An Abstract Harmonic Analysis Approach to Gabor Analysis

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A Starting Point

- One reason why Gabor theory has attracted so much attention is because of its practical applications.
- One recent (not so popular) application of Gabor theory is found in the analysis of certain nilpotent Lie groups.

Sampling on the Heisenberg group

Theorem

(H. Führ, B. Currey, and A. Mayeli) Let $\mathbb{H}$ *be the 3-dimensional Heisenberg group. Then there exist an integer lattice subgroup* Γ*, and a left-invariant closed subspace* H *of* $L^2(\mathbb{H})$ *consisting of continuous functions such that for all* $f \in H$*,*

1. $\sum_{\gamma\in\Gamma} |f(\gamma)|^2 = \|f\|_H^2$.
2. *There exists a function* $S \in H$ *such that*

$$f(x) = \sum_{\gamma\in\Gamma} f(\gamma)\, S\left(\gamma^{-1}x\right)$$

with convergence both in $L^2(\mathbb{H})$*-norm and uniformly.*

Sampling on the Heisenberg group

Theorem

(B. Currey, and A. Mayeli) The Heisenberg group admits sampling spaces with the interpolation property.

Theorem

(O) Let N be a simply connected, connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ such that $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{b} \oplus \mathfrak{a}$, $\mathfrak{z}$ is the center of $\mathfrak{n}$, $\mathfrak{b} = \mathbb{R}Y_d \oplus \mathbb{R}Y_{d-1} \oplus \cdots \oplus \mathbb{R}Y_1, \mathfrak{a} = \mathbb{R}X_d \oplus \mathbb{R}X_{d-1} \oplus \cdots \oplus \mathbb{R}X_1$, $\mathfrak{z} \oplus \mathfrak{b}$ is a maximal commutative ideal of $\mathfrak{n}$, $[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{z}$, and $\det([X_i, Y_j])_{1 \leq i,j \leq d}$ is a non-trivial homogeneous polynomial defined over the ideal $[\mathfrak{n}, \mathfrak{n}]$. There exist band-limited spaces which are sampling subspaces of $L^2(N)$ with respect to some discrete set Γ.

The proofs of all these results mentioned, rely on the well-known **Density Condition** of Gabor systems.

Question: Can we use abstract harmonic analysis tools to study time-frequency analysis?

Some Background

Larry Baggett decomposed the Stone-von Neumann representation of the discrete Heisenberg group acting on $L^2(\mathbb{R})$ into a direct integral of representations. As a result, he proved:

Theorem

(L. Baggett, 1990) If $\alpha\beta > 1$, there is no $\phi \in L^2(\mathbb{R})$ such that

$$\left\{ e^{2\pi i \alpha j} \phi(t - \beta k) : (j, k) \in \mathbb{Z}^2 \right\}$$

spans $L^2(\mathbb{R})$.

Tight Weyl-Heisenberg Frames with Integer Oversampling

Let L be an integer greater than one. Let $\Gamma = \mathbb{Z} \times \mathbb{Z} \times (\mathbb{Z}/L\mathbb{Z})$ with group law

$$\left(n, k, \overline{\ell}\right)\left(n', k', \overline{\ell'}\right) = \left(n + n', k + k', \overline{\ell} + \overline{\ell'} + \overline{k'n}\right).$$

Then Γ admits a unitary representation π acting in $L^2(\mathbb{R})$ such that

$$\pi\left(n, k, \overline{\ell}\right) = e^{2\pi i \ell / L} T_k M_{n/L}$$

where T_k is a translation operator and $M_{n/L}$ is a modulation operator. Define abelian normal subgroup

$$\Gamma_1 = L\mathbb{Z} \times \mathbb{Z} \times (\mathbb{Z}/L\mathbb{Z}).$$

Theorem

(H. Führ, 2002) The representation π admits the following irreducible direct integral decomposition:

$$\int_{[0,1)\times\left[0,\frac{1}{L}\right)}^{\oplus} \mathrm{Ind}_{\Gamma_1}^{\Gamma}\left(\chi_\omega\right) d\omega.$$

Using this decomposition, H. Führ characterized functions $g \in L^2(\mathbb{R})$ such that $\pi(\Gamma) g$ is a Parseval frame.

Central Theme

I will present a generalization of Führ's ideas to higher dimensions with some applications.

A Few Relevant Papers

- The content of this talk is taken from
  - V. Oussa, **An Abstract Harmonic Analysis Approach to Gabor Theory**, preprint (2014).
- The content of this talk extends results in
  - H. Führ, **Abstract harmonic analysis of continuous wavelet transforms. Lecture Notes in Mathematics**, 1863. Springer-Verlag, Berlin, 2005
  - A. Mayeli, V. Oussa, **Regular Representations of Time Frequency Groups**, Math. Nachr. 1–21 (2014) / DOI 10.1002/mana.201300019
  - L. Baggett, **Processing a radar signal and representations of the discrete Heisenberg group**, Colloq. Math. 60/61 (1990), no. 1, 195–203.

