Numerical Solutions to Gravitational Lensing

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1 Einstein and the General Theory of Relativity

Before the publication of Einstein's 1904 paper on special relativity it was believed that light travelled through a medium called the ether. This theory came about from the observation of the wavelike properties of light. But in order for light to "wave" it needed something to "wave" through, thus the ether. But since the ether was believed to be a substance it stood to reason that the earth travelled through it during its rotation about its axis and the sun.

An experiment was devised to test the existence of the ether using this idea. If light travelled through the ether and if the Earth moved with respect to the ether then light travelling against the ether would be observed by people on earth to move more slowly than light moving perpendicular to the movement of the Earth through the ether. A device called the Michelson - Morley Interferometer was constructed, but no matter how it was oriented it always failed to detect any light beam moving at any different speed from another. Einstein suggested that the idea of the ether be thrown out and that it should be accepted that light travels at the same speed in any inertial reference frame, thus preserving Maxwell's equations of Electricity and Magnetism at the expense of the Newtonian world view.

This, however, as a theory would have bizarre consequences. What this meant was that there was no preferred reference frame. That is that an observer at "rest" would observe a light beam to be moving at the speed c, while an observer in "motion" at a constant velocity with respect to the first observer would also observe the same light beam to be travelling at the same speed c. Carried to its logical conclusion this means there is no test that can be performed that will prove that either observer is really at "rest" or in "motion". And in fact it is meaningless to say that either of them is.

The new postulates of special relativity bring about a lot of bizarre and famous paradoxes. But all of these seeming contradictions are flawed in that they use non-inertial reference frames, that is frames that accelerate, and the Special Theory of Relativity covers only inertial (non-accelerating) reference frames. But the world is full of accelerating objects. So it was necessary to generalize the theory of relativity so that it applied to everything. This was done by stating that acceleration and gravity are identical and that space is not flat and Euclidian but curved. Light and geometry bend around gravitational fields.

This idea of curvature was tested in 1919 by Dyson, Eddington and Davidson. By measuring the angular separation between stars before and during an eclipse it was shown that while the the eclipsing sun was between two stars the stars appeared closer together. The light from the stars was being bent around the sun. Furthermore the observed bending angle was in close agreement with the angle predicted by Einstein's theories.

It is this idea of curvature that is central to my thesis. Very strong gravitational fields bend light to such a degree that they act in a similar manner to physical lenses. By measuring this lensing it is possible to determine the mass of the lensing object. Learning the mass of large distant astronomical objects sheds light on many important cosmological questions, such as the abundance of dark matter and the mass density of the universe.

2 The Object RXJ 1347.5-1145

The object RXJ 1347.5-1145 is the brightest known x-ray emitting cluster of galaxies. Models of lensing are available and some of these approximations have been applied to RXJ 1347.5 - 1145. Sahu et. al. used images from Space Telescope Imaging Spectrograph on board the Hubble Space Telescope to analyze RXJ. They determined the redshift of the cluster to be 0.451. Also two large symmetric arcs were observed at a radius of $35''$. Inside the arcs they observed approximately 100 galaxies with a combined luminosity of about $4 \times 10^{11} L_{\odot}$. One of the arcs had a measured redshift of 0.81. Applying the Einstein ring angle equation (to be derived later) they found the gravitating mass of the cluster to be $6.3 \times 10^{14} M_{\odot}$. By comparing the luminosity measurements and X-ray observations of gas to the gravitating mass they determined that at least 6 percent of the mass is baryonic. **(here i am a bit confused can we go over the Sahu paper together?)** Fischer and Tyson used the weak lensing approximation and found the mass to be $1.0 \times 10^{15} M_{\odot}$. This mass is for a larger volume of the cluster than the Sahu calculation($400''$ compared to $70''$). Ravindranath and Ho studied the spectroscopy of the lens arcs in more detail. They determined the redshift of one of the arcs to be 0.806. Since this object is so frequently studied it may be useful to measure the lensing by finding the numerically integrated path of the light rays as they travel from the source, bend around RXJ 1347.5-1145 and intersect with us.

3 Tensors and Metrics

Tensor calculus can be used to describe how spacetime is bent and thus gravitational lenses. A tensor mapping is transformation from one coordinate system to another or from one manifold to another. A metric is a tensor mapping that has the following properties:

1. $g_{ij} = g_{ji}$,
2. $\det g_{ij} \neq 0$.

The first condition states that the metric is a symmetric matrix, while the second condition implies that the metric has an inverse.

A metric is useful because it can provide a way to measure the infinitesimal distance given the measured coordinate distances:

$$ds^2 = g_{ab} dx^a dx^b. \tag{1}$$

The metric can be derived through application of General Relativity. **lots more stuff to go here as soon as i get it straight**

4 Cosmological Spaces

The most commonly used cosmological spacetimes are the Friedmann Robertson Walker (FRW) spacetimes. The geometry of these spacetimes influences the physics of how distance is determined.

To see how this affects the universe as a whole we can ignore local distortions and look at the large scale structure of space. According to observations distant astronomical objects in any direction recede from the observer. Given this and the high degree of homogeneity seen in the cosmic microwave background on a large scale we can say that the universe is homogenous and isotropic.

