1})^* \lambda$ is an immersion at the identity of H.

Proof. Let

$$G = \left\{ \begin{bmatrix} h & x \\ 0_n & \mathrm{Id}_n \end{bmatrix} : h \in K \text{ and } x \in \mathfrak{gl}(n,\mathbb{R}) \right\}.$$

Clearly, the map

$$h \mapsto \begin{bmatrix} h & 0_n \\ 0_n & \mathrm{Id}_n \end{bmatrix}$$

defines a Lie group isomorphism between K and H and we shall identify K with H via this isomorphism. By assumption, the dimension of N must be greater than or equal to that of H, and the Lie algebra of N takes the form

$$\mathfrak{n} = \left\{ \begin{bmatrix} 0_n & x \\ 0_n & 0_n \end{bmatrix} : x \in \mathfrak{gl}(n,\mathbb{R}) \right\}.$$

We identify $\mathfrak{n}$ with its dual such that for $\lambda \in \mathfrak{n}^*$ and $X \in \mathfrak{n}$, the pairing $\langle \lambda, X \rangle$ is given by the trace of the matrix $\lambda^T X$. For $\lambda \in \mathfrak{n}^*$, the map

$$\beta_\lambda : H \to \beta_\lambda(H) \subseteq \mathfrak{n}^*$$

described in (2.1) is computed as follows. Put

$$\lambda_\omega = \lambda = \begin{bmatrix} 0_n & \omega \\ 0_n & 0_n \end{bmatrix} \in \mathfrak{n}^*$$

where ω is a matrix of order n and let $h \in H$. Then

$$\left\langle \lambda, Ad\left(h^{-1}\right) X \right\rangle = \left\langle \begin{bmatrix} 0_n & \omega \\ 0_n & 0_n \end{bmatrix}, \begin{bmatrix} h^{-1} & 0_n \\ 0_n & \mathrm{Id}_n \end{bmatrix} \begin{bmatrix} 0_n & x \\ 0_n & 0_n \end{bmatrix} \right\rangle.$$

Suppose next, that ω is the identity matrix of order n. Then

$$\left\langle \lambda, Ad\left(h^{-1}\right) X \right\rangle = \mathrm{Tr}\left(\underbrace{\begin{bmatrix} 0_n & 0_n \\ \mathrm{Id}_n & 0_n \end{bmatrix}}_{\lambda^T} \underbrace{\begin{bmatrix} h^{-1} & 0_n \\ 0_n & \mathrm{Id}_n \end{bmatrix} \begin{bmatrix} 0_n & x \\ 0_n & 0_n \end{bmatrix}}_{Ad(h^{-1})X} \right).$$

Since

$$\begin{bmatrix} 0_n & 0_n \\ \mathrm{Id}_n & 0_n \end{bmatrix} \begin{bmatrix} h^{-1} & 0_n \\ 0_n & \mathrm{Id}_n \end{bmatrix} = \begin{bmatrix} 0_n & 0_n \\ h^{-1} & 0_n \end{bmatrix},$$

it follows that

$$\left\langle \lambda, Ad\left(h^{-1}\right) X \right\rangle = \mathrm{Tr}\left(\begin{bmatrix} 0_n & 0_n \\ h^{-1} & 0_n \end{bmatrix} \begin{bmatrix} 0_n & x \\ 0_n & 0_n \end{bmatrix} \right).$$

Consequently,

$$\left\langle Ad\left(h^{-1}\right)^* \lambda, X \right\rangle = \left\langle \begin{bmatrix} 0_n & \left(h^{-1}\right)^T \\ 0_n & 0_n \end{bmatrix}, \begin{bmatrix} 0_n & x \\ 0_n & 0_n \end{bmatrix} \right\rangle$$

and we obtain

$$Ad\left(h^{-1}\right)^* \lambda = \begin{bmatrix} 0_n & \left(h^{-1}\right)^T \\ 0_n & 0_n \end{bmatrix}.$$

By a slight abuse of notation, we can now say that $\beta_\lambda(h) = \left(h^{-1}\right)^T$ and clearly, this implies that β_λ is an immersion at the identity of H. □

Remark 27. *Note that the proof of Proposition 26 still works if we were to replace $\lambda_\omega \in \mathfrak{n}^*$ with a linear function $\lambda_{\omega'}$ such that ω' is any other invertible matrix.*

3.2. **Some explicit formulas for $\mathbf{W}_\lambda$ and applications.** Since $\mathbf{W}_\lambda$ plays a central role in this work, we exhibit below some examples in which $\mathbf{W}_\lambda$ is explicitly computed. In some cases, we also give a construction of frames and orthonormal bases.

