Light Rays in the Kerr solution to Einstein's Equation

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The following is an investigation into the paths of light rays around a black hole as described by the Kerr solution to Einstein's equation. Basic equations of motion were derived based on the Kerr metric, using numeric and non-numeric equation solvers. After complete equations of motion were determined, plots of light ray paths were made under a variety of different circumstances. To better understand the contributions made by each variable, plots of light ray paths were made varying the initial angle ψ and holding constant the mass m and angular acceleration a. Similar plots were made varying a and m. A final plot was made simulating a basic gravitational lensing scenario.

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1 Introduction

In 1916, Albert Einstein proposed a new theory of gravitation [1] - one that solved many problems in classical physics and better described the phenomenon of gravity than Newtonian theory. One of the more interesting aspects of general relativity is that it predicted the existence of black holes decades before evidence for them was observed.

The following paper will examine relativistic treatment of black holes and predicted paths of light rays around them. Before examining any results of general relativity, we will review the mathematical methods neccesary to derive equations from basic information about the black hole. This information about a black hole is contained in a metric - a solution to Einstein's equation in general relativity. Of the solutions found to Einstein's equations, the Kerr solution (applying to a rotating black hole) will be used, since it is expected that black holes are rotating [2]. We will be able to produce plots of light rays around black holes and will find it of particular interest to examine the contributions made to these plots by different variables. Finally, we will produce a plot that simulates a basic case of gravitational lensing, a field of study that exemplifies the practical use of general relativity.

Substantial work has been done on a variety of aspects of general relativity and gravitational lensing. An exact lens equation has been produced based on the Schwarzschild spacetime that proved to be substantially more accurate than thin-lens approximations [3]. Off-axis images called "relativistic images" have been studied in cases of gravitational lensing [4] in the Schwarzschild spacetime. These images are maps of sources from a variety of different directions and are only seen very close to the black hole ($r = 3m$). Both of these projects involved the Schwarzschild spacetime – a spacetime describing a non-rotating, uncharged black hole that we will discuss further in sections below. Our primary investigation will be of a Kerr black hole – one with no net charge, but is rotating.

1.1 Physics Before Relativity

At the end of the 19th century, physics was grouped into three fields: what we now refer to as classical mechanics, thermodynamics and electomagnetic theory. Electromagnetic theory had been reduced to Maxwell's equations, and the speed of light (c) had been determined to be some finite quantity. The question this presented was: who would observe this speed of c? If an observer at rest observed light travelling at c, then Newtonian mechanics predicted that an observer moving in the opposite direction of the light would measure its velocity as some quantity greater than c. Likewise, observers in other frames could say the speed of light is less than the c observed "at rest". The situation motivates us to ask what is meant by "at rest"? If some observers and some reference frames are said to be moving and others not moving, it would seem their motion must be relative to some "rest frame".

A proposed solution to this problem was the concept of ether. Ether was predicted to be a gas-like substance that permeated the universe, and was the medium through which light traveled. This ether was thought to be at rest, and all objects – observers, stars, planets – were all moving with respect to it. Accordingly, an observer at rest with respect to the ether would measure the speed of light as c, and observers in moving frames would measure some greater or lesser speed.

According to the ether hypothesis, an observer in the Earth's frame of reference should detect an "ether wind" passing over and around the planet. The ether hypothesis was tested by timing how long it took for light to travel a distance "into" the ether wind and how long it took for light to travel the same distance perpendicular to the wind. If the ether hypothesis was valid, the elapsed time into the wind should have been greater than the time perpendicular to the wind. The concept behind this experiment was used in the Michelson-Morely interferometer, where slight differences in travel time would produce deconstructive interference between two beams of light.

However, these experiments disproved the ether hypothesis - travel times for the two paths were found to be the same and deconstructive interference was not observed in the Michelson-Morely experiment. With the ether hypothesis shown to be invalid, the question of a preferred frame was left without an adequate answer for some time.

1.2 Special Relativity

In 1905, Albert Einstein published his work on what he called the special theory of relativity [5]. This theory was designed so that the laws of physics are the same for observers in different inertial frames, regardless of their velocities. In addition, observations from one frame could be transformed to get those from a second frame. This solved the problem of a "preferred frame", since these transformations were designed so that the speed of light would be measured as the same value in all inertial frames. These are called the Lorentz Transformations and are shown below for translation between two systems of coordinates $(x,\ y,\ z,\ t)$ and $(\tau,\ \xi,\ \eta,\ \zeta)$ moving at some velocity with respect to each other in the x direction [6]:

$$\tau = \beta(t - vx/c^2) \qquad \xi = \beta(x - vt) \tag{1}$$
$$\eta = y \qquad \zeta = z$$

where

$$\beta = 1/\sqrt{1 - v^2/c^2}.$$

In preserving c, these transformations predict different observers to measure length and time differently. As an observer's velocity increases with respect to an object, the object will be observed to contract along the direction of his motion. Similarly, the same observer moving at a significant fraction of c with respect to a clock will observe time passing more slowly. These two phenomenon are described in expressions for length contraction and time dilation:

$$(Length\ \ Contraction) \qquad L = L_0\sqrt{1 - v/c^2} \tag{2}$$

$$(Time\ \ Dilation) \qquad T = \frac{T_0}{\sqrt{1 - v/c^2}}. \tag{3}$$

This was a shocking development for other physicists, and it took some convincing for people to accept the predictions made by special relativity. The idea that time (heretofore thought of as a reliable constant) was now observer-dependent was central to special relativity. This leads to a second new concept presented by Einstein: simultaneity is no longer universally agreed upon. Two events that are simultaneous in one inertial frame are not simultaneous in another that is moving with respect to the first [7]. As with most of the unusual predictions made by special relativity, this effect is only observable if the reference frames in question are moving at some significant fraction of the speed of light with respect to each other.

As crucial as this theory was, special relativity begged the question “how can we relate observations from two different *non-inertial* frames?” The answer would come ten years later in the form of general relativity.

1.3 General Relativity

In a general sense, special relativity proved that physics "worked" the same way in different inertial frames. If observations between frames could not be related, it could be said that different laws govern motion depending on the nature of the observation frame. This conclusion had been avoided for inertial frames, and Einstein hoped it could be likewise avoided for non-inertial frames [1].

Einstein's general theory of relativity was based on his idea that being in a gravitational field is equivalent to and indistinguishable from being accelerated. If an observer is standing in an elevator on Earth and subject to Earth's gravity, he will feel the same sensations as an observer deep in space in an elevator with upward acceleration g. To clarify this example, let us consider the case of each observer dropping a ball in his elevator.