First, we define cosmic time t, clocks at rest with respect to the cosmic fluid of the homogenous model and that remain at those same coordinates measure cosmic time. Then, we take a slice through space-time at some time t, the slice will also be homogenous and isotropic. Take any gaussian surface in this hypersurface, the curvature K of this surface will be dependent upon t and defined as:

$$K(t) = \frac{k}{R^2(t)}$$

where for a spherical geometry; $k = +1$. Zero for flat and -1 for hyperbolic and the four dimensional definition of the scale factor $R(t)$ in spherical coordinates,

$$R^2(t) = x^2 + y^2 + z^2 + w^2 = r^2 + w^2$$

given this and the equation for the separation of points on a hypersurface

$$dl^2 = dr^2 + r^2 d\Omega^2 + dw^2,$$

we can write out

$$ds^2 = dt^2 - R^2(t)\left(\frac{d\sigma^2}{1 - k\sigma^2} + \sigma^2 d\Omega^2\right).$$

This is the general equation for the FRW metric where $\sigma = r/R$ for spherical coordinates. By making the substitution $d\sigma = \cos\psi d\psi$ and $S^2(\psi) = \sin^2(\psi)$ for the case of a hypersphere:

$$ds^2 = dt^2 - R^2(t)\left(\frac{\cos^2(\psi)}{1 - \sin^2(\psi)} d\psi^2 + S^2(\psi) d\Omega^2\right), \quad (2)$$

and by recalling that $\cos^2(\psi) + \sin^2(\psi) = 1$

$$ds^2 = dt^2 - R^2(t)(d\psi^2 + S^2(\psi) d\Omega^2). \quad (3)$$

This also holds for $\sigma^2 = \psi^2$ for a flat cosmology and $\sinh^2(\psi)$ for hyperbolic.

We can also write the FRW Metric in terms of the conformal time

$$d\eta = dt/R(t).$$

as,

$$ds^2 = R^2(\eta)(d\eta^2 - d\psi^2 - S^2(\psi)d\Omega^2).$$

In the FRW metric, if the geometry is spherical this implies that the expansion of the universe will slow, eventually stop entirely and then implode into what is called the big crunch. If the geometry is flat, the universe will expand indefinitely while asymptotically approaching some limit. Also in the flat case the scale factor is the only difference between physical and coordinate distance. If spacetime is hyperbolic the universe will expand and there will be no limit on how large the physical distance between coordinate points will grow.

The Hubble constant can be written in terms of the scale factor as $H = \dot{R}/R$ and the deceleration parameter as $q = \ddot{R}/RH^2$. By series expanding the expression for the scale factor we can write it in terms of these quantities:

$$R_e = R(1 - \Delta tH + \frac{\Delta t^2}{2} qH^2 ...)$$

where R_e is the value of the scale factor at the time of emission of a light ray and R the value at the time of reception.

5 Derivation of the Lagrangian

A spacetime by definition is a manifold with a metric g_{ab}. We will assume that our spacetime is a four dimensional manifold. Take two points on this spacetime,

$$\begin{aligned} X^a_A &= (t_A, x_A, y_A, z_A) \\ X^a_B &= (t_B, x_B, y_B, z_B). \end{aligned}$$

A curve through this spacetime is defined as

$$X^a(\lambda) = (t(\lambda),\, x(\lambda),\, y(\lambda),\, z(\lambda)),$$

where λ is an affine parameter and $X^a(0) = X^a_A$. In order to find the length of this curve the metric of the spacetime must be known,

$$ds^2 = g_{ab}dx^a dx^b.$$

now by the following steps,

$$ds = \sqrt{g_{ab}dx^a dx^b} = \sqrt{g_{ab}dx^a dx^b}\left(\frac{d\lambda}{d\lambda}\right) = \sqrt{g_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda}}d\lambda$$

and the by definition of the affine parameter we have

$$ds = \sqrt{g_{ab}\dot{x}^a\dot{x}^b}d\lambda.$$

Integrating this leaves,

$$S = \int ds = \int_0^{\lambda_{max}} \sqrt{g_{ab}\dot{x}^a\dot{x}^b}d\lambda,$$

an equation for the length S. By identifying this with the path integral

$$J = \int_0^{t_{max}} \mathcal{L}\, dt,$$

where $\mathcal{L}$ is the Lagrangian. And by Hamilton's Variational Principle we can say for a spacetime that:

$$\mathcal{L} = \sqrt{g_{ab}\dot{x}^a\dot{x}^b}.$$

This will obey the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial x^c} - \frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^c} = 0.$$

Applying these two equations we obtain:

$$\frac{1}{2}\frac{1}{\sqrt{g_{de}\dot{x}^d\dot{x}^e}}\left(\frac{\partial g_{ab}}{\partial x^c}\dot{x}^a\dot{x}^b\right) - \frac{d}{d\lambda}\left(\frac{1}{\sqrt{g_{de}\dot{x}^d\dot{x}^e}}g_{cb}\dot{x}^b\right) = 0.$$

Using the chain rule on the second term,

$$\frac{1}{2}\frac{1}{\sqrt{g_{de}\dot{x}^d\dot{x}^e}}\left(\frac{\partial g_{ab}}{\partial x^c}\dot{x}^a\dot{x}^b\right) - \frac{1}{\sqrt{g_{de}\dot{x}^d\dot{x}^e}}\frac{d}{d\lambda}\left(g_{cb}\dot{x}^b\right) + \frac{g_{cb}\dot{x}^b}{2\left(g_{de}\dot{x}^d\dot{x}^e\right)^{3/2}}\frac{d}{d\lambda}\left(g_{lm}\dot{x}^l\dot{x}^m\right) = 0.$$

Since we are looking at light rays we can say that the Lagrangian is equal to zero. This is because, given SR units, light travels at the same rate in the time direction as it does in the spatial direction. For example in Minkowski space

$$ds^2 = dt^2 - dx^2 = (a)^2 - (a)^2 = 0.$$

This implies that the length of the null tangent vector is equal to zero so:

$$g_{ab}\dot{x}^a\dot{x}^b = 0.$$

Therefore the third term disappears leaving

$$\frac{1}{2}\frac{1}{\sqrt{g_{de}\dot{x}^d\dot{x}^e}}\left(\frac{\partial g_{ab}}{\partial x^c}\dot{x}^a\dot{x}^b\right) - \frac{1}{\sqrt{g_{de}\dot{x}^d\dot{x}^e}}\frac{d}{d\lambda}\left(g_{cb}\dot{x}^b\right) = 0.$$

Multiplying through by $\sqrt{g_{de}\dot{x}^d\dot{x}^e}$ we are left with

$$\frac{1}{2}\left(\frac{\partial g_{ab}}{\partial x^c}\dot{x}^a\dot{x}^b\right) - \frac{d}{d\lambda}\left(g_{cb}\dot{x}^b\right) = 0.$$

These Euler-Lagrange equations can be directly obtained by using

$$\widetilde{\mathcal{L}} = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b \tag{4}$$

as a Lagrangian, as can be seen from direct inspection. Hence, when one is interested in finding the null geodesics (light rays) through the spacetime *(but not timelike or spacelike geodesics)* equation (4) can be used.

