

The Bianchi identity and weak gravitational lensing

Thomas P. Kling* and Brian Keith†

Dept. of Physics, Bridgewater State College, Bridgewater, MA 02325

(Dated: June 27, 2005)

We consider the Bianchi identity as a field equation for the distortion of the shapes of images produced by weak gravitational lensing. Using the spin coefficient formalism of Newman & Penrose [1], we show that certain complex components of the Weyl and Ricci curvature tensors are directly related to fundamental observables in weak gravitational lensing. In the case of weak gravitational fields, we then show that the Bianchi identity provides a field equation for the Ricci tensor assuming a known Weyl tensor. From the Bianchi identity, we derive the integral equation for weak lensing presented by Miralda-Escude [2], thus making the Bianchi identity a first principles equation of weak gravitational lensing. This equation is integrated in the important case of an axially symmetric lens and explicitly demonstrated in the case of a point lens and a SIS model.

PACS numbers: 98.62.Sb, 95.30.-k, 95.30.Sf, 04.90.+e

I. INTRODUCTION

The Bianchi identity in general relativity is usually expressed as

$$\nabla_{[a}R_{bc]d}{}^e = 0 \quad (1)$$

and is considered to be a consequence of the geometrical properties of the covariant derivative operator. Most students of relativity come across the Bianchi identity in a contracted form that is quite useful in discussions of various properties of the Einstein tensor.

The decomposition of the Riemann tensor into the Ricci (R_{ab}) and Weyl (C_{abcd}) tensors,

$$R_{abcd} = C_{abcd} + (g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) - \frac{1}{3}Rg_{a[c}g_{d]b}, \quad (2)$$

presents a powerful use of the Bianchi identity. As Hawking & Ellis [3] point out, the Ricci tensor is determined by the Einstein field equation, while the Weyl tensor is the part of the curvature not determined locally by the matter distribution. However, the application of the Bianchi identity to Eq. 2 yields a constraint on the Weyl tensor once the Ricci tensor is found from the Einstein field equations. By treating the Bianchi identity as a field equation for the Weyl tensor given a known Ricci tensor, Newman & Penrose [1] (hereafter NP) developed a consistent methodology for the study of gravitational radiation.

In this paper, we consider the Bianchi identity in the reverse manner. Specifically, we assume that the Weyl tensor is known and treat the Bianchi identity as a partial differential equation for the Ricci tensor.

Our motivation for this approach is the study of weak gravitational lensing, where measurements are made of the distortion of images due to the presence of gravitational fields. From the observed distortion of images, a matter distribution can be inferred, so weak gravitational lensing is now a very active area of observational research with many papers published each year. Pivotal early work relating the observed distortion and inferred mass distribution was done by Miralda-Escude [4] and Kaiser & Squires [5] amongst others. Substantial reviews of the subject are provided by Mellier [6] and Schneider & Bartelmann [7]. For this paper, we are most interested in the presentation given in Miralda-Escude [2].

In this paper, we present four results. First, we show that the observed distortion and inferred matter distributions of weak lensing are equal to projected components of the Ricci and Weyl tensors as expressed in the formalism introduced by Newman & Penrose [1]. Next, we demonstrate that the integral equation used in observational weak lensing to determine the projected mass distribution from observed image distortions is actually an integral relation between the Ψ_0 and Φ_{00} components of the Weyl and Ricci tensors.

*Electronic address: tkling@bridgew.edu

†Sponsored by the Adrian Tinsley Program for Undergraduate Research, BSC

Our most important result is that we prove that a particular component of the Bianchi identity yields a partial differential equation for the Φ_{00} component of the Ricci tensor given a known Ψ_0 Weyl tensor. Using a Green's function, we derive the standard equation used in observational weak lensing as formulated in Miralda-Escude [2] from the Bianchi identity. Thus, we have developed a PDE approach to weak lensing that may provide a new calculational approach to the field that allows for more accurate determinations of mass mappings.