If Einstein is correct in his proposition, each ball must fall to the bottom of the elevator in the same manner for both observers. In the elevator at rest on Earth, the motion of the ball is going to involve the gravitational mass of the ball. The gravitational mass is the mass referred to in considering potential energy due to gravity ($U = m_g gh$) or gravitational force ($F = G\frac{m_1 m_2}{r^2}$). It can be thought of as a measure of an object's susceptibility to gravity. The motion of the ball in the elevator being accelerated upward in space involves the inertial mass of the ball. The inertial mass is the mass referred to in equations concerning kinetic energy ($K = \frac{1}{2}m_i v^2$) and force ($F = m_i a$). Postulating $m_g = m_i$ is referred to as the Weak Equivalence Principle (WEP). Previously, values for these two masses were experimentally determined to be extremely close, if not equal [8].

In order for Einstein's proposed equivalence to hold, the two balls must behave in the same manner, obey the same laws, and the WEP must hold. If the WEP did not hold, each ball would behave differently due to a difference in the two types of mass. Because the WEP is valid, Einstein was able to make a stronger statement: Bodies in a gravitational field behave as in the absence of a gravitational field if, in the latter case, the system of reference used is a uniformly accelerated coordinate system (instead of an inertial system) [9]. This is referred to as the Strong Equivalence Principle (SEP).

Let us consider a case where the man in the elevator in space walks to one side and shines a flashlight on the opposite wall. Maxwell's equations and special relativity established c as a finite constant, so we know it will take some amount of time for the light to travel from one side of the elevator to another. In this length of time, the elevator will move up some distance, causing the photons from the flashlight to hit the wall at a height lower than where they left. From the man's point of view, the light travels along a downward curved path. An observer at rest outside the elevator would see the light following a straight line to the opposite wall (see figure 1). If we recall that Einstein has asserted that experiencing an acceleration is the same as being in a gravitational field, then light should experience the same distortion of its path in the elevator at rest on earth. This means that the path of light will actually be bent by a gravitational field.

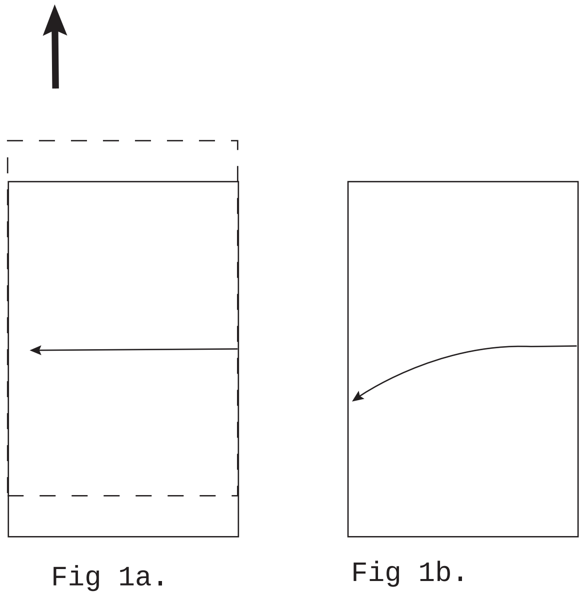

Figure 1: The gravitational effect on light rays. Fig 1a: Light is emitted from the right side of the elevator and moves to the left, as the elevator moves upward. From the perspective of an observer in the elevator, the light follows a curved path downward. Fig 1b: Einstein's Strong Equivalence Principle indicates that the same observation will be made in an elevator at rest in a gravitational field.

Now that we can see that light is subject to gravity in the same way as massive particles, its easy to see how the path of light can be "bent" in different amounts. Just as a comet's path is curved by the presence of the Sun, light will bend toward (and sometimes around) a massive object. The first example of this was observed and recorded by Arthur Eddington in 1919

[10]. Eddington took advantage of a solar eclipse taking place and was able to observe light from stars that passed very close to the Sun, and was effected by its gravitational field. This "bending" causes the observed stars to seem as if they are in a different location than would ordinarily be observed. By comparing his observations of the stars with star charts, Eddington could verify that the light from these stars was in fact being bent in exactly the way predicted by general relativity.

Further evidence of general relativity's validity was discovered in relation to Mercury's orbit. The planet's elliptical orbit is perturbed slightly by Venus, which causes the axis of the ellipse to rotate a certain amount every year. This is referred to as the precession of the perihilion, in reference to the perihilion, or point at which the planet is closest to the Sun. However, observations showed the precession to be substantially more than what the effect from Venus could cause. General relativity predicts that Mercury will follow a geodesic – a path through relativity's curved space-time – that matches the planet's observed path to within 1% error [11].

1.4 Black Holes

If a person on Earth throws a ball straight up in the air at some moderate velocity, the ball will decelerate, and then fall back toward the person. If he throws the ball with a greater velocity, the ball will take longer to slow and fall back down and it will reach a greater height before it falls back to Earth. If the ball is thrown hard enough, that is, with sufficient velocity, it will be able to break free of the Earth's gravitational field and continue off into space. The velocity necessary for this to occur is called the escape velocity. For all velocities less than this, the object will fall back to the source of gravity (here, the Earth).

If we were to make the Earth so massive that the ball could never escape, even if it were thrown at the speed of light, it could be called a black hole. A black hole is an object so massive that light within a certain distance (the event horizon) cannot escape its gravitational field. The escape velocity for a black hole would be greater than the speed of light, and therefore cannot be achieved.

By their very nature, black holes present a difficult problem: if light can never leave them, they can never be directly observed. However, the effects of a black hole on the region of space around it can be seen in several different ways. Black holes often draw in large amounts of interstellar material, forming an accretion disk. The accretion disk surrounds the black hole, with the material closest to the center moving the fastest as it nears the event horizon. As this material is accelerated, it gains energy, increases in temperature, and emits light. It is this light that can be observed to understand the nature of a black hole.

Perhaps the best data collected has been from a black hole at the center of galaxy NGC 4258. The 1.3 maser-emission line of H_2O has been observed precisely enough to calculate the velocity of the H_2O disc. The velocities are so high that they could only be caused by a black hole of mass 3.6 x $10^7 M_\odot$ [12, 13].

In addition, a black hole can be indirectly observed through its gravitational distortion of the light from nearby stars. Just as Eddington observed starlight being distorted by the Sun, information on black holes can be gathered by recording the way light is curved in its vicinity.

The region around a black hole can be simulated using different sets of equations that are solutions to Einstein's equations. There are four widely used solutions: one for non-rotating, uncharged black holes (Schwarzschild), one for non-rotating but charged (Reisner-Nordstrom), one for rotating but uncharged (Kerr), and one for rotating and charged (charged Kerr or Kerr-Newman). Because it is reasonable to expect that black holes will have no significant net charge and will be rotating [2, page 602], the Kerr solution provides the most realistic simulation of a black hole and will be the one we examine.