Taking the FRW Metric

$$ds^2 = R^2(\eta)(d\eta^2 - d\psi^2 - S^2(\psi)d\sigma^2),$$

and perturbing it linearly for a gravitating mass at the origin **(section on how this is done here)** we obtain

$$ds^2 = R^2(\eta)\left[\left(1 - \frac{2M}{rR(t)}\right)d\eta^2 - \left(1 + \frac{2M}{rR(t)}\right)d\sigma^2\right].$$

Where $r = \sqrt{x^2 + y^2 + z^2}$. Since the lensing takes place at a cosmologically significant distance it is necessary to multiply the radius element by the scale factor $R(t)$. Note that the perturbation goes as the mass over the radius. This is similar to the Newtonian potential for gravity. Using this and the functional form of the Lagrangian we can write

$$\mathcal{L} = \frac{R^2(\eta)}{2}\left[\left(1 - \frac{2M}{rR(t)}\right)\dot{\eta}^2 - \left(1 + \frac{2M}{rR(t)}\right)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right].$$

If two spacetimes differ only by a common factor $g^A_{ab} = \Omega^2(x)g^B_{ab}$ where $\Omega^2(x)$ is always nonzero and positive (i.e. real), they are said to be conformal. Light cones in conformaly related spacetimes are identical because the infinitesimal unit vector of light rays are equal to zero in both. So take the following Lagrangian characterizing a spacetime:

$$\mathcal{L} = \frac{1}{2}\left[\left(1 - \frac{2M}{rR(t)}\right)\dot{\eta}^2 - \left(1 + \frac{2M}{rR(t)}\right)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right]. \quad (5)$$

This Lagrangian's spacetime is conformal to the spacetime just derived so the paths of light rays are identical in both. This allows us to eliminate the scale factor.

6 Parameterization

Now we must consider how we will express units in our calculation. We will be numerically integrating a light ray which we will "shoot" from the observer (us) to the lensed object. Using the observed lensing angle and the redshifts of the lensed galaxy and the cluster we will determine the mass of the cluster. Since the measurements to be used in the equations of motion are known only in terms of redshift it is sensible to have all equations in a form conducive to these types of values. First take the equation for the scale factor in a flat dust filled universe:

$$R = \left(\frac{9C}{4}\right)^{1/3} t^{2/3},$$

where C is a constant and t is cosmic time. Now taking the expression for conformal time,

$$d\eta = \frac{dt}{R}.$$

Substituting and taking the integral we obtain:

$$\int d\eta = \left(\frac{9C}{4}\right)^{-1/3} \int t^{-2/3} dt.$$

Solving this gives us

$$\eta = 3\left(\frac{9C}{4}\right)^{-1/3} t^{1/3} = At^{1/3}, \tag{6}$$

where A is the collection of constants. We fix the constant of integration by defining $\eta - t = 0$ at the big bang.

The expression for redshift may be written as

$$1 + z = \frac{R_{now}}{R_{then}}.$$

Now taking the original equation for the scale factor substituting in equation (6) and entering this into the equation for redshift we are left with

$$\eta_e = \eta_c \sqrt{\frac{1}{1+z}}, \tag{7}$$

where $\eta_e = \eta_{then}$ for emission time and $\eta_c = \eta_{now}$ for total cosmic time.

Now in order to scale the model in a uniform and convenient manner we can choose the way we express the constants. So,

$$R = \gamma t^{2/3},$$

where γ is defined as a constant that makes the following expression correct:

$$R(t_c) = 1.$$

So it follows that

$$\gamma = \frac{1}{t_c^{2/3}}. \tag{8}$$

From equation (6)

$$t = \frac{C}{12}\eta^3 \tag{9}$$

and by the definition of γ and the expression for the scale factor we can write that:

$$1 = \left(\frac{9C}{4}\right)^{1/3} t_c^{2/3}.$$

This can be simplified and combined with equation (8) to yield,

$$C = \frac{4}{9}\gamma^3.$$

Now re-substituting this expression for C back into equation (9) we are left with:

$$t = \frac{\gamma^3}{27}\eta^3, \tag{10}$$

which can be written as:

$$\eta_c = 3t_c, \tag{11}$$

in terms of the current cosmic and conformal times.

Now we can substitute equation (11) into equation (7) to obtain:

$$\eta_e = 3t_c\sqrt{\frac{1}{1+z}}.$$

The redshift observed for the Galactic Cluster RXJ 1347.5-1145 is 0.45 and the redshift for the lensed galaxy is 0.80. Applying the above equation to these values we find that the emission time for the cluster is $2.4914t_c$ and $2.2361t_c$ for the lensed object.

Take the FRW Metric for a flat spacetime:

$$ds^2 = R^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2),$$

and remove the x and y coordinates:

$$ds^2 = R^2(\eta)(d\eta^2 - dz^2).$$

Recall that for light rays $ds = 0$,

$$d\eta = dz,$$

and integrate leaving:

$$\eta_c - \eta_e = z_c - z_e. \tag{12}$$

Using this and recalling equation (11) we can solve for positions of the observer and lensed object along the z-axis, given that the cluster is at the origin. We obtain $-0.5086t_c$ for the observer and $0.2553t_c$ for the object.

Having all measurements as multiples of t_c allows us to separate out any error due to uncertainties in Hubble Constant. The model will only involve the use of the Hubble Constant at the last step of calculation. So the results can be multiplied by a scalar to correct for a new value of the Hubble Constant.