Example 28. *Let $G = \mathbb{R}^3 \rtimes \mathbb{R}$ be a semidirect product equipped with the product*

$$(x,t)(y,s) = \left(x + \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & e^t \end{bmatrix} y, t+s \right).$$

Here

$$N = \mathbb{R}^3 \times \{0\} \text{ and } H = \{0\} \times \mathbb{R}.$$

The group $G = NH$ is simply connected and solvable but it is not completely solvable. Next, let $\lambda \in \mathfrak{n}^*$ such that $\lambda = (0,1,0)$. The corresponding induced representation π is neither irreducible nor square-integrable neither. The non irreducibility of π follows from the fact that the stabilizer subgroup H_λ in H is not trivial [11, Chapter 6]. Next,

$$Ad\left(h^{-1}\right)^* \lambda = (-\sin t, \cos t, 0)$$

and letting $P^{\ast}:\mathfrak{n}^{\ast}\rightarrow\mathfrak{n}^{\ast}$ *such that*

$$P^{\ast}\left(\lambda_{1},\lambda_{2},\lambda_{3}\right)=\left(\lambda_{1},0,0\right),$$

we obtain $\Theta_{\lambda}\left(t\right)=-\sin t$ *which is clearly a local diffeomorphism at zero (clearly not a global diffeomorphism.) Its local inverse is given by* $\Theta_{\lambda}^{-1}\left(\xi\right)=-\arcsin\left(\xi\right)$*. As a result,*

$$\mathbf{W}_{\lambda}\left(\xi\right)=\frac{1}{\sqrt{1-\xi^{2}}}.$$

Staying away from the singular points of Θ_{λ}*, fix* $\mathcal{O}=\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$*. Then* Θ_{λ} *defines a diffeomorphism between* $\mathcal{O}$ *and its image. Let* $\mathbf{f}$ *be a continuous function supported on* $\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$ *such that*

$$m_{H,\mathbf{f}}=\inf_{t}\left(\sum_{\ell\in\alpha\mathbb{Z}}\left|\frac{\mathbf{f}\left(t+\ell\right)}{\sqrt{1-\sin\left(t+\ell\right)^{2}}}\right|^{2}\right)>0$$

and

$$M_{H,\mathbf{f}}=\sup_{t}\left(\sum_{\ell\in\alpha\mathbb{Z}}\left|\frac{\mathbf{f}\left(t+\ell\right)}{\sqrt{1-\sin\left(t+\ell\right)^{2}}}\right|^{2}\right)<\infty$$

for some positive real number α*. Take for example,*

$$\mathbf{f}\left(t\right)=\sqrt{1-\sin\left(t\right)^{2}}\left(1_{\left(-\frac{1}{2},\frac{1}{2}\right)}\ast1_{\left(-\frac{1}{2},\frac{1}{2}\right)}\right)^{1/2}\left(t\right)\text{ and }\alpha=1.$$

Note also that $\left\{\frac{e^{2\pi i\langle\xi,Y\rangle}}{2^{1/4}}:Y\in\frac{1}{\sqrt{2}}\mathbb{Z}\right\}$ *is an orthonormal basis for* $L^{2}\left(C,d\xi\right)$ *where* $C=\sin\left(-\frac{\pi}{4},\frac{\pi}{4}\right)=\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$*. Let*

$$\Gamma=\left\{\left(0,0,\ell\right)\left(\kappa,0,0\right):\ell\in\alpha\mathbb{Z},\kappa\in\frac{1}{\sqrt{2}}\mathbb{Z}\right\}\subset G.$$