1.5 Gravitational Lensing

Because the path of a light ray can be "bent" in general relativity, the assumption that light travels in a straight line must be modified. Part of this change involves the way in which we gather information about an object by observing light coming from it. If the path of light can be bent, then the direction from which light is coming is not neccesarily in the direction of the source. This phenomenon of mass distorting an observed image is called gravitational lensing.

Although gravitational lensing can be deceiving in causing objects to seem to be in different locations than they really are (as Eddington showed), it can also help in making some observations. If a body of mass is exactly aligned with both the observer and source, the mass can function as a lens, and bend more of the light from the source toward the observer.

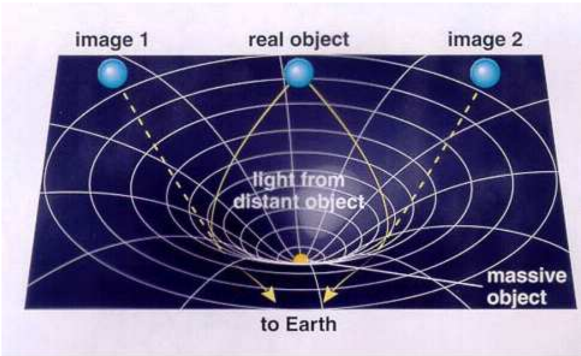

Figure 2: An example of simple gravitational lensing [15].

Because the alignment and mass of a gravitational lensing scenario can vary, different classes of phenomena are produced [14]. Depending on how close the mass is to the line of sight between observer and image, possible effects include observation of arcs (image, mass, and observer not aligned), multiple images (somewhat aligned), or rings (well-aligned). In the case where multiple images are observed, their separation in arcseconds is the basis for further categorization. Those instances where images are seperated by an angle on the order of an arcsecond are referred to as "macro-lensing" – the separation here is relatively large. Cases where the separation angle is on the order of microarcseconds are referred to as "micro-lensing" – the separation here is relatively small. In all cases of gravitational lensing, more light from the image reaches the observer than if the mass was not present. This will be demonstrated in section 6.

2 Mathematical Techniques

2.1 Metrics and Solutions to Einstein's Equation

In 1916, Einstein connected gravitational theory with geometry through a set of differential equations now referred to as Einstein's Equations. Each solution to Einstein's equation is a spacetime geometry in four dimensions, composed of a different surface, or manifold, on which motion takes place. These spacetimes can be uniquely described by expressions for the separation between two points on their manifolds. These expressions are called metrics.

In the cartesian xy plane, the metric is [11]

$$ds^2 = dx^2 + dy^2. \tag{4}$$

Special relativity first introduced the concept of spacetime, where we now have dimensions x, y, z, and t. Here, the metric is expressed as

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{5}$$

or, in general:

$$ds^2 = \sum_{\mu}\sum_{\nu} \eta_{\mu\nu} dx^{\mu} dx^{\nu}. \tag{6}$$

By convention, the sumation signs are often removed, and summation is implied over the repeated indices. The term $\eta_{\mu\nu}$ is a matrix element called the metric tensor, whose elements are the from the given expression for ds^2. In the case of special relativity (shown below), the scalars in front of c^2dt^2, dx^2, dy^2, and dz^2 become entries in the matrix. By convention, the symbol η is used for Minkowski space, a flat spacetime, and g is used for curved spacetimes like Kerr. For the special relativity metric, the metric tensor is of the form

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{7}$$

In addition to defining the distance between two points, physical metrics are also solutions to Einstein's equation, the end result of general relativity [16]:

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab} \tag{8}$$

Here, we introduce several new terms. G_{ab} is referred to as the Einstein tensor, and is defined to its right. R_{ab} and R are functions of the first and second derivative of the metric, g_{ab}. T_{ab} is called the stress-energy tensor, and represents the source of the curvature (energy or mass). We will examine the Minkowski metric – for a "flat" spacetime with zero mass and angular momentum, and the Kerr metric – for a spacetime with mass and angular momentum.

2.2 Lagrangian Techniques

In studying the geometry of black holes, we will find it helpful to use a physical quantity called the Lagrangian. The Lagrangian is defined as the difference between the kinetic and potential energy

$$\mathcal{L} = K - U. \tag{9}$$

We will find this expression can be manipulated to find basic equations of motion by using the Euler-Lagrange equation of motion [18, page 65]:

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0. \tag{10}$$

The Euler-Lagranage equation can be derived by examining some J that is a function of $y(x)$ and $y'(x)$:

$$J = \int_{x_1}^{x_2} f\{y(x), y'(x); x\} dx. \tag{11}$$

Later, we will use a Lagrangian that is a function of both coordinates (x, y, z, t) and their derivatives (x', y', z', t'). If we suppose that J is a function that connects two points x_1 and x_2 on a manifold, it would be useful to know which conditions yield a function of the shortest distance between the two points. Since Fermat's principle states that light will follow the path of shortest distance [19], it will follow the path of J with the shortest distance. The minimum distance along J can be found by determining where $\delta J = 0$.

$$\delta J = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial x}\delta x + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x}\delta x\right) = 0 \tag{12}$$

We will find it useful to make a substitution for the second term:

$$\frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\frac{\partial y}{\partial x}\right) - \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial x} \tag{13}$$

$$\Downarrow$$

$$\partial J = \int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\frac{\partial f}{\partial y'}\right]\frac{\partial y}{\partial x}\partial x + \int_{x_1}^{x_2} dx \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\frac{\partial y}{\partial x}\right). \tag{14}$$

Here, the second term is referred to as the "surface term", and is assumed to be zero. Since the entire right hand side must be zero, and neither $\frac{\partial y}{\partial x}$ nor ∂x can be zero, the term in brackets must be equal to zero. This leaves us with a general form of the Euler-Lagrange equation:

$$\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\frac{\partial f}{\partial y'} = 0. \tag{15}$$

The function we will apply this to will be $\frac{1}{2}g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$ - essentially the metric with a factor of one half inserted for convienience. Because this function is minimized in a way similar to the Lagrangian in particle motion, we will refer to it as the Lagrangian for our black hole systems. In the derivation above, y was a function of x. Since we will use x as a coordinate, we will take the derivatives in the Euler-Lagrange equation with respect to some parameter λ. Notation will be employed that use the term $\dot{t}$ as the derivative of t with respect to λ. It should be noted that only the potential and kinetic energy need to be known - information about specific forces or accelerations is unnecessary. As is shown in the following sections, the Euler-Lagrange equation can be used to find expressions for motion in each dimension.

