Lecture 21: Robertson-Walker Dynamics

Note Title 4/18/2011

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dx^2}{1 - \frac{kx^2}{R^2}} + x^2\, d\Omega^2\right] \qquad k = +1, 0, -1$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

$$x = S_k(r) = \begin{cases} R\sin\left(\frac{r}{R}\right) & k = +1 \\ r & k = 0 \\ R\sinh\left(\frac{r}{R}\right) & k = -1 \end{cases}$$

or $$ds^2 = -dt^2 + a^2(t)\left[dr^2 + S_k^2(r)\, d\Omega^2\right]$$

are the Robertson-Walker metrics - the homogeneous and isotropic solutions to the Einstein field Equations.

- $a(t)$ is the scale factor, whose functional form depends on the contents of the universe

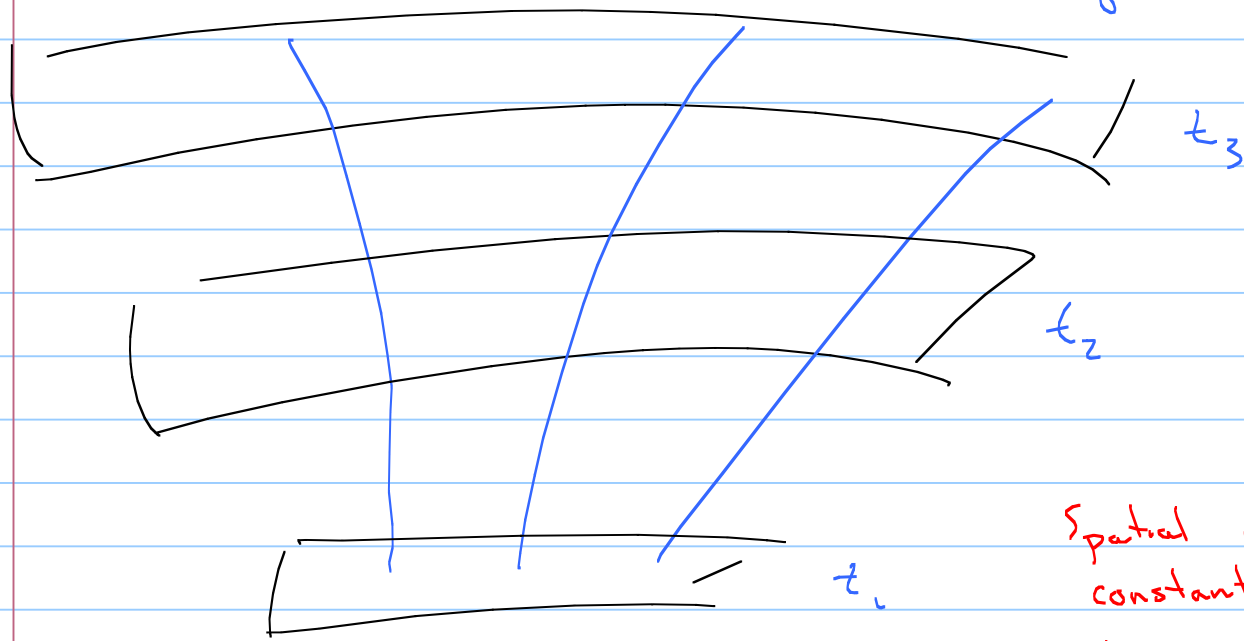
$$k = 0 \Rightarrow \quad ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\right]$$

This looks like Minkowski space-time (flat) except for the $a^2(t)$ in front of the spatial part.

$k = +1 \Rightarrow$ 3-sphere metric (closed)

$k = -1 \Rightarrow$ Hyperbolic geometry - constant negative curvature

(1) What does "t" mean? The co-moving time...



Spatial slices at constant co-moving times

The trajectories given by constant r, θ, ϕ are geodesics. Freely falling observers travel along paths

$$(\underbrace{t}_{\text{varies}}, \underbrace{r_0, \theta_0, \phi_0}_{\text{constant.}}) \quad \text{or} \quad (t, r_1, \theta_1, \phi_1)$$

(2) How far apart are 2 freely falling galaxies?

At a given time (on one of the sheets), put one galaxy at $r=0$, the other at r...

Metric proper distance $\rightarrow$ $$d_p(t) = a(t)\int_0^r dr = a(t)\, r$$

But $r(x)$ ...

$$d_p(t) = \begin{cases} a(t)\, R_0 \sin^{-1}\left(\frac{x}{R_0}\right) & k=+1 \\ a(t)\, x & k=0 \\ a(t)\, R_0 \sinh^{-1}\left(\frac{x}{R_0}\right) & k=-1 \end{cases}$$

where R_0 = Curvature radius of universe today at time t.