Finally, we consider the important case of axially symmetric lenses and show that in this case the Bianchi identity is easily integrated. Two explicit examples are shown: a point lens and a singular isothermal sphere.

Our results on weak lensing extend a new way of thinking about gravitational lensing based on a space-time perspective. Strong lensing and general image distortion have been studied by several researchers including Frittelli & Newman [8] and two connected papers by Frittelli et al. [9, 10] with a substantial review provided by Perlick [11]. This paper is the first paper that seeks a first principles equation for observational weak lensing.

II. COMPUTATION OF RICCI AND WEYL TENSORS

In this section, we compute the complex Ricci and Weyl tensor components and the spin coefficients introduced in Newman & Penrose [1]. Our goal is to determine the spin coefficient form of the Ricci and Weyl tensor components to first order in a static metric perturbation off flat space.

While modern evidence strongly indicates that the universe is accelerating and dominated by dark energy, it is reasonable to chose to perturb off flat space for several reasons. First, we wish to directly compare our derivation of the equations of weak lensing with the standard derivations using the thin-lens approximation. Because all the “lensing activity” in the thin-lens approximation happens when light rays passes the lens, the choice of cosmology does not directly enter the equations. In fact, both in the literature and in actual application, the only place the cosmological model enters weak lensing calculations is in the conversion between redshift and angular diameter distance. Hence, one of the strengths of lensing studies is that the basic equations of the theory are relatively independent of cosmological model.

Furthermore, since the basic cosmological models are conformally flat, our basic results can be translated to them. Beginning with a more “realistic” metric does not add to the calculation and introduces a number of conceptual issues best handled in a different paper.

The physical situation that we wish to study is the appearance and distortion of a small patch of the sky containing extended galaxies that has been weakly lensed by some matter distribution. Both the astronomer in question and the sky of galaxies are assumed to be situated in reasonably isolated regions of space-time where the space-time is flat.

The observed galaxies are connected to our astronomer by light rays along her past light cone. In fact, we may think of pencils of light rays that travel from each extended galaxy to the telescope. The path of these light rays is to be determined by integrating the null geodesic equations of the space-time metric, so that we may find a tangent vector, ℓ^a , to the light rays.

The matter distribution that lenses the light rays is encoded into a space-time metric. For our purposes, we may assume a weakly perturbed Minkowski metric,

$$ds^2 = (1 + 2\varphi)dt^2 - (1 - 2\varphi)(dx^2 + dy^2 + dz^2), \quad (3)$$

where φ is a static gravitational potential satisfying

$$\nabla^2\varphi = 4\pi\rho(x, y, z). \quad (4)$$

As the astronomer and observed galaxies are to be isolated, we assume that the matter distribution, $\rho(x, y, z)$, is contained in some region of space far from the astronomer and galaxies. In practice, this is always the case.

Following the program of NP, we chose a tetrad of null vectors,

$$\lambda_i^a = (\lambda_1^a, \lambda_2^a, \lambda_3^a, \lambda_4^a) = (\ell^a, n^a, m^a, \bar{m}^a), \quad (5)$$

associated with the pencil of light rays connecting our astronomer and each individual observed galaxy, where ℓ^a is the real vector tangent to the pencil. By convention, the other vectors are chosen such that

$$\ell^a n_a = 1 \quad \& \quad m^a \bar{m}_a = -1, \quad (6)$$

with all other products zero. m^a and \bar{m}^a are spatial, complex null vectors parallel propagated along the pencil.

In principle, the space-time metric can be written in terms of a general null tetrad as $g^{ab} = \eta^{ij} \lambda_i^a \lambda_j^b$, although this is not central to our presentation. Further, we will make physical restrictions on the tetrad vectors tied to our goal of finding the Weyl and Ricci tensor components to first order in the perturbation.