7 One Lens Equations of Motion

Recall equation (10) and substitute this into the expression for the scale factor in terms of gamma (γ) to obtain:

$$R(\eta) = \frac{1}{9}t_c^2\eta^2.$$

Next take t_c to be equal to one. This is again resetting the scale. But because this will be the scale for r and for M it cancels out and makes no numerical difference. Substituting what is left back into equation (5) we obtain the lagrangian's final form:

$$\mathcal{L} = \frac{1}{2}\left[\left(1 - \frac{18M}{r\eta^2}\right)\dot{\eta}^2 - \left(1 + \frac{18M}{r\eta^2}\right)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right] \tag{13}$$

With this we can derive equations of motion by applying the Euler-Lagrange Equations and suppressing the y coordinate since it is not needed:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \eta} &= \frac{1}{2}\frac{\partial}{\partial \eta}\left[\left(1-\frac{18M}{\eta^2 r}\right)\dot{\eta}^2-\left(1+\frac{18M}{\eta^2 r}\right)(\dot{x}^2+\dot{z}^2)\right] \\
&= \frac{18M}{\eta^3 r}(\dot{\eta}^2+\dot{x}^2+\dot{z}^2) \\
\frac{\partial \mathcal{L}}{\partial \dot{\eta}} &= \dot{\eta}\left(1-\frac{18M}{\eta^2 r}\right) \\
\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{\eta}} &= \ddot{\eta}\left(1-\frac{18M}{\eta^2 r}\right)+\frac{36M}{r\eta^3}\dot{\eta}^2+\frac{18M(x\dot{x}+z\dot{z})}{\eta^2 r^3}\dot{\eta}
\end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial \eta}-\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{\eta}}=0$$

$$\frac{18M}{\eta^3 r}(\dot{\eta}^2+\dot{x}^2+\dot{z}^2)=\ddot{\eta}\left(1-\frac{18M}{\eta^2 r}\right)+\frac{36M}{r\eta^3}\dot{\eta}^2+\frac{18M(x\dot{x}+z\dot{z})}{\eta^2 r^3}\dot{\eta}. \tag{14}$$

And for x:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{9M}{r\eta^2}\dot{\eta}^2-\frac{9M}{r\eta^2}(\dot{x}^2+\dot{z}^2)\right) \\
&= \frac{9M(\dot{\eta}^2+\dot{x}^2+\dot{z}^2)x}{r^3\eta^2} \\
\frac{\partial \mathcal{L}}{\partial \dot{x}} &= -\left(1+\frac{18M}{r\eta^2}\right)\dot{x} \\
\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= -\ddot{x}\left(1+\frac{18M}{r\eta^2}\right)+\dot{x}\left(\frac{36M}{r\eta^3}\dot{\eta}+\frac{18M(x\dot{x}+z\dot{z})}{\eta^2 r^3}\right)
\end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial x}-\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)=0$$

$$\frac{9M(\dot{\eta}^2+\dot{x}^2+\dot{z}^2)x}{r^3\eta^2}=-\ddot{x}\left(1+\frac{18M}{r\eta^2}\right)+\dot{x}\left(\frac{36M}{r\eta^3}\dot{\eta}+\frac{18M(x\dot{x}+z\dot{z})}{\eta^2 r^3}\right). \tag{15}$$

The equation for z will follow the same form as x so we can say by intuition that:

$$\frac{9M(\dot{\eta}^2+\dot{x}^2+\dot{z}^2)z}{r^3\eta^2} = -\ddot{z}\left(1+\frac{18M}{r\eta^2}\right)+\dot{z}\left(\frac{36M}{r\eta^3}\dot{\eta}+\frac{18M(x\dot{x}+z\dot{z})}{\eta^2 r^3}\right).$$

This gives us three second order differential equations of motion to plot the paths of the light rays. To make this task easier we can use simple substitution to reduce this to six first order differential equations. So:

$$\begin{aligned} v_x &= \dot{x}, \quad \dot{v}_x = \ddot{x}, \\ v_\eta &= \dot{\eta}, \quad \dot{v}_\eta = \ddot{\eta}. \end{aligned}$$

8 Mass Varying Program

Now in order to find the paths of these light rays we can numerically integrate the equations. I have used *Mathematica* to write a program that will for, the distances derived above, numerically integrate the functions, varying the mass as a bracketing step function until it finds a mass that creates a light ray that will, originating from the observer position, intersect with the source. That code is given in full in the appendix. Basically there are several nested loops. These are represented below:

Clear variables
Set initial conditions
Set equations of motion
Begin loop 1: varies the mass until light ray intersects with lensed object
    Solve differential equations, integrate light ray
    Begin loop 2
        Vary affine parameter until y-intercept is found
    End loop 2
    Print graph of light ray
    Test to see if intercept is correct, if too high, decrease mass
    Solve differential equations

Begin loop 3
Find intercept again
End loop 3
Test to see if intercept is correct, if too low, increase mass
If it is returned that intercept is first too high and then too low decrease the amount it is varied by one decimal place
End loop 1

Figure (1) is an example of the graph this method produces.

9 One-Lens Comparisons

It may be useful to compare these measurements from the numerical integration to the results obtained from the Einstein ring angle formula. This equation is a result of the thin lens approximation of gravitational lensing.

This approximation states that all the bending of the light ray takes place at one sharp angle in the plane of the gravitating mass. This angle called the bending angle is derived through application of general relativity. The rest of the equation is the result of simple geometry. The thin lens approximation involves three parallel planes: the source plane (which contains the source of the bent light), the lens plane (which contains the gravitating object) and the observer plane (which contains the observer).