Unitary Representations

- Let G be a locally compact group. A **unitary representation** of G is a homomorphism π from G into the group of unitary operators $\mathcal{U}(\mathbf{H})$ such that

$$x \longmapsto \pi(x)\, u, \;\; u \in \mathbf{H}$$

  is strongly continuous, and $\mathbf{H}$ is a Hilbert space.
- If π admits an invariant subspace that is non-trivial, then π is **reducible**, otherwise π is **irreducible**.

Induced Representations Part 1

- Let G be a locally compact group, and let K be a closed subgroup.
- Let $q : G \to G/K$ be the canonical quotient map, let φ be a unitary representation of the group K acting in a Hilbert space $\mathbf{H}$.
- Let

$$\mathsf{K}_1 = \left\{ \begin{array}{c} f : G \to \mathbf{H} : f \text{ is cont} \\ q\left(\text{support } f\right) \text{ is compact} \\ f\left(gk\right) = \varphi\left(k\right)^{-1} f\left(g\right) \text{ where } g \in G, k \in K \end{array} \right\}.$$

Induced Representations Part 2

- G acts on the space $\mathbf{K}_1$ by left translation.
- Assume G/K admits an **invariant measure** $d(gK)$. We construct a unitary representation of G by endowing $\mathbf{K}_1$ with

$$\langle f, f' \rangle = \int_{G/K} \langle f(g), f'(g) \rangle_{\mathbf{H}} \, d(gK) \text{ for } f, f' \in \mathbf{K}_1.$$

- Let $\mathbf{K}$ be the Hilbert completion of the space $\mathbf{K}_1$. The translation operators extend to unitary operators on $\mathbf{K}$ inducing a unitary representation: $\mathrm{Ind}_K^G(\varphi)$

$$\mathrm{Ind}_K^G(\varphi)(x) f(g) = f\left(x^{-1} g\right) \text{ for } f \in \mathbf{K}.$$

Direct Integrals

- Let $\{H_\alpha\}_{\alpha \in A}$ be a family of separable Hilbert spaces. The direct integral of this family the $\{H_\alpha\}_{\alpha \in A}$ with respect to μ is the space of functions f defined on A such that $f(\alpha)$ is an element of H_α and

$$\int_A \|f(\alpha)\|_{H_\alpha}^2 \, d\mu(\alpha) < \infty$$

  with additional *measurability conditions.*
- $f : A \to \bigcup_\alpha H_\alpha$ such that $f(\alpha) \in H_\alpha$ is called a **vector field** on A.
- The direct integral of the spaces H_α with respect to the measure μ is denoted by

$$\int_A^{\oplus} H_\alpha d\mu(\alpha).$$

Properties of Induction

- Inducing in stages: If G_1 is a subgroup of G containing a subgroup K then

$$\mathrm{Ind}_K^G(\varphi) \simeq \mathrm{Ind}_{G_1}^G\left(\mathrm{Ind}_K^{G_1}(\varphi)\right).$$

- The induction of a direct integral is an direct integral of induced representations

$$\mathrm{Ind}_K^G\left(\int_A^{\oplus} \varphi_x \, dx\right) \simeq \int_A^{\oplus} \left[\mathrm{Ind}_K^G(\varphi_x)\right] dx.$$

Gabor Representation

Let B be a non-singular matrix with real entries. Define the shift and frequency-shift operators acting in $L^2\left(\mathbb{R}^d\right)$

$$T_k f\left(t\right) = f\left(t-k\right) \text{ for } k \in \mathbb{Z}^d$$
$$M_l f\left(t\right) = e^{-2\pi i \langle l,t\rangle} f\left(t\right) \text{ for } l \in B\mathbb{Z}^d$$

and define $\Gamma < \mathcal{U}\left(L^2\left(\mathbb{R}^d\right)\right)$

$$\Gamma = \left\langle T_k, M_l : (k,l) \in \mathbb{Z}^d \times B\mathbb{Z}^d \right\rangle.$$

Some Facts

- Γ is a nilpotent group with $2d$ generators (existence of lower central series of finite length).
- $[\Gamma, \Gamma] = \left\{ e^{2\pi i \langle l, Bk \rangle} \in \mathbb{T} : (k, l) \in \mathbb{Z}^d \times B\mathbb{Z}^d \right\} \subset \mathbb{T}$.
- If B has irrational entries, $[\Gamma, \Gamma]$ is an infinite central subgroup of Γ.
- If B has rational entries then $[\Gamma, \Gamma]$ is a finite subgroup of the torus.