We orient our spatial coordinates, (x, y, z) , such that the astronomer is located at $x = y = 0$ and $z = z_a$ with the “center” of the matter distribution at the spatial origin. With this placement, our astronomer’s telescope points straight down the \hat{z} axis.

In principle, each null tetrad associated with an individual pencil of light connecting the astronomer to a lensed galaxy varies along the pencil. However, in our calculations, we are justified in considering each tetrad to be a constant null tetrad given by

$$\ell^a = \frac{1}{\sqrt{2}}(1, 0, 0, 1), \quad n^a = \frac{1}{\sqrt{2}}(1, 0, 0, -1) \quad \& \quad m^a = \frac{1}{\sqrt{2}}(0, 1, i, 0), \quad (7)$$

for two reasons. First, the wide field telescopes used today see a very small portion of the sky. For example, the proposed 8.4 m Dark Matter Telescope would have a field of view of only 260 square milli degrees [12]. Hence, with our orientation, all the pencils of light are parallel to the \hat{z} axis. Second, even though the individual pencils will be deflected by the lens, the deflection of the light ray will be proportional to the 2-dimensional (x, y) gradient of φ . Since we will be contracting the tetrad vectors with the Ricci and Weyl tensors, to work to first order in φ , we must neglect any variation in the null tetrad.

The basis of the NP formalism is twelve complex spin coefficients that are analogous to the twenty-four real Ricci rotation coefficients defined by

$$\gamma_{jk}^i = \lambda_j^a \lambda_k^b \nabla_b \lambda_a^i. \quad (8)$$

Because our choice of tetrad involves vectors whose components are all constant, all the NP spin coefficients are zero for our physical situation (to zeroth order in the metric perturbation).

In the NP formalism, tetrad components of the Ricci and Weyl tensors are computed by contracting the coordinate components with the tetrad vectors. This yields five complex Weyl tensor and ten complex Ricci tensor components (see appendix A).

Using the null tetrad above, the non-zero NP formalism Ricci curvature components are

$$\Phi_{00} = \Phi_{22} = 2\Phi_{11} = \frac{1}{2}\nabla^2\varphi. \quad (9)$$

The first order Weyl tensor components are

$$\begin{aligned} \Psi_0 = \Psi_4 &= \frac{1}{2}(\varphi_{xx} - \varphi_{yy} + 2i\varphi_{xy}), \\ \Psi_1 = \bar{\Psi}_3 &= -\frac{1}{2}(\varphi_{xz} + i\varphi_{yz}), \\ \Psi_2 &= \frac{1}{2}\left(\varphi_{zz} - \frac{1}{3}\nabla^2\varphi\right). \end{aligned} \quad (10)$$

We note that up to numerical factors, Φ_{00} is equal to the matter density $\rho(x, y, z)$ that determines the gravitational perturbation φ .

The four directional derivatives associated with the null tetrad play an important role in the NP formalism and are given special names:

$$D = \ell^a \partial_a, \quad \Delta = n^a \partial_a, \quad \delta = m^a \partial_a, \quad \& \quad \bar{\delta} = \bar{m}^a \partial_a. \quad (11)$$

Using our choice of tetrad, and the fact that the metric perturbation is static, these derivative operators will act as

$$\begin{aligned} D &= \frac{1}{\sqrt{2}} \frac{\partial}{\partial z}, & \Delta &= -\frac{1}{\sqrt{2}} \frac{\partial}{\partial z}, \\ \delta &= \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), & \bar{\delta} &= \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right). \end{aligned} \quad (12)$$

III. OBSERVATIONAL WEAK LENSING IN PRACTICE

In this section, we briefly outline the usual presentation of the equations of weak lensing by a thin lens so that we can draw parallels to our presentation that uses the Weyl and Ricci tensor. The derivation presented here relies heavily on that of Miralda-Escude [2].

The practicing astrophysicist assumes a thin (or two dimensional) lens that divides a background space-time into an observer and a source side. The lens lies in a two (spatial) dimensional “lens plane” that is perpendicular to the line of sight of the telescope. The observed background galaxy is said to lie in a two dimensional “source plane,” also perpendicular to the line of sight.