Now several quantities are defined. Draw a line from the observer through the mass to the source plane, this is D_s, the distance from the observer to the source plane; it is perpendicular to all three planes. D_l is the segment of this line which is the distance from the observer to the lens plane. D_{ls} is the segment of this line which is the distance from the lens plane to the source plane. All of these distances are angular diameter distances. To find an angular diameter distance the Dyer-Roeder equation must be solved:

$$D(z) = 2\frac{(1+z)^{1/4} - (1+z)^{-1/4}}{(1+z)^{5/4}}.$$

In this equation z is the redshift. To find the distance between two distant objects, each with their own redshift, a different form of the equation must be used:

$$D(z_1, z_2) = 2\left[\frac{(1+z_2)^{-1}}{(1+z_1)^{1/2}} - \frac{1}{(1+z_2)^{3/2}}\right].$$

Now that the distances are concretely defined we can begin deriving the ring angle equation. Let x be the distance from where D_s intercepts the source plane to where the source object is physically located. Let y be the distance from where D_s intercepts the source plane to where the source is *apparently* located. Draw a line from the observer to the apparent position of the source, the angle this line makes with D_s will be θ. Also where this line intersects with the lens plane will be b the distance from this point to the object will also be called b. Now draw one line from b to the apparent position and another line from b to the physical position. The angle between these two lines will be called α. **(seems there should be a picture here, how do you recommend i make one?)** An expression for this angle can be derived through application of general relativity:

$$\alpha = \frac{4GM}{c^2 b}. \tag{16}$$

We can set $G/c^2 = 1$ if we use meters as the units for M: $\alpha = 4M/b$. Also let β be the angle between D_l and a line drawn from the observer to x. Through the small angle approximation we can say that $y = D_s\theta$, $x = D_s\beta$ and $y - x = D_{ls}\alpha$. By combining these equations we arrive at:

$$D_s\theta = D_s\beta + D_{ls}\alpha. \tag{17}$$

Again by the small angle approximation: $b = D_l\theta$. Substituting this into equation (17) yields

$$\theta = \beta + \frac{4M}{\theta}\frac{D_{ls}}{D_l D_s}. \tag{18}$$

If the source is directly behind the object then $\beta = 0$ and a phenomenon known as an "Einstein ring" is visible. An Einstein ring appears as a ring of light from the source at and angle θ_E around the lensing object. Though a complete ring is not present around RXJ, there are two large symmetric arcs to either side of it, so it is a good approximation. A complete ring would hardly be expected for such a diffuse and uneven cluster. The final form of the Einstein ring angle equation is

$$\theta_E = 2\sqrt{\frac{MD_{ls}}{D_l D_s}}. \tag{19}$$

This is probably the most common method of estimating mass through gravitational lensing. So it may be useful to compare my results to what would be obtained through use of these equations. I wrote two *Mathematica* programs these programs were similar to the one described above with an added routine to calculate the mass with the ring equation. All of this was nested in a loop that varied either the distance from the source to the mass or the distance from the observer to the mass. The results of each were recorded and compared. Figure 2 is a representation of these comparisons where the distance to the lensed object is varied and for each different distance the value given by the numerical integration is in red and the value given by the ring angle is in blue. The x-axis is the calculated mass and the y-axis is the distance given in **(i need to go back over the program to figure out what units are being used here**. The same thing is done for varying the observer in figure 3. In both cases higher masses lead to more divergence between the two methods. However even in these cases the discrepancy is not large.

10 One Moving Lens

Now it may be useful to consider a lens that is moving. We can take equation (13) and take the radius element r to be

$$r = \sqrt{(x_m - x)^2 + (z)^2}. \tag{20}$$

For a lens moving along the x-axis. Where the m subscripts denote the position of the lens (or mass). But we must consider that we will know the motion in terms of our time and the velocity will be expressed in *distance/second*. So we need to convert from cosmic time to conformal time. Combine equations (8) and (10) and recall that we have set t_c to unity:

$$t = \frac{\eta^3}{27}. \tag{21}$$

So the position will just be this multiplied by the measured velocity and the velocity will be the derivative of that:

$$x_m = x_0 - \frac{\beta}{27}\eta^3, \quad \dot{x}_m = \frac{\beta}{9}\eta^2 \tag{22}$$

Where x_0 is the initial position and β is in units of fractions of the speed of light. The lens is moving in the negative x direction. Then we substitute this back into equation (20) to obtain

$$r = \sqrt{(x_0 - \frac{\beta}{27}\eta^3 - x)^2 + (z)^2}. \tag{23}$$

The metric for a moving lens will be different from the metric for a stationary lens. To see why, take the the metric for a stationary lens in its array form:

$$g_{ab} = \begin{pmatrix} 1+2\varphi & 0 & 0 & 0 \\ 0 & -(1-2\varphi) & 0 & 0 \\ 0 & 0 & -(1-2\varphi) & 0 \\ 0 & 0 & 0 & -(1-2\varphi) \end{pmatrix}. \tag{24}$$

Where φ is the linear mass perturbation and is defined as $\varphi = \frac{M}{Rr}$. Now we need to perform a coordinate transform on this metric. The equations that govern the coordinate transform are the Lorentz transforms:

$$t = \gamma(t' + \beta x'),$$
$$x = \gamma(x' + \beta t').$$

Where the prime coordinates are the new moving coordinates. If we assume that β is small compared to the speed of light then we can eliminate γ:

$$t = t' + \beta x', \tag{25}$$
$$x = x' + \beta t'. \tag{26}$$

We use the matrix coordinate transform equation

$$g'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b} g_{cd}$$

on equation (24) using equations (25) and (26). Also it is important to note that $R(t) = t^{2/3} = (t' + \beta x')^{2/3}$. So the metric for a moving lens is obtained:

$$g'_{ab} = \begin{pmatrix} (1+2\varphi) - \beta^2\xi & \beta(1+2\varphi) - \beta\xi & 0 & 0 \\ \beta(1+2\varphi) - \beta\xi & \beta^2(1+2\varphi) - \xi & 0 & 0 \\ 0 & 0 & -\xi & 0 \\ 0 & 0 & 0 & -\xi \end{pmatrix}. \tag{27}$$

Where $\xi = (t' + \beta x')^{4/3}(1 - 2\varphi) = R^2(1 - 2\varphi)$. Keeping with our assumption that β is a small fraction of the speed of light we can ignore second and higher orders of β, so equation (27) becomes:

$$g'_{ab} = \begin{pmatrix} 1+2\varphi & \beta(1+2\varphi) - \beta\xi & 0 & 0 \\ \beta(1+2\varphi) - \beta\xi & -\xi & 0 & 0 \\ 0 & 0 & -\xi & 0 \\ 0 & 0 & 0 & -\xi \end{pmatrix}. \tag{28}$$