Note: $\dot{d}_p = \dot{a}\, r = \frac{\dot{a}}{a} d_p$

proper velocity of galaxy moving away $\rightarrow$ $v_p(t_0) = H_0\, d_p(t_0)$ $\boxed{t_0 = \text{today}}$

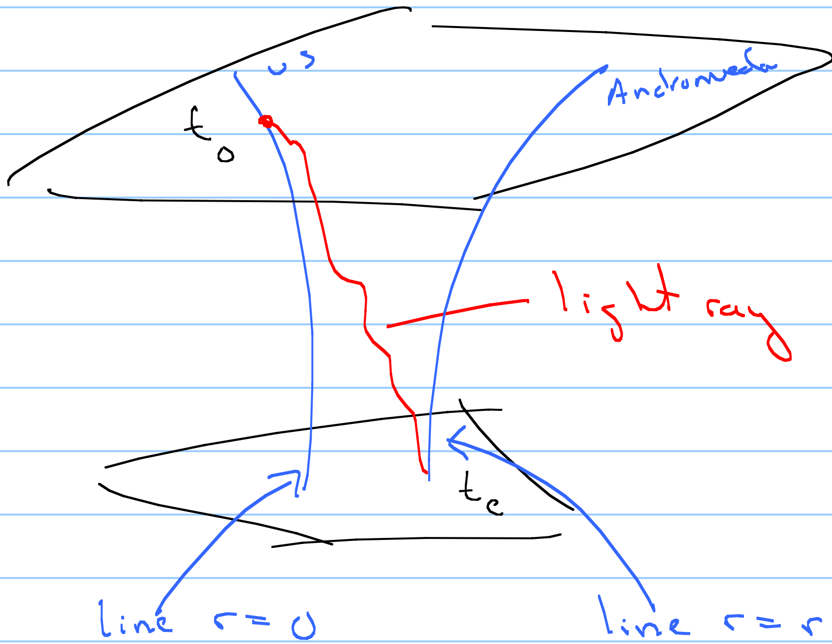
$H_0 = \left.\left(\frac{\dot{a}}{a}\right)\right|_{t=t_0}$ $\leftarrow$ Hubble constant.

If we define $d_H(t_0) \equiv \frac{c}{H_0} \approx 4300 \pm 400\ \text{Mpc}$

$\hookrightarrow$ objects w/ $d_p > d_H$ have $v_p > c$ ...

But, nothing is actually moving, this is a false speed due to the stretching of space.

③ What happens to light signals?



$ds^2 = 0 \ldots$

$c^2 dt^2 = a^2(t)\, dr^2$

$$\underbrace{\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_r^0 dr}$$

once you know $a(t)$, can do a lot w/ this.

Now take a 2nd light signal... emitted at $t = t_e + \frac{\lambda_e}{c}$

Because of $a(t)$, $\lambda_0 \neq \lambda_e$... the light signal will be received at $t_0 + \frac{\lambda_0}{c}$

$$\int_r^0 dr = \int_{t_e + \frac{\lambda_e}{c}}^{t_0 + \frac{\lambda_0}{c}} \frac{dt}{a(t)} = \int_{t_e}^{t_0} \frac{dt}{a(t)}$$

$$\left.\begin{aligned} - \int_{t_e + \frac{\lambda_e}{c}}^{t_0} \frac{dt}{a(t)} = - \int_{t_e + \frac{\lambda_e}{c}}^{t_0} \frac{dt}{a(t)} \end{aligned}\right\}$$ Subtract from both sides of above

$$\int_{t_e}^{t_e + \frac{\lambda_e}{c}} \frac{dt}{a(t)} = \int_{t_0}^{t_0 + \frac{\lambda_0}{c}} \frac{dt}{a(t)}$$

← over short time $\frac{\lambda_0}{c}$ or $\frac{\lambda_e}{c}$

$a(t) = \text{constant}$

$$\frac{1}{a(t_e)} \int_{t_e}^{t_e+\frac{\lambda_e}{c}} dt = \frac{\lambda_e}{c\, a(t_e)} = \frac{1}{a(t_0)} \int_{t_0}^{t_0+\frac{\lambda_0}{c}} dt = \frac{\lambda_0}{c\, a(t_0)}$$

$$\frac{\lambda_e}{a(t_e)} = \frac{\lambda_0}{a(t_0)} \; ; \quad z \equiv \frac{\lambda_0 - \lambda_e}{\lambda_e}$$

$$\boxed{1 + z = \frac{a(t_0)}{a(t_e)} = \frac{1}{a(t_e)}}$$

where $a(t_0) = 1$ is custom.