Using dimensionless cartesian coordinates \vec{r}_s in the source plane and $\vec{r}_l = (x, y)$ for the lens plane, the starting point for thin-lens lensing is a mapping from the lens plane to the source plane given by

$$\vec{r}_s = \vec{r}_l - \vec{\nabla}_2 \psi(\vec{r}_l). \quad (13)$$

The two-dimensional gravitational potential ψ is determined from an ordinary three-dimensional potential φ by projection,

$$\psi(\vec{r}_l) = \int_{z_a}^{z_s} dz \varphi(x, y, z), \quad (14)$$

where z_a and z_s are the z coordinates of the observing astronomer and distant source, respectively. The two dimensional potential ψ will satisfy

$$\frac{1}{2} \Delta^2 \psi = \frac{1}{2} (\partial_x^2 + \partial_y^2) \psi = \frac{\Sigma(\vec{r}_l)}{\Sigma_{crit}}, \quad (15)$$

where $\Sigma(\vec{r}_l)$ is a projected mass density and Σ_{crit} is a critical density for strong lensing, or the appearance of multiple images.

To consider weak lensing, one forms a Jacobian matrix

$$\mathcal{J} = \frac{\partial \vec{r}_s}{\partial \vec{r}_l} = \begin{pmatrix} 1 - \psi_{xx} & -\psi_{xy} \\ -\psi_{xy} & 1 - \psi_{yy} \end{pmatrix} = \begin{pmatrix} 1 - \kappa - \lambda & -\mu \\ -\mu & 1 - \kappa + \lambda \end{pmatrix}, \quad (16)$$

where

$$\kappa = \frac{1}{2} (\psi_{xx} + \psi_{yy}) = \frac{1}{2} \Delta^2 \psi = \frac{\Sigma}{\Sigma_{crit}} \quad (17)$$

is referred to as the “convergence,” and the two quantities

$$\lambda = \frac{1}{2} (\psi_{xx} - \psi_{yy}) \quad \& \quad \mu = \psi_{xy} \quad (18)$$

are called the “shears” in the thin-lens literature. Frittelli et al. [10] shows how these quantities are related to the convergence (ρ) and shear (σ) in general relativity. For this paper, it is critical to note the similarities that κ has with Φ_{00} and that Ψ_0 has with λ and μ .

If the outer surface of a circular, extended source is parameterized by $\vec{t}_s = R(\cos \delta, \sin \delta)$, then under the thin lens mapping the inverse of the Jacobian matrix in Eq. 16 maps $\vec{t}_s \rightarrow \vec{t}_l$, a new elliptical curve in the lens plane. The orientation and ellipticity of the resulting ellipse is determined by λ and μ – see Schneider et al. [13] for the details.

Since the orientation and ellipticity is observable, the goal in observational weak lensing is to measure the shears (λ, μ) and infer the projected mass density, κ . To do this, one inverts Eq. 17 using the two dimensional Green’s function and applies the differentiations in Eq. 18 to the result. This yields

$$\begin{aligned} \lambda &= \int d\vec{r}' \frac{\kappa(\vec{r}')}{\pi} \frac{(-\cos 2\eta)}{|\vec{r} - \vec{r}'|^2}, \\ \mu &= \int d\vec{r}' \frac{\kappa(\vec{r}')}{\pi} \frac{(-\sin 2\eta)}{|\vec{r} - \vec{r}'|^2}, \end{aligned} \quad (19)$$

where η is the angle in the lens plane between $\vec{r} - \vec{r}'$ and the \hat{x} axis.