Going back to equation (1) we can use equation (28) to find:

$$ds^2 = (1+2\varphi)dt^2 - R^2(1-2\varphi)(dx^2+dy^2+dz^2) + 2\beta[(1+2\varphi) - R^2(1-2\varphi)]dxdt.$$

Since $dt = Rd\eta$ and the y coordinate will not be used we can rewrite the above equation as:

$$\begin{aligned} ds^2 = R^2(1+2\varphi)d\eta^2 &- R^2(1-2\varphi)(dx^2+dz^2) \\ &+ 2R\beta[(1+2\varphi) - R^2(1-2\varphi)]dxd\eta. \end{aligned} \tag{29}$$

Substituting equation (29) into equation (4) we obtain an expression for the Lagrangian of a single moving lens:

$$\mathcal{L} = (1+2\varphi)\dot{\eta}^2 - (1-2\varphi)(\dot{x}^2 + \dot{z}^2) + \frac{2\beta}{R}[(1+2\varphi) - R^2(1-2\varphi)]\dot{x}\dot{\eta}. \tag{30}$$

The Lagrangian is also equal to zero so the factor of one half can be removed. Before applying the Euler-Lagrange equation it should be noted that:

$$R(t) = t^{2/3} = (t' + \beta x')^{2/3} = t'^{2/3}\left(1 + \frac{\beta x'}{t'}\right)^{2/3}.$$

The factor of $\left(1+\frac{\beta x'}{t'}\right)^{2/3}$ can be ignored if β is small so,

$$R = t'^{2/3} = \frac{\eta^2}{9},$$

due to equation (21).

It will be useful to take derivatives in preparation for applying the Euler Lagrange equations:

$$\frac{\partial\varphi}{\partial\eta} = \frac{\beta\eta^2 M}{9r^3 R}\left(x_0 - \frac{\beta}{27}\eta^3 - x\right) - \frac{18M}{\eta^3 r}.$$

Keeping mind our assumption that β is small this becomes

$$\frac{\partial\varphi}{\partial\eta} = \frac{\beta\eta^2 M}{9r^3 R}(x_0 - x) - \frac{18M}{\eta^3 r}. \tag{31}$$

The derivative of the Lagrangian with respect to η is

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\eta} &= 2\frac{\partial\varphi}{\partial\eta}(\dot{x}^2+\dot{z}^2+\dot{\eta}^2) - \frac{36\beta}{\eta^3}(1+2\varphi-R^2(1-2\varphi))\dot{x}\dot{\eta} \\ &+ \frac{2\beta}{R}\left[2\frac{\partial\varphi}{\partial\eta}+2R^2\frac{\partial\varphi}{\partial\eta}-\frac{4}{81}\eta^3(1-2\varphi)\right]\dot{x}\dot{\eta}.\end{aligned} \tag{32}$$

Taking into account small β, equation (32) becomes

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\eta} &= 2\frac{\partial\varphi}{\partial\eta}(\dot{x}^2+\dot{z}^2+\dot{\eta}^2) - \frac{36\beta}{\eta^3}(1+2\varphi-R^2(1-2\varphi))\dot{x}\dot{\eta} \\ &- \left[\frac{72M}{R\eta^3 r}+\frac{72MR}{\eta^3 r}+\frac{8\beta\eta^3}{81R}(1-2\varphi)\right]\dot{x}\dot{\eta}.\end{aligned} \tag{33}$$

Somewhat more involved is the other half of this Euler Lagrange equation. It will be useful to first calculate the derivative of φ with respect to the affine parameter (λ):

$$\frac{d\varphi}{d\lambda} = -M\left(\frac{18\dot{\eta}}{\eta^3 r}+\frac{1}{r^3 R}(z\dot{z}-\frac{1}{9}\beta x_0\eta^2\dot{\eta}+\frac{\beta}{9}x\eta^2\dot{\eta}+\dot{x}x_0+\frac{\beta}{27}\dot{x}\eta^3+\dot{x}x)\right). \tag{34}$$

We take the derivative of the Lagrangian with respect to $\dot{\eta}$,

$$\frac{\partial \mathcal{L}}{\partial \dot{\eta}} = 2(1+2\varphi)\dot{\eta} + \frac{2\beta}{R}[(1+2\varphi) - R^2(1-2\varphi)]\dot{x},$$

take the derivative of this equation with respect to the the affine parameter and make all possible simplifications, to obtain:

$$\begin{aligned} \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}}\right) &= 2(1+2\varphi)\ddot{\eta} + 4\dot{\eta}\frac{d\varphi}{d\lambda} + \left\{\frac{2\beta}{R}\ddot{x} - \frac{36\beta}{\eta^3}\dot{x}\dot{\eta}\right\}[(1+2\varphi) - R^2(1-2\varphi)] \\ &- \frac{2\beta}{R}\dot{x}\left\{2\left(\frac{18M\dot{\eta}}{\eta^3 r} + \frac{M}{r^3 R}(z\dot{z} + \dot{x}x_0 + \dot{x}x)\right) + \frac{4\eta^3\dot{\eta}}{81}(1-2\varphi)\right. \\ &+ \left. 2R^2\left(\frac{18M\dot{\eta}}{\eta^3 r} + \frac{M}{r^3 R}(z\dot{z} + \dot{x}x_0 + \dot{x}x)\right)\right\}. \end{aligned} \quad (35)$$