Gabor Representation

Assume B is a **rational matrix**. So, $[\Gamma, \Gamma]$ is isomorphic to some finite cyclic group $\mathbb{Z}_m$. Then, there is a semi-direct product group $\left(\mathbb{Z}_m \times B\mathbb{Z}^d\right) \rtimes \mathbb{Z}^d$ such that the representation

$$\pi : \left(\mathbb{Z}_m \times B\mathbb{Z}^d\right) \rtimes \mathbb{Z}^d \to \Gamma$$
$$(j, Bl, k) \mapsto e^{\frac{2\pi j i}{m}} M_{Bl} T_k$$

is unitary, faithful and acts in $L^2\left(\mathbb{R}^d\right)$.

A Special Subgroup

Let $B^\star$ denote the inverse transpose of the matrix B. Put

$$\Gamma_0 = \left\langle \begin{array}{c} [\Gamma, \Gamma], M_l, T_k : \\ l \in B\mathbb{Z}^d, k \in B^\star\mathbb{Z}^d \cap \mathbb{Z}^d \end{array} \right\rangle$$
$$\simeq \mathbb{Z}_m \times B\mathbb{Z}^d \times \left(B^\star\mathbb{Z}^d \cap \mathbb{Z}^d\right).$$

- Γ_0 is a normal abelian subgroup of Γ of finite index.
- There exists an invertible matrix A such that $A\mathbb{Z}^d = B^\star\mathbb{Z}^d \cap \mathbb{Z}^d$.

Left Regular Representation

The left regular representation is a unitary representation which acts in $l^2(\Gamma)$ as follows:

$$L(\gamma)a(\alpha) = a(\gamma^{-1}\alpha)$$

for any sequence $a \in l^2(\Gamma)$.

Theorem

Assume that B is a rational matrix. The left regular representation of $\left(\mathbb{Z}_m \times B\mathbb{Z}^d\right) \rtimes \mathbb{Z}^d$ is decomposed as follows:

$$L \simeq \oplus_{k=0}^{m-1} \int_{\frac{\mathbb{R}^d}{B^\star\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \chi_{(k,t)} \, dt. \tag{1}$$

Moreover, the measure dt in (1) is a Lebesgue measure class, and in general (1) is ***not an irreducible*** *decomposition.*

Proof.

Let e be the identity element in Γ.

$$\begin{aligned} L &\simeq \mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\mathrm{Ind}_{\{e\}}^{\Gamma_0}(1)\right) \\ &\simeq \mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(\int_{\widehat{\Gamma}_0}^{\oplus} \chi_t \; dt\right) \\ &\simeq \int_{\widehat{\Gamma}_0}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\chi_t) \; dt. \end{aligned}$$

Next, identifying $\widehat{\Gamma}_0$ with $\mathbb{Z}_m \times \frac{\mathbb{R}^d}{B^\star\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}$, it follows that

$$L \simeq \oplus_{k=0}^{m-1} \int_{\frac{\mathbb{R}^d}{B^\star\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\chi_{(k,t)} \; dt.$$

$\square$

Theorem

Let us suppose that B is a rational matrix. Then

$$\pi \simeq \int_{\frac{\mathbb{R}^d}{\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{B^\star \mathbb{Z}^d}}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \chi_{(1,\sigma)} \, d\sigma.$$

This is an irreducible decomposition which is realized as acting in

$$L^2\left(\frac{\mathbb{R}^d}{\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{B^\star \mathbb{Z}^d}, \mathrm{l}^2\left(\frac{\Gamma}{\Gamma_0}\right)\right).$$

Sketching the Proof

1. Let $\Gamma_1 = \left\langle [\Gamma, \Gamma], M_l : l \in B\mathbb{Z}^d \right\rangle$.
2. $\pi \simeq \int_{\frac{\mathbb{R}^d}{\mathbb{Z}^d}}^{\oplus} \left[\mathrm{Ind}_{\Gamma_1}^{\Gamma} \chi_{(1,-t)}\right] dt$ (the intertwining map here is a periodization operator)
3. $$\begin{aligned} \mathrm{Ind}_{\Gamma_1}^{\Gamma} \chi_{(1,-t)} &\simeq \mathrm{Ind}_{\Gamma_0}^{\Gamma} \left[\mathrm{Ind}_{\Gamma_1}^{\Gamma_0} \left(\chi_{(1,-t)}\right)\right] \\ &\simeq \mathrm{Ind}_{\Gamma_0}^{\Gamma} \left[\int_{\frac{\widehat{\Gamma_0}}{\Gamma_1}}^{\oplus} \chi_{(1,-t,\xi)} \, d\xi\right] \\ &\simeq \int_{\frac{\widehat{\Gamma_0}}{\Gamma_1}}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \chi_{(1,-t,\xi)} d\xi. \end{aligned}$$
4. $\pi \simeq \int_{\frac{\mathbb{R}^d}{\mathbb{Z}^d} \times \frac{\mathbb{R}^d}{B^\star \mathbb{Z}^d}}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \chi_{(1,\sigma)} d\sigma.$