By Proposition 18, the system $\mathcal{S}\left(\mathbf{f},\Gamma\right)$ *is a frame for* $L^{2}\left(\mathbb{R}\right)$ *with frame bounds*

$$0<m_{H,\mathbf{f}}\cdot\sqrt{2}\leq M_{H,\mathbf{f}}\cdot\sqrt{2}<\infty.$$

Example 29. *(The standard two-dimensional shearlet group, [7]) Let* $G=\mathbb{R}^{2}\rtimes\mathbb{R}^{2}$ *be a semi-direct product group with multiplication law given by*

$$\left(v,t\right)\left(w,s\right)=\left(v+t\diamond w,t+s\right)$$

where

$$t\diamond w=\left[\begin{array}{cc}e^{t_{2}} & t_{1}e^{t_{2}}\\ 0 & e^{t_{2}}\end{array}\right]\left[\begin{array}{c}w_{1}\\ w_{2}\end{array}\right].$$

For a fixed linear functional λ of $\mathbb{R}^2$, let

$$\pi = \mathrm{ind}_{\mathbb{R}^2}^{\mathbb{R}^2 \rtimes \mathbb{R}^2} (\chi_\lambda)$$

be a unitary representation of G realized as acting on $L^2(\mathbb{R}^2)$ as follows. For a square-integrable function $\mathbf{f}$ over $\mathbb{R}^2$, the action of π is described as follows

$$[\pi(v,s)\mathbf{f}](t) = e^{2\pi i \left\langle \left[\begin{array}{c} \lambda_1 e^{-t_2} \\ e^{-t_2}(\lambda_2 - \lambda_1 t_1) \end{array} \right], \begin{array}{c} v_1 \\ v_2 \end{array} \right\rangle} \cdot \mathbf{f}(t-s) \text{ for } (v,t) \in \mathbb{R}^2 \rtimes \mathbb{R}^2.$$

Take $\lambda_1 = 1$ and $\lambda_2 = 0$, and J to be the set containing 1 and 2. By identifying G with its Lie algebra, φ is just the identity map. Next, let $\Theta_\lambda : \mathbb{R}^2 \to (0,\infty) \times \mathbb{R}$ such that

$$\Theta_\lambda(t_1,t_2) = \left(e^{-t_2}, -e^{-t_2} t_1\right).$$

The inverse of Θ_λ is given by $\Theta_\lambda^{-1} : (0,\infty) \times \mathbb{R} \to \mathbb{R}^2$ such that

$$\Theta_\lambda^{-1}(\xi_1,\xi_2) = \left(-\frac{\xi_2}{\xi_1}, \ln\left(\frac{1}{\xi_1}\right)\right)$$

and the Jacobian of Θ_λ^{-1} is

$$\mathrm{Jac}_{\Theta_\lambda^{-1}}(\xi_1,\xi_2) = \left[\begin{array}{cc} \xi_2 \xi_1^{-2} & -\xi_1^{-1} \\ -\xi_1^{-1} & 0 \end{array} \right].$$

Since H is commutative and is identified with its Lie algebra,

$$\mathbf{W}_\lambda(\xi_1,\xi_2) = \left| \det \mathrm{Jac}_{\Theta_\lambda^{-1}}(\xi_1,\xi_2) \right| = \xi_1^{-2}.$$

Next, Θ_λ defines a global diffeomorphism between $\mathbb{R}^2$ and $(0,\infty)\times\mathbb{R}$. According to Theorem 1 for any continuous function $\mathbf{f}$ supported on some compact set $K \subset \mathbb{R}^2$, there exists a discrete set $\Gamma_{K,\mathbf{f}} \subset G$ such that $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma_{K,\mathbf{f}}\}$ is a frame for $L^2(\mathbb{R}^2, dx)$.

Example 30. *(A case where H is $SL(2,\mathbb{R})$) Following the construction given in Proposition 26, let*

$$G = \left\{ \left[\begin{array}{cccc} 1 & 0 & x_1 & x_2 \\ 0 & 1 & x_3 & x_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \left[\begin{array}{cccc} a\cos\theta & as\cos\theta - \frac{1}{a}\sin\theta & 0 & 0 \\ a\sin\theta & \frac{1}{a}\cos\theta + as\sin\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] : x_1, \cdots, x_4, \theta, a, s \in \mathbb{R} \right\}$$

be a matrix group with Lie algebra $\mathfrak{g}=\mathfrak{n}+\mathfrak{h}$ *such that* $\mathfrak{n}$ *is spanned by*

$$\left[\begin{array}{cccc}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{array}\right],\left[\begin{array}{cccc}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{array}\right],\left[\begin{array}{cccc}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{array}\right],\left[\begin{array}{cccc}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{array}\right]$$

and $\mathfrak{h}$ *is spanned by the matrices*

$$\left[\begin{array}{cccc}0&-1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{array}\right],\left[\begin{array}{cccc}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{array}\right],\left[\begin{array}{cccc}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{array}\right].$$