Example: A One Dimensional Spring

We can show that the Lagrangian and Euler-Lagrange equation can be used to determine equations of motion in the following example. By starting with the Lagrangian of a spring oscillating in one dimension, we can derive the Hooke's Law for the spring's motion. We begin with the Lagrangian as defined above (equation 2.2). Classically, the kinetic and potenial energies for a spring are defined as: $K = \frac{1}{2}mv^2$ and $U = \frac{1}{2}kx^2$. This gives us the Lagrangian of a spring:

$$\mathcal{L} = \frac{1}{2}mv^2 - \frac{1}{2}kx^2. \tag{16}$$

By applying the Euler-Lagrange equation (15), we have:

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\dot{x}}\right) = -kx - \frac{d}{dt}(m\dot{x}) = 0, \tag{17}$$

and with a few algebraic steps, we are able to derive Hooke's Law:

$$-kx - m\ddot{x} = 0 \quad \Rightarrow \quad m\ddot{x} = -kx \tag{18}$$

$$F_{spring} = -kx. \tag{19}$$

3 Working in Spacetime Geometries

3.1 Minkowski Space

The simplest solution to Einstein's equation is Minkowski space. In Minkowski space, there is no mass and therefore no curvature. The metric here is [11]

$$ds^2 = c^2\dot{t}^2 - \dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2. \tag{20}$$

To simplify calculations, we will use units of c and G, that is, c and G will both be equal to one. This gives us a Lagrangian of

$$\mathcal{L} = +\frac{1}{2}\left(\dot{t}^2 - \dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2\right). \tag{21}$$

To make our coordinate system easier to work with and more useful, standard spherical coordinates can be altered using the following:

$$u = t - r \qquad l = 1/r. \tag{22}$$

These coordinate changes will he helpful in specifically looking at black holes. The use of l allows us to examine the case where $r = \infty$. In this case, l goes to zero, which is more managable than dealing with simply r going to ∞. By substituting into the spherical Lagrangian for t and r, we have

$$\mathcal{L} = \frac{1}{2}\left(\dot{u}^2 - 2\frac{\dot{u}\dot{l}}{l^2} - \frac{\dot{\theta}^2}{l^2} - \frac{1}{l^2}\sin\theta\dot{\phi}^2\right). \tag{23}$$

For our purposes, we will restrict our Lagrangian to the xy plane. By setting $\theta = \pi/2$ and simplifying, we have

$$\mathcal{L} = \frac{1}{2}\left(\dot{u}^2 - 2\frac{\dot{u}\dot{l}}{l^2} - \frac{\dot{\phi}^2}{l^2}\right). \tag{24}$$

In the case of light rays, which have zero mass, the Lagrangian is equal to zero. This is convienient, since it enables us to divide or multiply the Lagrangian by different terms, and it will still be equal to zero. We can simplify our Lagrangian by multiplying through by l^2:

$$\mathcal{L} = \frac{1}{2}\left(\dot{u}^2 l^2 - 2\dot{u}\dot{l} - \dot{\phi}^2\right). \tag{25}$$

In order to derive expressions for the coordinates l, u and ϕ, we will use the Euler-Lagrange equations of motion (15), as described above. Applying this to our Lagrangian, we can quickly find that there is some constant change in ϕ:

$$0 - \frac{d}{d\lambda}(-\dot{\phi}) = 0 \quad \Rightarrow \quad \dot{\phi} = c_1. \tag{26}$$

We can repeat this process to find a similar, although slightly more complicated expressions for l:

$$\dot{u}^2 l - \frac{d}{d\lambda}\left(-\dot{u}\right) = 0 \tag{27}$$

$$\dot{u}^2 l + \ddot{u} = 0, \tag{28}$$

and u:

$$0 - \frac{d}{d\lambda}\left(\dot{u}l^2 - \dot{l}\right) = 0 \tag{29}$$

$$\dot{u}l^2 - \dot{l} = c_2 \;\Rightarrow \dot{u} = \frac{c_2 + \dot{l}}{l^2}. \tag{30}$$

In order to understand the relationship between $\dot{u}$ and $\dot{\phi}$, we can insert our expression for $\dot{u}$ back into our Lagrangian (25):

$$\dot{u}l^2 = c_2 + \dot{l} \tag{31}$$

$$\mathcal{L} = \frac{1}{2}\left(\dot{u}\left(c_2 + \dot{l}\right) - 2\dot{u}\dot{l} - \dot{\phi}^2\right) = 0, \tag{32}$$

and solve for $\dot{u}$:

$$\dot{u} = \frac{\dot{\phi}^2}{c_2 - \dot{l}} \tag{33}$$

By equating our two expressions for $\dot{u}$, we can determine an expression for $\dot{l}$.

$$\frac{\dot{\phi}^2}{c_2 - \dot{l}} = \frac{c_2 + \dot{l}}{l^2} \tag{34}$$

$$\dot{l} = \pm\sqrt{{c_2}^2 - {c_1}^2 l^2} \tag{35}$$

Returning to $\dot{\phi}$, we can now express ϕ in terms of l and $\dot{l}$:

$$\dot{\phi} = \frac{d\phi}{d\lambda} \Rightarrow \frac{d\phi}{d\lambda}\frac{d\lambda}{dl} = \frac{d\phi}{dl} = \pm\frac{c_1}{\sqrt{{c_2}^2 - {c_1}^2 l^2}}. \tag{36}$$

This can also be written in integral form, considering some path from l_0 to l_f:

$$\phi = \phi_0 \pm \int_{l_0}^{l_f} \frac{c_1 dl}{\sqrt{{c_2}^2 - {c_1}^2 l^2}}. \tag{37}$$

We can manipulate $\dot{u}$ in a similar manner:

$$\dot{u} = \frac{du}{d\lambda} = \frac{du}{d\lambda}\frac{d\lambda}{dl} = \frac{du}{dl} = \pm\frac{c_2 + \dot{l}}{l^2\sqrt{{c_2}^2 - {c_1}^2 l^2}}, \tag{38}$$

and write this as an integral:

$$u = u_0 \pm \int_{l_0}^{l_f} \frac{c_2 \pm \sqrt{{c_2}^2 - {c_1}^2 l^2} dl}{l^2\sqrt{{c_2}^2 - {c_1}^2 l^2}}. \tag{39}$$

In evaluating these integrals, some attention should be paid to the restrictions on c_1 and c_2. In order for $\dot{l}$ to be a real number, the quantity $\pm\sqrt{{c_2}^2 - {c_1}^2 l^2}$ must be real, and so ${c_2}^2 - {c_1}^2 l^2$ must be positive. The relationships between c_1 and c_2 follow:

$${c_2}^2 - {c_1}^2 l^2 \geq 0 \qquad \Rightarrow \qquad c_1 \leq \frac{c_2}{l}. \tag{40}$$

The $\pm$ sign in the expression for $\dot{l}$ indicates the direction of a light ray: $+$ for moving away from the origin and $-$ for moving toward it. Our equation for ϕ can be adjusted accordingly, and different equations can now be used for incoming and outgoing rays.