Theorem

Assume that B is an rational matrix. L is equivalent to a direct sum of $\mathrm{card}\left([\Gamma,\Gamma]\right)$*-many non-equivalent highly reducible representations such that*

$$L \simeq \bigoplus_{k=0}^{\mathrm{card}([\Gamma,\Gamma])-1} L_k \text{ where } L_k \simeq \int_{\frac{\mathbb{R}^d}{B^\star\mathbb{Z}^d}\times\frac{\mathbb{R}^d}{A^\star\mathbb{Z}^d}}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\chi_{(k,t)}\, dt.$$

Moreover, the representations L_k are disjoint from π whenever $k \neq 1$, and the Gabor representation π is equivalent a subrepresentation of the subrepresentation L_1 of L if and only if $|\det B| \leq 1$.

Application to Time-Frequency Analysis

Let π be a unitary representation of a locally compact group G, acting in $\mathcal{H}$. We say that π is **admissible**, if and only if there exists some vector $\phi \in \mathcal{H}$ such that

$$W_\phi^\pi : \mathcal{H} \to L^2(G), \; W_\phi^\pi \psi(x) = \langle \psi, \pi(x)\phi \rangle$$

is an isometry of $\mathcal{H}$ into $L^2(G)$.

Notice that if G has the discrete topology, then π is admissible if and only if there exists some vector $\phi \in \mathcal{H}$ such that $\pi(G)\phi$ is a Parseval frame.

Application to Time-Frequency Analysis

Theorem

A representation of the group Γ is admissible if and only if the representation is equivalent to a subrepresentation of the left regular representation of Γ.

Application to Time-Frequency Analysis

Theorem

Let B be a rational matrix. There exists a vector $g \in L^2\left(\mathbb{R}^d\right)$ such that the system $\left\{M_l T_k g : l \in B\mathbb{Z}^d, k \in \mathbb{Z}^d\right\}$ is a Parseval frame if and only if $\left|\det B\right| \leq 1$.

Application to Time-Frequency Analysis

There exists a unitary operator

$$\mathfrak{A} : \int_{\Sigma}^{\oplus} \left[\oplus_{k=1}^{\ell(\sigma)} l^2 \left(\frac{\Gamma}{\Gamma_0} \right) \right] d\sigma \to L^2 \left(\mathbb{R}^d \right)$$

which intertwines the representations

$$\int_{\Sigma}^{\oplus} \left[\oplus_{k=1}^{\ell(\sigma)} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \left(\chi_{(1,\sigma)} \right) \right] d\sigma \text{ with } \pi$$

Σ is a subset of a measurable transversal of $\frac{\mathbb{R}^d}{B^\star \mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^\star \mathbb{Z}^d}$.

Characterization of Parseval frames

Theorem

Let us suppose that $|\det B| \leq 1$. *Then*

$$\pi \simeq \int_{\Sigma}^{\oplus} \left[\oplus_{k=1}^{\ell(\sigma)} \mathrm{Ind}_{\Gamma_0}^{\Gamma} \left(\chi_{(1,\sigma)} \right) \right] d\sigma$$

with $\ell(\sigma) \leq |\det A|$ $d\sigma$*-almost everywhere. Moreover,* $\pi(\Gamma) f$ *is a Parseval frame in* $L^2(\mathbb{R}^d)$ *if and only if* $f = \mathfrak{A}\left(a(\sigma)_{\sigma \in \Sigma}\right)$ *such that for* $d\sigma$*-almost every* $\sigma \in \Sigma$, *and for* $i, j \in \{1, \cdots, \ell(\sigma)\}$ *we have*

$$\|a(\sigma)(i)\|_{l^2\left(\frac{\Gamma}{\Gamma_0}\right)} = 1 \text{ and } \langle a(\sigma)(i), a(\sigma)(j) \rangle_{l^2\left(\frac{\Gamma}{\Gamma_0}\right)} = 0.$$

Concluding Remarks

- For the case where B has some irrational entries, the representation theoretic tools used to obtain our results break down.
- One major problem is that the underlying group displays some major pathological behavior (G. Folland gave a talk on these groups during the last FT)

Thank you for your attention