Let $\varphi:SL\left(2,\mathbb{R}\right)=H\rightarrow\mathbb{R}^{3}$ *such that*

$$\varphi\left(\left[\begin{array}{cccc}a\cos\theta&as\cos\theta-\frac{1}{a}\sin\theta&0&0\\a\sin\theta&\frac{1}{a}\cos\theta+as\sin\theta&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\right)=\left(\theta,a,s\right).$$

Then φ *defines a local diffeomorphism at the identity of* H *and there exists an open set* $\mathcal{O}$ *around the identity of* H *such that* $(\mathcal{O},\varphi)$ *is a smooth chart. Next, we fix a linear functional*

$$\lambda=\left[\begin{array}{cccc}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{array}\right]\in\mathfrak{n}^{*}.$$

To compute $Ad\left(h^{-1}\right)^{*}\lambda$*, we proceed as follows. Let* $M\left(\theta,a,s\right)$ *be the inverse transpose of the following matrix.*

$$\left[\begin{array}{cccc}\cos\theta&-\sin\theta&0&0\\\sin\theta&\cos\theta&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\left[\begin{array}{cccc}a&0&0&0\\0&1/a&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\left[\begin{array}{cccc}1&s&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right].$$

With straightforward calculations, we obtain

$$Ad\left(h^{-1}\right)^{*}\lambda = M\left(\theta,a,s\right)\lambda = \begin{bmatrix} 0 & 0 & a^{-1}\cos\theta + as\sin\theta & -a\sin\theta \\ 0 & 0 & a^{-1}\sin\theta - as\cos\theta & a\cos\theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The map $h \mapsto Ad\left(h^{-1}\right)^{*}\lambda$ *is clearly an immersion at the identity of* H. *Next, let* $P^{*} : \mathfrak{n}^{*} \to \mathfrak{n}^{*}$ *be a linear projection satisfying*

$$P^{*}\left(\begin{bmatrix} 0 & 0 & \frac{1}{a}\cos\theta + as\sin\theta & -a\sin\theta \\ 0 & 0 & \frac{1}{a}\sin\theta - as\cos\theta & a\cos\theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 & -a\sin\theta \\ 0 & 0 & \frac{1}{a}\sin\theta - as\cos\theta & a\cos\theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

With respect to this projection, we define Θ_{λ} *(in local coordinates) as follows*

$$\Theta_{\lambda}\left(\theta,a,s\right) = \left(\frac{-a^{2}s\cos\theta + \sin\theta}{a}, -a\sin\theta, a\cos\theta\right).$$

The Jacobian of Θ_{λ} *is given by*

$$\begin{bmatrix} \frac{1}{a}\cos\theta + as\sin\theta & -s\cos\theta - \frac{1}{a^{2}}\sin\theta & -a\cos\theta \\ -a\cos\theta & -\sin\theta & 0 \\ -a\sin\theta & \cos\theta & 0 \end{bmatrix}$$

and its determinant $a^{2}\cos\theta$ *is nonzero at* $(\theta,a,s) = (0,1,0)$. *Thus,* Θ_{λ} *is a local diffeomorphism at* $(0,1,0)$. *Using the following Haar measure* $a^{-3}d\theta da ds$ *on* H, *we obtain*

$$\mathbf{W}_{\lambda}\left(\xi\right) = \frac{1}{|\xi_{3}|\cdot\left(\xi_{2}^{2}+\xi_{3}^{2}\right)^{2}}.$$

Let $\mathcal{O}$ *be a relatively compact subset of* $SL\left(2,\mathbb{R}\right)$ *such that* Θ_{λ} *defines a diffeomorphism between* $\mathcal{O}$ *and its image. According to Theorem 1, for any continuous function* $\mathbf{f}$ *supported on* $\mathcal{O}$, *there exists a discrete set* $\Gamma_{\mathbf{f},\mathcal{O}} \subset G$ *such that* $\left\{\pi\left(\gamma\right)\mathbf{f} : \gamma \in \Gamma_{\mathbf{f},\mathcal{O}}\right\}$ *is a frame for* $L^{2}\left(SL\left(2,\mathbb{R}\right)\right)$.