Outgoing:

$$\phi = \phi_0 + \int_{l_0}^{l_f} \frac{c_1 dl}{\sqrt{{c_2}^2 - {c_1}^2 l^2}}, \tag{41}$$

Incoming:

$$\phi = \phi_0 - \int_{l_0}^{\tilde{l}} \frac{c_1 dl}{\sqrt{{c_2}^2 - {c_1}^2 l^2}} + \int_{\tilde{l}}^{l_f} \frac{c_1 dl}{\sqrt{{c_2}^2 - {c_1}^2 l^2}}. \tag{42}$$

The equation for an incoming ray has three parts: some initial value for ϕ, some portion as the ray approaches the center, and some portion after the ray has passed as close as it will to the center and is now outgoing. The quantity $\tilde{l}$ is the value for l closest to the origin, an can be calculated by solving for where $\dot{l} = 0$. The quantities l_o and l_f are initial and final values for l.

3.2 The Kerr Geometry

The black hole solution to Einstein's equations we will examine more closely is the Kerr, or rotating, uncharged geometry. The Kerr geometry is more likely to simulate an actual black hole, as astronomers are all but certain that they are rotating but have no significant net charge. The metric given by the Kerr geometry is written as [16, page 313]

$$ds^2 = -\left(\frac{\Delta - a^2 \sin\theta^2}{\Sigma}\right) dt^2 - \frac{2a\sin\theta^2(r^2 + a^2 - \Delta)}{\Sigma} dt d\phi \tag{43}$$

$$+ \left[\frac{(r^2 + a^2)^2 - \Delta a^2 \sin\theta^2}{\Sigma}\right] \sin\theta^2 d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2,$$

where

$$\Delta = r^2 + a^2 - 2mr,$$

$$\Sigma = r^2 + a^2 \cos\theta^2.$$

The Lagrangian we will be using will be the metric above, with some modification (as described in 2.2), and have the condition $\theta = \pi/2$.

$$\mathcal{L} = -\left(\frac{\Delta - a^2\sin\theta^2}{2\Sigma}\right)\dot{t}^2 - \frac{2a\sin\theta^2(r^2+a^2-\Delta}{2\Sigma}\dot{t}\dot{\phi} \tag{44}$$

$$+\left[\frac{(r^2+a^2)^2 - \Delta a^2\sin\theta^2}{2\Sigma}\right]\sin\theta^2 d\phi^2 + \frac{\Sigma}{2\Delta}\dot{r}^2 + \frac{\Sigma}{2}\dot{\theta}^2.$$

Since the Kerr solution is the geometry of a rotating black hole, the Lagrangian involves an angular momentum variable a and mass m. The process of determining equations of motion for light rays in the Kerr geometry once again begins with the Lagrangian.

The Euler-Lagrange equation (15) is used to determine equations for $\dot{t}$ and $\dot{\phi}$:

$$c_1 = -\frac{\Delta - a^2\sin\theta^2}{\Sigma}\dot{t} - \frac{a\sin\theta\,(r^2+a^2-\Delta)}{\Sigma}\dot{\phi}, \tag{45}$$

$$c_2 = -\frac{\left(r^2+a^2\right)^2 - \Delta a^2\sin\theta^2}{\Sigma}\sin\theta^2\dot{\phi} - \frac{a\sin\theta^2\,(r^2+a^2-\Delta)}{\Sigma}\dot{t}, \tag{46}$$

where c_1 and c_2 are contants. These equations can be solved for $\dot{\phi}$ and $\dot{t}$:

$$\dot{t} = -\frac{4\Sigma(a^4c_1 + a^3c_2 + 2a^2c_1r^2 + c_1r^4ac_2(r^2-\Delta) - a^2c_1\delta\sin\theta^2)}{\Delta(a^2+2r^2+a^2\cos 2\theta)^2} \tag{47}$$

$$\dot{\phi} = -\frac{4\Sigma(a(a^2c_1 + ac_2 + c_1(r^2-\Delta)) - c_2\Delta\csc\theta^2)}{\Delta(a^2+2r^2+a^2\cos 2\theta)^2}. \tag{48}$$

We will continue to work in a plane of the Kerr geometry, using $\theta = \pi/2$. An equation for $\dot{r}$ can be obtained by solving the Lagrangian for $\dot{r}$ and substituting in for $\dot{t}$ and $\dot{\phi}$:

$$\dot{r} = \sqrt{\frac{(a(ac_1+c_2)+c_1r^2)^2 - (ac_1+c_2)^2\Delta}{r^4}}. \tag{49}$$

All three of these equations are functions of s, the affine parameter represented by λ used in the Euler-Lagrange equations (10). A numerical differential equation solver be used to find expressions for r, t, and ϕ based on the equations for $\dot{r}$, $\dot{\phi}$, and $\dot{t}$. Due to the complexity of these functions, it is neccesary to use a numerical differential equation solver, and therefore the expressions produced for ϕ, r and t are interpolating functions. These interpolating functions closely resemble the actual function over a specified interval of s.

3.3 Orientation of Light Rays

In creating plots of the paths of light rays, we will need to control the direction in which the rays are initially travelling. As could be seen in 3.1, c_1 and c_2 are related to the initial values for $\dot{t}$ and $\dot{\phi}$. By changing c_2, we can change the initial direction of a light ray, but the exact relationship between some initial angle of orientation ψ and c_2 must be understood.

If we consider a unit vector $\hat{r}$ pointing toward the origin of a light ray and a unit vector $\hat{L}$ pointing in the initial direction of the light ray, the angle ψ between the two is the angle of initial orientation for the ray [17]. The two vectors and angle are related by the dot product $\hat{r} \cdot \hat{L}$. The dot product for the vectors L and r is defined in general as

$$\cos\psi = g_{ij}\hat{r}^i\hat{L}^j, \tag{50}$$

where g_{ij} is the metric. As mentioned before, summation is implied over repeated indices. In cartesian coordinates, the metric is a matrix with ones along the diagonal and zeroes elsewhere, so the dot product is simplified to the more familiar expression

$$\cos\psi = r_x L_x + r_y L_y + r_z L_z. \tag{51}$$

However, we will be working in Kerr space with a more complicated metric. Because we are concerned with the orientation of the light ray at $t = 0$, we will only need the spatial part of the metric, written as

$$g_{ij} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix}, \tag{52}$$

where

$$g_{11} = \frac{\Sigma}{\Delta},$$

$$g_{22} = \Sigma,$$

$$g_{33} = \left(\frac{(r^2 + a^2)^2 - \Delta a^2 \sin\theta^2}{\Sigma} \right).$$