(4) What is $a(t)$?

All the "stuff" in the universe will have an energy density $\varepsilon(t)$ & Pressure $P(t)$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{\kappa c^2}{R_0^2 a^2}$$ ← Friedmann equation

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0$$ ← fluid eqn

$$P = \omega \varepsilon$$ ← Eqn of State

Range of w ...

$w = 0$ - matter

$w = -1$ - cosmological constant

$w = \frac{1}{3}$ - radiation

Energy densities scale w/ $a(t)$

$$\varepsilon_m = \frac{\varepsilon_{m,0}}{a^3} \quad \text{- for matter}$$

$$\varepsilon_r = \frac{\varepsilon_{r,0}}{a^4} \quad \text{- for radiation}$$

$$\varepsilon_\Lambda = \varepsilon_{\Lambda,0} \text{ (constant)} \quad \text{- for cosmological constant}$$

Define $H \equiv \frac{\dot{a}}{a}$ ... Friedmann eqn is

$$H^2(t) = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2 a^2(t)}$$

Now: $\frac{\kappa}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_0 - 1)$

where Ω_0 is the density parameter

$$\Omega(t) = \frac{\varepsilon(t)}{\varepsilon_c(t)} \quad \text{today } (t_0, \Omega_0)$$

for critical density $\varepsilon_c(t) = \frac{3c^2}{8\pi G} H^2(t)$

(If $\Omega = 1$, $\kappa = 0$, or universe is flat. $\Omega > 1$, universe is closed.)

So

$$H^2(t) = \frac{8\pi G}{3c^2}\,\varepsilon(t) - \frac{H_0^2}{a^2(t)}\,(\Omega_0 - 1)$$

$$\vdots$$

$$\frac{\dot{a}^2}{a^2 H_0^2} = \frac{H^2(t)}{H_0^2} = \frac{\varepsilon(t)}{\varepsilon_{c,0}} + \frac{1-\Omega_0}{a^2(t)} \quad +$$

use $\varepsilon = \varepsilon_m + \varepsilon_r + \varepsilon_\Lambda = \dfrac{\varepsilon_{m,0}}{a^3} + \dfrac{\varepsilon_{r,0}}{a^4} + \varepsilon_{\Lambda,0}$

and define $\Omega_{r,0} = \dfrac{\varepsilon_{r,0}}{\varepsilon_{c,0}}$, $\Omega_{m,0} = \dfrac{\varepsilon_{m,0}}{\varepsilon_{c,0}}$

$\Omega_{\Lambda,0} = \dfrac{\varepsilon_{\Lambda,0}}{\varepsilon_{c,0}}$

Also $\Omega_0 = \Omega_{m,0} + \Omega_{r,0} + \Omega_{\Lambda,0}$

$+$ becomes... $H_0^{-1}\,\dot{a} = \left[\dfrac{\Omega_{r,0}}{a^2} + \dfrac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0}\,a^2 + (1-\Omega_0)\right]^{1/2}$

or

$$\boxed{H_0 t = \int_0^a \frac{da}{\left[\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0}a^2 + (1-\Omega_0)\right]^{1/2}}}$$

↳ integrate it, invert it for $a(t)$

See Ryden "Introduction to Cosmology" for examples

Today's evidence suggests universe is matter-Λ dominated and nearly flat...

$$\Omega_{\Lambda,0} = 1 - \Omega_{m,0} \qquad \Omega_{r,0} \approx 0$$

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + (1 - \Omega_{m,0})$$

← can replace $\frac{1}{a^3} = \frac{1}{(1+z)^3}$

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(5) Which universe is ours?

Pretty solid evidence

$$\underline{\Omega_{r,0} \simeq 8.4 \times 10^{-5}}$$

$$\left.\begin{array}{l} \Omega_{bary,0} \approx .04 \\ \Omega_{dm,0} = .26 \end{array}\right\} \underline{\Omega_{m,0} = .3}$$

$$\underline{\Omega_{\Lambda,0} = .7}$$

(6) So dynamics of universe is what?

<u>Note</u> $\varepsilon_\Lambda = \varepsilon_{\Lambda,0}$ is constant

$\varepsilon_m = \frac{\varepsilon_{m,0}}{a^3}$ so it is getting smaller

$\varepsilon_r = \frac{\varepsilon_{r,0}}{a^4}$ so it is already really small.

The <u>future</u> is dominated by ε_Λ

The far past is dominated by radiation

$t_{rm} = 4.7 \times 10^4$ yr ← when radiation & matter had equal contributions

$t_{m\Lambda} = 9.8$ Gyr ← when matter & Λ had equal contributions

$t_0 = 13.5$ Gyr ← Now