3.3. **Construction of orthonormal bases.** We present below some examples illustrating Proposition 21. Our first example shares a striking resemblance with the Heisenberg group which plays a important role in Gabor analysis.

Example 31. *Let*

$$\mathfrak{n}=\sum_{k=1}^{3}\mathbb{R}X_k,\mathfrak{h}=\sum_{k=1}^{2}\mathbb{R}A_k$$

such that

$$\sum_{k=1}^{3}x_kX_k=\begin{bmatrix}0&0&0&x_1\\0&0&0&x_2\\0&0&0&x_3\\0&0&0&0\end{bmatrix}\text{ and }\sum_{k=1}^{2}a_kA_k=-\begin{bmatrix}0&a_1&a_2&0\\0&0&a_1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

Every element of $G=NH$ can be uniquely factored as

$$\exp\left(\sum_{k=1}^{3}x_kX_k\right)\exp\left(\sum_{k=1}^{2}a_kA_k\right)=\begin{bmatrix}1&0&0&x_1\\0&1&0&x_2\\0&0&1&x_3\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&-a_1&\frac{1}{2}a_1^2-a_2&0\\0&1&-a_1&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

In fact G is a step-three metabelian nilpotent Lie group which was also studied in [29] in the context of sampling theory for left-invariant spaces. Next, let $\lambda=X_1^*$ and define the linear projection $P:\mathfrak{n}\to\mathfrak{n}$ of rank two such that

$$P\left(\sum_{k=1}^{3}x_kX_k\right)=\sum_{k=2}^{3}x_kX_k.$$

Note that the center of the algebra $\mathfrak{g}=\mathfrak{n}+\mathfrak{h}$ is contained in the null-space of P and

$$\Theta_\lambda(t_1,t_2)=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\t_1&1&0\\\frac{1}{2}t_1^2+t_2&t_1&1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\t_1\\\frac{1}{2}t_1^2+t_2\end{bmatrix}.$$

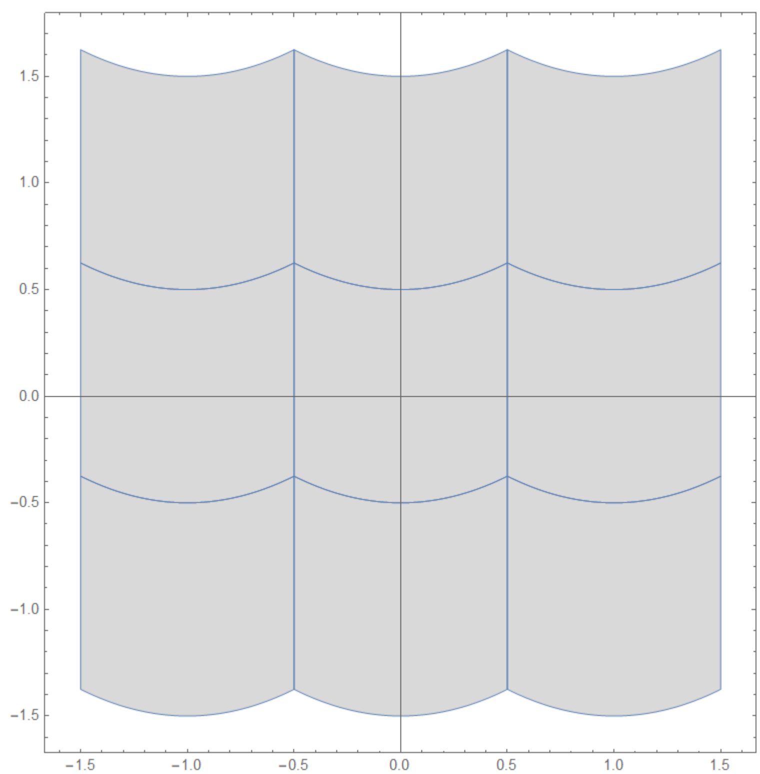
To simplify our presentation, we just write

$$\Theta_\lambda(t_1,t_2)=\left[t_1,\frac{1}{2}t_1^2+t_2\right]^T.$$

It is easy to verify that Θ_λ defines a diffeomorphism between $\mathbb{R}^2$ and its image. In fact, since