The vector $\vec{L}$ represents the motion of a light ray - the combined effects of changes in the ϕ, θ, and r coordinates. Since we are working in the $\theta = \pi/2$ plane, we will consider the contributions from θ to be zero. This can be expressed as, $\vec{L} = (\dot{r}, 0, \dot{\phi})$. By dividing each vector component by $\mid L \mid$, we obtain the unit vector $\hat{L} = (\frac{\dot{r}}{|L|}, 0, \frac{\dot{\phi}}{|L|)}$. Because $\hat{r} = (\frac{\sqrt{\Delta}}{\sqrt{\Sigma}}, 0, 0)$, the first term in the dot product will be the only non-zero term. Our dot product now takes the form

$$\cos\psi = g_{11}\sqrt{\frac{\Delta}{\Sigma}}\frac{\dot{r}}{\mid \vec{L} \mid} = \sqrt{\frac{\Sigma}{\Delta}}\frac{\dot{r}}{L}. \tag{53}$$

By inserting our expressions for $\dot{\phi}$ and $\dot{r}$, ψ becomes a function of c_1, c_2, a, m and the initial values for θ, r, and ϕ. When we create create plots of the paths of light rays, we can specify an angle ψ along with values for m, a and c_1 to produce the appropriate value for c_2. We will find this to be critical in creating plots where ψ must stay the same, but a or m must vary. Under these conditions, any difference in a or m will cause the value for c_2 associated with the angle ψ to change.

3.4 Incoming and Outgoing Rays

The radical in the expression for $\dot{r}$ allows for the specification of either incoming (r decreasing and $\dot{r}$ negative) or outgoing (r increasing and $\dot{r}$ positive) light rays. In order to combine these two as a complete expression for r, we must determine what value of s corresponds with a switch from an incoming ray to an outgoing one. This value of s was found using a numerical differential equation solver to find the point of discontinuity. Once this was found, the solver could be run twice: once up to the determined value for s with $\dot{r}$ positive (incoming), and once from that s-value out to a predetermined limit of s with $\dot{r}$ negative (outgoing).

4 Graphical Analysis of Light Ray Paths

4.1 Light Ray Paths in Minkowski Space

Creating plots of light ray paths in Minkowski space can be done one of two ways: by using the results of the derivation for Minkowski space shown in 3.1 or by using the results of 3.2 and setting the mass and angular momentum to zero. The later was used in creating Figure 3 in section 6. This figure shows how varying c_2 has an effect on the direction of each light ray. Here, m and a have been set to zero, reducing the Kerr space time to Minkowski space. Each light ray follows a straight line, since there is no mass to provide curvature.

4.2 Light Ray Paths in Kerr Space

In the Kerr geometry a and m have non-zero values, causing light rays passing close to the origin to experience some deviation from a straight line. The angular momentum and mass of the black hole pull nearby light rays closer to the origin. This is demonstrated in Figure 4 – with a single light ray passing close to a black hole.

In order to understand the contribution that each variable makes to a light ray's path, plots can be generated that keep all but one variable constant. In Figure 5, the initial angle of orientation, ψ, is varied, while m is kept as a constant and $a = 0$. The angle ψ is changed using its relationship between m, a, c_1 and c_2 as outlined in 3.3. Although the rays in Figure 5 look somewhat like those in Figure 3, the effects of the mass are now clear. While the rays for ψ close to zero are still close to straight lines, those significantly above or below zero are being "pulled" back toward the origin.

Before examining the effects of the mass and angular momentum, a mathematical restriction on their relationship should be considered. In order for the Kerr solution to accurately represent a black hole, the mass must be greater than or equal to the angular momentum [16]. By keeping the angular momentum as some constant and letting m vary, the effects of a black hole's mass can be seen in Figure 6.

As might be expected, increasing m increases the curvature of the space time, and causes light rays to pass more closely to the origin. If we now keep the mass as some constant and let a vary between zero and that constant, the effects of a black hole's angular momentum can be seen, as in Figure 7.

By varying only a, it appears as if light rays are pulled around the origin in a counter-clockwise direction, with this effect increasing as a increases. We can also see the effects of mass and angular momentum competing with one another. For the rays above the x-axis, the mass is pulling inward and the rotation is pulling them down toward the origin. However, for those below the x-axis, the mass is pulling upward while the rotation pushes them downward. This difference between mass and angular momentum working either against one another or togther is visible in the separation between different rays. Above the axis, where a and m are having similar effects, the spacing is large. Below the axis, where a and m are competing, the separation is smaller.

As we said previously, the study of light rays near black holes has significant application to the field of gravitational lensing. One of the most basic phenomenon of gravitational lensing is massive object functioning to concentrate the light from a distant object. This can be seen in Figure 8, with the object to the right, observer to the left, and black hole at the origin. A unique c_2 was calculated for each ray, based on each r_0, and the mass and angular momentum of the system. These values are shown in Table 1. Light from the object coming directly toward the black hole will fall into the event horizon, and therefore not be observed. However, this loss is minimal compared to the benefits of the lensing effect.

5 Summary

We have shown that techniques used to derive equations of motion in classical physics can be used to derive those for relativistic scenarios. This process can be repeated for more complicated systems – complex distributions of mass for example – provided the metric is known. The resultant equations of motion can be used to predict behavior as we did for a black hole.

This analysis of light rays in both Minkowski and Kerr space has significant application in astronomy and gravitational lensing, as described in sections 1.5 and 4.2. In addition to using information about a black hole to learn more about an object it is distrorting, if we know information about the object being distorted, we can learn more about the black hole. In deriving equations of motion for light in Kerr space, we can see that the only variables are angular momentum (a) and mass (m) [1]. It will be of particular impor-

[1]Black holes are also described by a third quantity: charge. But, as was said before,

tance to study how a black hole's mass and angular momentum change over time. In developing theory describing any loss of mass or angular momentum a black hole might experience, observations being interpreted as described above will be critical.

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black holes most likely do not have any net charge.