By applying the Euler Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial \eta} - \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{\eta}} = 0,$$

we can find the η equation of motion

$$\begin{aligned} 0 &= \frac{36\beta}{\eta^3}(1+2\varphi - R^2(1-2\varphi))\dot{x}\dot{\eta} - 2\frac{\partial\varphi}{\partial\eta}(\dot{x}^2 + \dot{z}^2 + \dot{\eta}^2) \\ &+ \left[\frac{72M}{R\eta^3 r} + \frac{72MR}{\eta^3 r} + \frac{8\beta\eta^3}{81R}(1-2\varphi)\right]\dot{x}\dot{\eta} + 2(1+2\varphi)\ddot{\eta} \\ &+ 4\dot{\eta}\frac{d\varphi}{d\lambda} + \left\{\frac{2\beta}{R}\ddot{x} - \frac{36\beta}{\eta^3}\dot{x}\dot{\eta}\right\}[(1+2\varphi) - R^2(1-2\varphi)] \\ &- \frac{2\beta}{R}\dot{x}\left\{2\left(\frac{18M\dot{\eta}}{\eta^3 r} + \frac{M}{r^3 R}(z\dot{z} + \dot{x}x_0 + \dot{x}x)\right) + \frac{4\eta^3\dot{\eta}}{81}(1-2\varphi)\right. \\ &+ \left. 2R^2\left(\frac{18M\dot{\eta}}{\eta^3 r} + \frac{M}{r^3 R}(z\dot{z} + \dot{x}x_0 + \dot{x}x)\right)\right\}. \end{aligned} \quad (36)$$

to be continued...

11 Appendix 1: *Mathematica* Program for Exact Integration

```
H = 1.613*10^(-18); t = (2/(3*H))(3*10^5); Print["Scale=", t];
theta = ((4.848137*10^(-6))34.9; xi = 0; zi = -0.5086; zf =0.2553;
```

Sets the scale (t_c), firing angle, initial position and final position.

```
vxi = Sin[theta]; vzi = Cos[theta]; vni=-(1+2M/Sqrt(xi^2 +zi^2));
r[s]= Sqrt{x[s]^2 + z[s]^2};
```

Sets the initial values for the first order derivatives and the expression for the radius element.

```
nterm1 = vn'[s] (1-18M/((n[s]^{2})r[s]))
  +(36M*(vn[s]^{2}))/((n[s]^{3})r[s])
  +18M(x[s]*vx[s]
  +z[s]*vz[s])vn[s]/((n[s]^{2})(r[s]^{3}));
nterm2 = 18M*vn[s]^{2}/((n[s]^{3})r[s])
  +18M(vx[s]^{2} + vz[s]^{2})/((n[s]^{3})r[s]);
xterm1=9M(vn[s]^2 + vx[s]^2 + vz[s]^2)x[s]/((r[s]^3)
  (n[s]^2));
xterm2=-vx'[s] (1 + 18  M/(r[s] n[s]^{2}))
  +vx[s] (36 M*vn[s]/(r[s] n[s]^{3})
  +18  M (x[s] vx[s]+ z[s]vz[s])/(n[s]^{2}r[s]^{3}));
zterm1 = 9  M (vn[s]^2 + vx[s]^2 + vz[s]^2)
  z[s]/((r[s]^3)(n[s]^2));
zterm2 = -vz'[s] (1 + 18  M/(r[s] n[s]^{2}))
  +vz[s] (36  M*vn[s]/(r[s] n[s]^{3})
  + 18  M (x[s] vx[s] + z[s] vz[s])/(n[s]^{2} r[s]^{3}));
```

The six first order differential equations to be used, imputed one side at a time.

```
M = 0.94000000000000000000*10^(-8);
u = 0;
```

Sets the initial guess for the mass and sets the variable u, which will be used later to determine which decimal place of the mass is being varied, to zero.

```
For[j = 0, j < 1,{ Clear[Solution, q, s, x, z, vx, vz, l, k];
```

Begins the *For* loop which will vary the mass until the z-intercept is equal to the coordinate of the observer, and clears the variable which will be used each iteration of this loop.

```
Solution = NDSolve[{vx[s] == x'[s], vz[s]==z'[s],
  vn[s]==n'[s], xterm1 == xterm2, zterm1==zterm2,
  nterm1==nterm2, x[0]==xi, z[0]==zi, vx[0]==vxi,
  vz[0] == vzi, n[0] == 3, vn[0] == vni}, {x, z, n, vx, vz,vn },
  {s,  0,  1}, WorkingPrecision -> 20,   AccuracyGoal ->16,
  PrecisionGoal -> 16,  MaxSteps -> 10^6];
```

Solves the equations of motion for the current mass.

```
q =0; For[k = 0.20000000000000000000, k < 1,{
  l = x[k] /. First[Solution];
  If[(l <10^(-19))&& (l > -10^(-19)),intercept =z[k]/.First[Solution];k = 1,
    StepDown =0;
    StepUp = 0;
    If[l > 0,k = k + 0.1*10^(-q);StepUp = 1];
    l = x[k] /. First[Solution];
    If[l < 0, k = k - 0.1*10^(-q);StepDown = 1];
    If[(StepUp > 0) && (StepDown >0), q++];
  ]
}]; Print["x=",l]; Print["z=",intercept]; Print["Mass=",M];
```

This *For* loop finds the x value of the numerical integration of the light ray at a parameter value k starting with a value of 0.2. It then increasers k until x changes from positive to negative, this increments the value q which lowers the amount k is incremented by a factor of 10. The loop continues with q being incremented until a value for k is found for which x is very small and effectively zero. In essence this finds the z intercept of the light ray. The program as a whole uses this same process to vary the value of M until an intercept equivalent to the observer position is arrived at.

```
ParametricPlot[{x[s] /. First[Solution],z[s]
  /.First[Solution]},{s, 0, 1}];
StepDown2 = 0; StepUp2 = 0;
```

This prints out the graph of the light curve and sets the variables StepDown2 and StepUp2 to zero, which are used in the same way as StepDown and StepUp.

```
If[intercept < zf, M = M - 0.1*10^(-8 - u); Print["mass is too
  high"]; \StepDown2 = 1];
Print["*********************************************"];
Clear[Solution, q, s, x, z, vx, vz, l, k];
Solution=NDSolve[{vx[s]== x'[s],  vz[s] == z'[s],
  vn[s] == n'[s],xterm1 == xterm2, zterm1 == zterm2,
  nterm1 == nterm2,  x[0] == xi, z[0] == zi, vx[0] == vxi,
  vz[0] == vzi, n[0] == 3,  vn[0] == vni}, {x, z, n, vx, vz,
  vn},{s,0,1},WorkingPrecision -> 20, AccuracyGoal -> 16,
  PrecisionGoal -> 16,  MaxSteps -> 10^6];
```