$$\det\begin{bmatrix}\dfrac{\partial(t_1)}{\partial t_1}&\dfrac{\partial(t_1)}{\partial t_2}\\\dfrac{\partial\left(\frac{1}{2}t_1^2+t_2\right)}{\partial t_1}&\dfrac{\partial\left(\frac{1}{2}t_1^2+t_2\right)}{\partial t_2}\end{bmatrix}=\det\begin{bmatrix}1&0\\t_1&1\end{bmatrix}=1,$$

it follows that $\mathbf{W}_\lambda(\xi) = 1$. *Moreover, each of the following sets:* $[-1/2, 1/2)^2$ *and* $\Theta_\lambda\left([-1/2,1/2)^2\right)$ *tiles* $\mathbb{R}^2$ *translationally by* $\mathbb{Z}^2$.



Tiling the plane by integral translation of $\Theta_\lambda\left([-1/2,1/2)^2\right)$

Put $\Gamma = \exp\left(\sum_{k=1}^{2} \mathbb{Z}A_k\right)\exp\left(\sum_{k=2}^{3}\mathbb{Z}X_k\right)$. *Appealing to Proposition 21, we conclude that*

$$\left\{\operatorname{ind}_N^{NH}\left(\chi_{X_1^*}\right)(\gamma)\,1_{[-1/2,1/2)^2} : \gamma \in \Gamma\right\}$$

is an orthonormal basis for $L^2\left(\mathbb{R}^2, dt\right)$.

Our next example gives a construction of orthonormal bases on a class of nilpotent Lie groups.

Example 32. *Following the notation of Proposition 26, we let*

$$G = \left\{\begin{bmatrix} h & x \\ 0_n & \mathrm{Id}_n \end{bmatrix} : h \in K \text{ and } x \in \mathfrak{gl}(n,\mathbb{R})\right\}$$

such that

$$K=\left\{\left[\begin{array}{cccc}1 & a_{11} & \cdots & a_{1n}\\ & 1 & \ddots & \vdots\\ & & \ddots & a_{n1}\\ & & & 1\end{array}\right]:(a_{11},\cdots a_{n1})\in\mathbb{R}^{\frac{n^2-n}{2}}\right\}.$$

Here K is a simply connected nilpotent Lie group. In fact, the group G is also a simply connected nilpotent Lie group. Next, we consider the following linear functional

$$\lambda=\left[\begin{array}{cc}0_n & \mathrm{Id}_n\\ 0_n & 0_n\end{array}\right]\in\mathfrak{n}^*.$$

The map $h\mapsto Ad\left(h^{-1}\right)^*\lambda$ defines an immersion at the identity of H and the representation obtained by inducing χ_λ is irreducible. Let $P^*:\mathfrak{n}^*\to\mathfrak{n}^*$ be the linear projection given by

$$P^*\left(\left[\begin{array}{cc}0_n & \omega\\ 0_n & 0_n\end{array}\right]\right)=\left[\begin{array}{cc}0_n & \Pi(\omega)\\ 0_n & 0_n\end{array}\right]$$

where

$$\Pi\left(\left[\begin{array}{cccc}\lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n}\\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn}\end{array}\right]\right)=\left[\begin{array}{ccccc}0 & & & & \\ \lambda_{21} & 0 & & & \\ \lambda_{31} & \cdots & 0 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn-1} & 0\end{array}\right].$$

We describe $\Theta_\lambda:H\to\mathfrak{n}^*$ with respect to the projection P^* as follows

$$\Theta_\lambda(h)=\left[\begin{array}{cc}0_n & \Pi\left(\left(h^{-1}\right)^T\right)\\ 0_n & 0_n\end{array}\right].$$

Θ_λ is a polynomial function in the coordinates of H. Moreover, Θ_λ defines a global diffeomorphism between H and its image, and $\mathbf{W}_\lambda$ is merely the constant function 1. Appealing to the results in [28], it is not hard to verify that there exist $C,\mathbf{f},\Gamma_H,\Gamma_N$ satisfying Conditions 2.1 (2) (6) (7) and (8). By Proposition 21 Part (2), the representation $\pi=\mathrm{ind}_N^{NH}(\chi_\lambda)$ admits an orthonormal basis of the type $\mathcal{S}\left(|C|^{-1/2}\mathbf{f},\Gamma_H^{-1}\Gamma_N\right)$ for $L^2(H)$.

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