6 Light Ray Plots

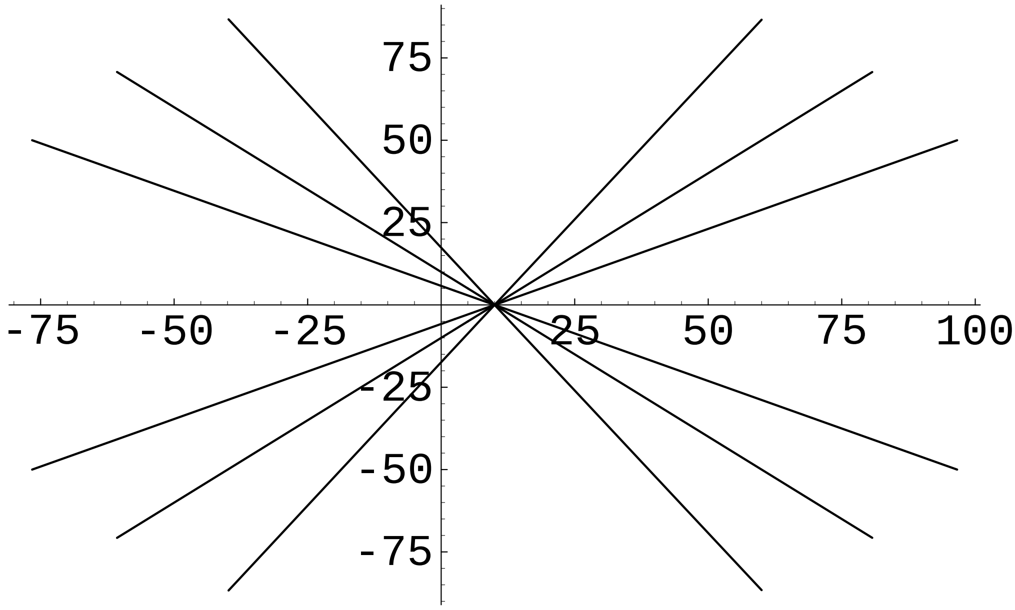

Figure 3: Light rays in Minkowski space ($m = 0$, $a = 0$). The constant c_2 has been varied so that light rays have initial angles of 0, $\pm\pi/6$, $\pm\pi/4$, $\pm\pi/3$, $\pm 2\pi/3$, $\pm 3\pi/4$ and $\pm 5\pi/6$. Because Minkowski space has zero mass or angular acceleration and is therefore flat, each ray follows a straight line away from the source.

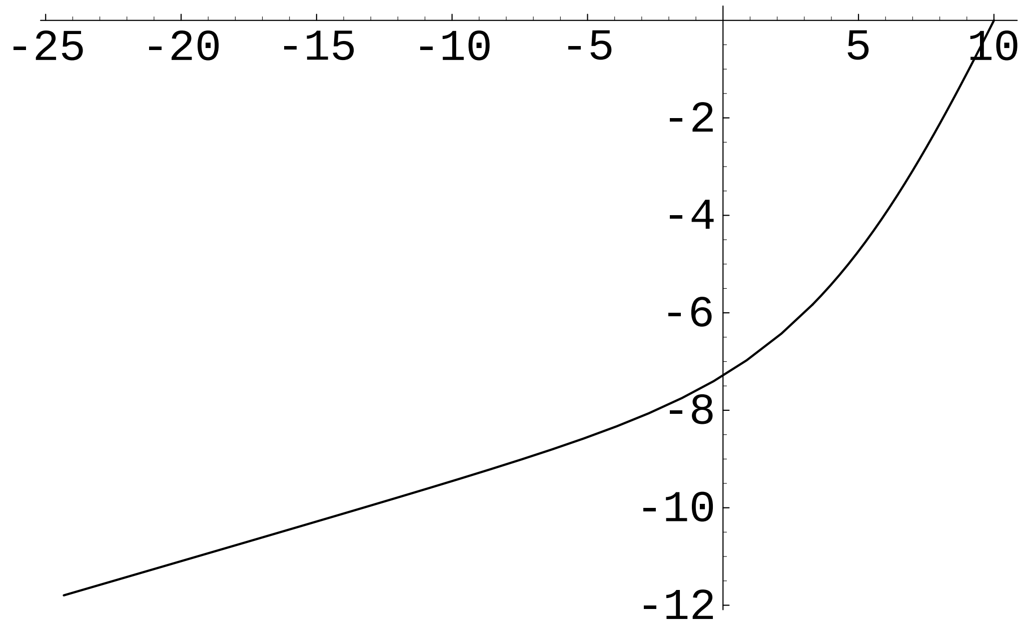

Figure 4: Light rays in Kerr space: single ray. Here, $m = 1$, $a = .9$, and the light ray has been given an initial angle of $2\pi/3$. The effects of both the mass and angular acceleration are now visible. While the light ray initially follows a somewhat straight path, the mass pulls it toward the origin as it gets closer. The angular acceleration has the effect of sweeping the ray around in a counter-clockwise manner. Since the light ray begins below the x-axis, the effects of the mass and angular acceleration are competing with one another.

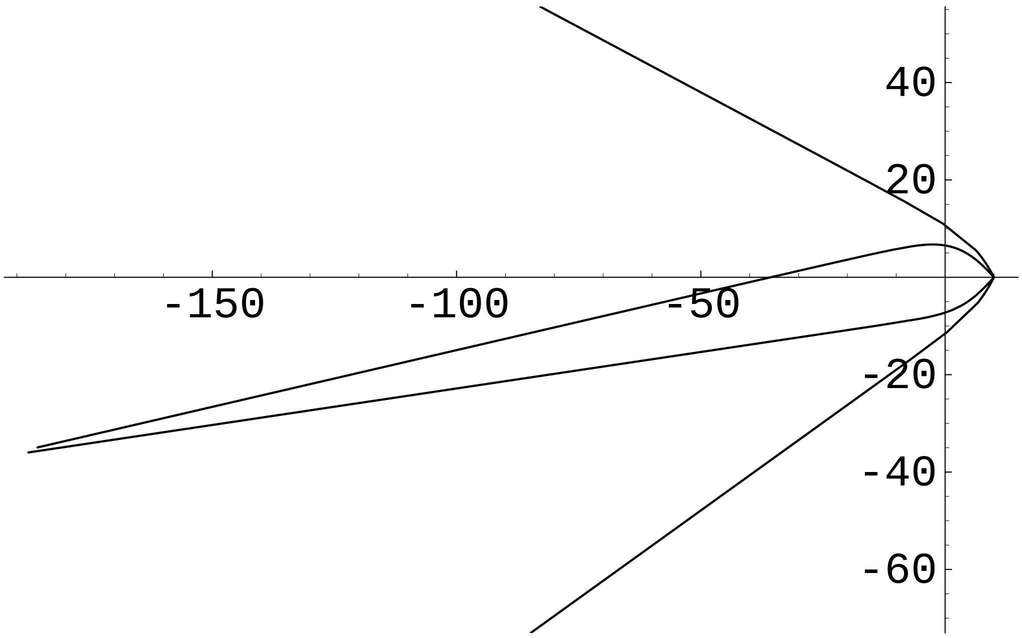

Figure 5: Light rays in Kerr space: varying angle of orientation ψ. Here, $a = .8$, $m = 1$, and ψ was given values of $\pm\pi/4$ and $\pm\pi/3$. In comparison with Figure 3, mass and angular momentum are now curving the spacetime and altering the light ray paths.