Checks to see if the mass is to high, if it is it readjusts it by an amount dependent upon u, which is used in the same way that q is in the previous for loop. Then it reevaluates the functions of the light curve for the new mass.

```
q =0; For[k = 0.20000000000000000000, k < 1,{
  l = x[k] /. First[Solution];
  If[(l <10^(-19))&& (l > -10^(-19)),intercept =z[k]/.First[Solution];k = 1,
    StepDown =0;
    StepUp = 0;
    If[l > 0,k = k + 0.1*10^(-q);StepUp = 1];
    l = x[k] /. First[Solution];
    If[l < 0, k = k - 0.1*10^(-q);StepDown = 1];
    If[(StepUp > 0) && (StepDown >0), q++];
  ]
}]; Print["x=",l]; Print["z=",intercept]; Print["Mass=",M];
```

Find the intercept again, identical to the other loop.

```
ParametricPlot[{x[s] /. First[Solution], z[s] /.
  First[Solution]},{s, 0, 1}];
If[intercept > zf, M = M + 0.1*10^(-8 - u);
  Print["mass is too low"];
  StepUp2 =1];
If[(StepUp2 > 0) && (StepDown2 > 0), u++]; Print["u=",u];
If[(intercept < (zf + 10^(-10))) && (intercept > (zf - 10^(-10))),
  j = 1; Print["mass just right=",M]];
Print["*********************************************"];
```

Checks to see if the mass is too low. If it is then it changes the mass. If the mass came out to high and then too low it means that the incrementation value is too large and it decresses it by increasing u which works in the same way as q. If the mass turns out right then it breaks the loop and reports the value.

```
  j = j + 0.001
}]
```

Limits the loop to a thousand iterations. Close loop.

Figures

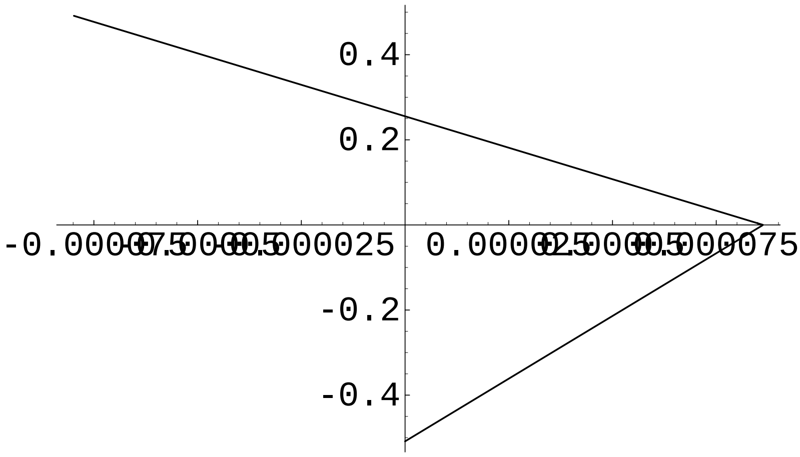

Figure 1: final light curve

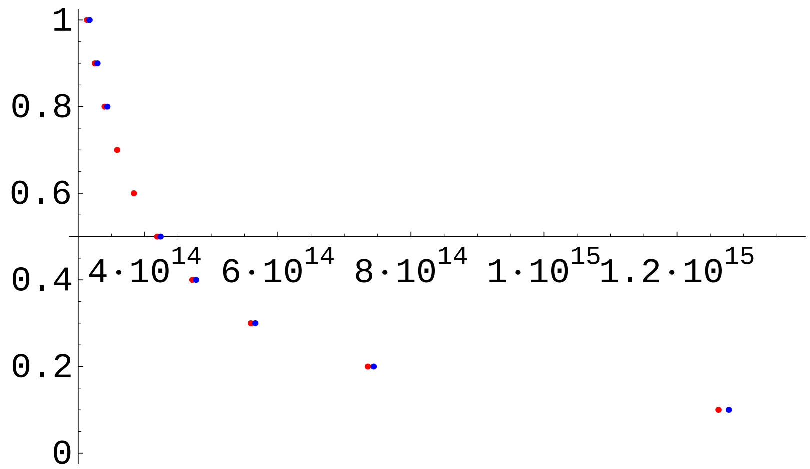

Figure 2: varying the source

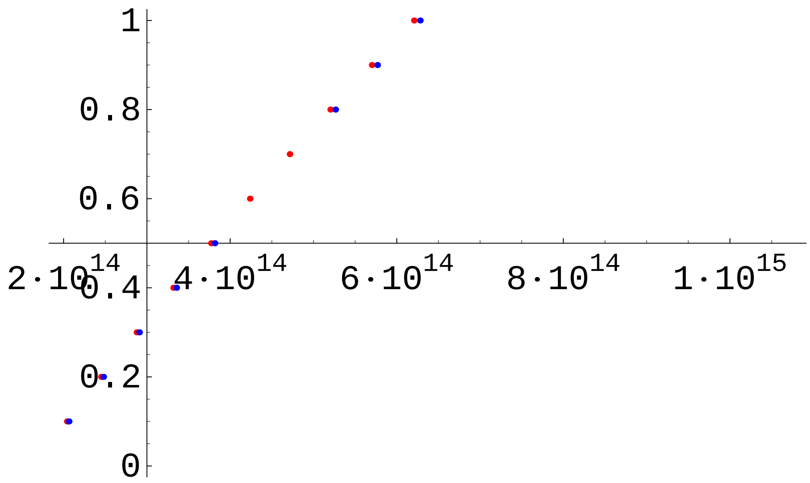

Figure 3: varying the observer

12 References