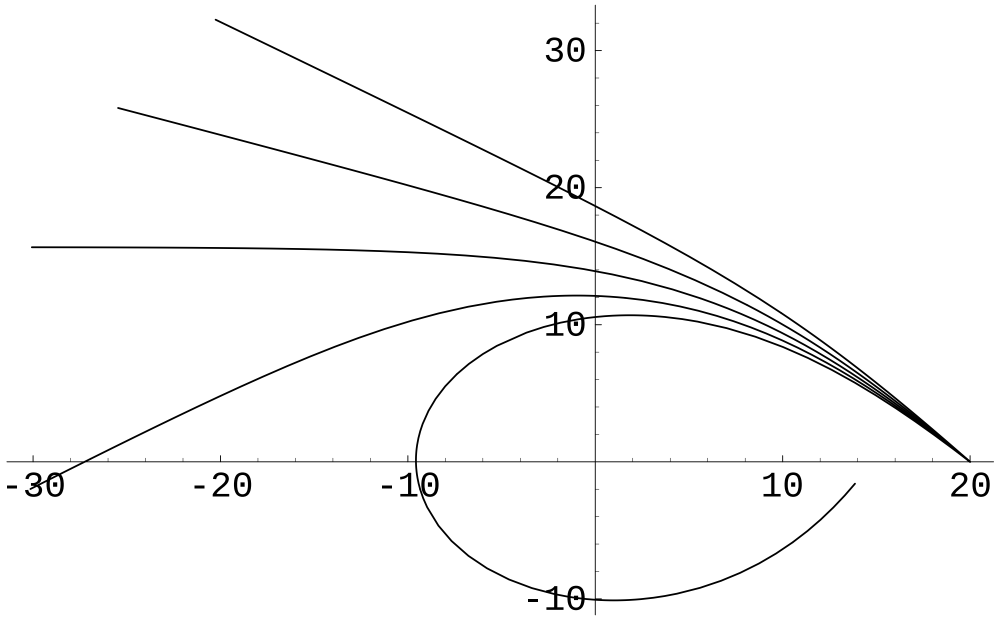

Figure 6: Light rays in Kerr space: varying mass. Here, ψ was kept at a constant $3\pi/4$, $a = .1$, and the mass was given values of 1, 1.5, 2, 2.5, and 3. In addition to demonstrating the effect of increasing the mass, the counter-clockwise sweeping due to a mentioned before is made visible. As m increases, the light ray is pulled closer to the black hole, where the angular acceleration has a greater effect.

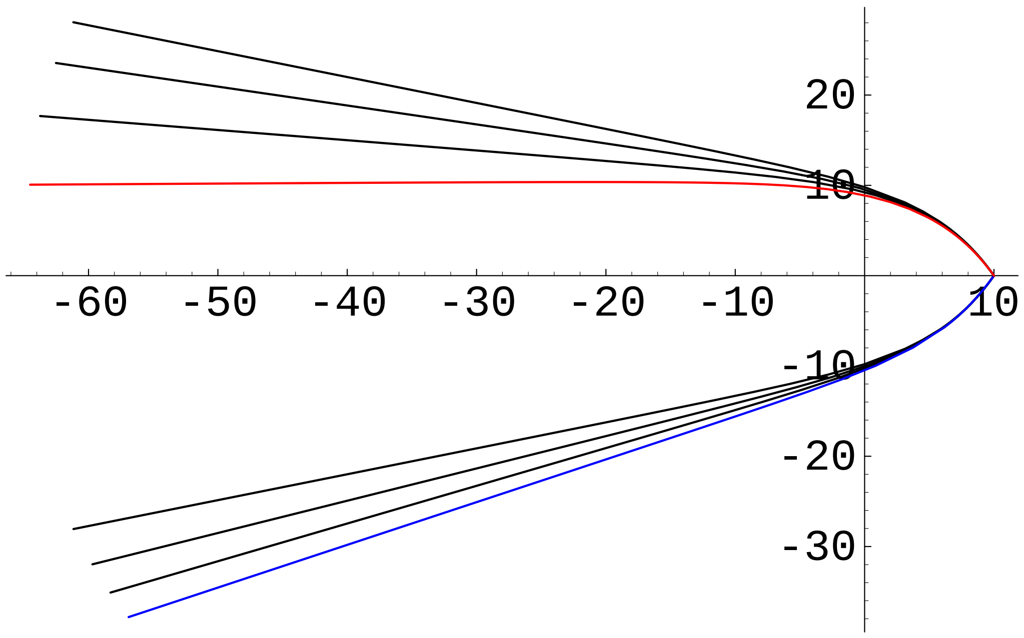

Figure 7: Light rays in Kerr space: varying angular momentum (a). Here, ψ was given values of $\pm 2\pi/3$, $m = 1.35$, and a was given values of 0, $.3m$, $.6m$, and $.9m$. Below the x-axis, as was the case in Figure 4, the effects of the mass and angular acceleration are competing with each other. Above the x-axis, m and a are effectively working together. The difference between these two subsets of rays can be seen in the seperation between rays in each group. Above the axis, where a and m worked together, the distance between rays is greater than below the axis, where they worked against each other.

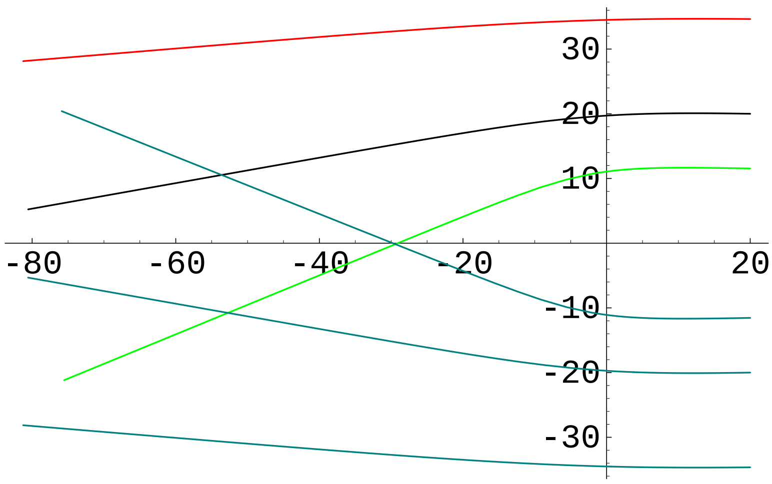

Figure 8: Light rays in Kerr space: gravitational lensing. Here, the source is at the positive end of the x-axis and the observer is at the negative end. The black hole was given a mass of 1 and an angular momentum of .1. The array of light rays was started at $x = 20$ and at angle above and below the x-axis of $\pi/6$, $\pi/4$, and $\pi/3$.

Ray	ψ	c_2
1	$\pi/3$	35.5463
2	$\pi/4$	20.7547
3	$\pi/6$	12.0916
4	$-\pi/6$	-12.0916
5	$-\pi/4$	-20.7547
6	$-\pi/3$	-35.5463

Table 1: Intial ψ and c_2 values for each member of the array of light rays in Figure 8.

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