By employing a Fourier transform technique, Miralda-Escude [2] shows that one can invert Eq. 19 to obtain

$$\kappa(\vec{r}) = \int d\vec{r}' \frac{[\lambda(\vec{r}'), \mu(\vec{r}')] }{\pi} \cdot \frac{[\cos 2\eta, \sin 2\eta]}{|\vec{r} - \vec{r}'|^2}. \quad (20)$$

Equation 20 is the primary equation for observational weak lensing. It is an integral relation that allows one to determine the projected mass density at every point if one has measured the “shears.”

IV. WEAK LENSING WITH CURVATURE TENSORS

We may exploit the similarities between the definitions of the thin-lens convergence and shears and Φ_{00} and Ψ_0 to recast Eq. 20 in different language. The first step is to project our gravitational potential and curvature tensors into a “thin-lens” form. Equation 14 projects our metric perturbation $\varphi(x, y, z)$ to a two dimensional $\psi(x, y)$. Then we define

$${}_L\Phi_{00} \equiv \int_{z_a}^{z_s} dz \Phi_{00} = \frac{1}{2} (\psi_{xx} + \psi_{yy}), \quad (21)$$

where we use Eq. 9 with Eq. 14 and the property that φ_z is zero far from the lens. Likewise, we can define

$${}_L\Psi_0 \equiv \int_{z_a}^{z_s} dz \Psi_0 = \frac{1}{2} (\psi_{xx} - \psi_{yy} + 2i\psi_{xy}). \quad (22)$$

We note that ${}_L\Phi_{00} = \kappa$ and ${}_L\Psi_0 = \lambda + i\mu$, so that the fundamental observable in weak gravitational lensing is a projected component of the Weyl tensor, while the fundamental inferred quantity of observational interest (the projected mass density) is a projected component of the Ricci tensor.

Then exactly as in section III, one can use a two-dimensional Green’s function to invert Eq. 21 for ψ and plug this relation into Eq. 22 to obtain an integral equation specifying ${}_L\Psi_0$ given a known ${}_L\Phi_{00}$. Taking the Fourier transform yields a relation between the transforms of ${}_L\Psi_0$ and ${}_L\Phi_{00}$ which is easily rearranged such that the inverse Fourier transform produces the desired integral equation for ${}_L\Phi_{00}$,

$${}_L\Phi_{00}(\vec{r}) = - \int d\vec{r}' \frac{{}_L\Psi_0(\vec{r}')}{\pi} \frac{e^{-2i\eta}}{|\vec{r} - \vec{r}'|^2}, \quad (23)$$

where η again is the angle between $\vec{r} - \vec{r}'$ and the \hat{x} axis in the lens plane.

Equation 23 demonstrates that the fundamental equation of weak lensing is an integral relation between components of the Weyl and Ricci curvature tensors. Specifically, a projected version of Ψ_0 is known through observation and used to determine a projected mass density ${}_L\Phi_{00}$.

V. THE BIANCHI IDENTITY

As discussed in section I, applying the Bianchi identity, Eq. 1, to the decomposition of the Riemann tensor into the Ricci and Weyl curvature tensors produces a differential constraint on the Weyl tensor. To obtain the Bianchi identity in the NP spin coefficient formalism, one contracts the Bianchi identity with all possible combinations of the tetrad vectors and writes the resulting equations out using the definitions of the spin coefficients and Ricci and Weyl tensor components. In the non-vacuum case, this results in twelve equations, which are listed in a number of references including Newman & Tod [14].

The first full Bianchi identity equation is

$$\begin{aligned} \bar{\delta}\Psi_0 - D\Psi_1 + D\Phi_{01} - \delta\Phi_{00} = & (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2 + (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} \\ & + 2(\epsilon + \bar{\rho})\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02}. \end{aligned} \quad (24)$$

Due to our choice of tetrad, all the spin coefficients are zero, so the entire right hand side of Eq. 24 vanishes. Also, Φ_{01} is zero, so to first order in the gravitational perturbation, we have

$$\bar{\delta}\Psi_0 - D\Psi_1 - \delta\Phi_{00} = 0. \quad (25)$$

Equation 25 is a partial differential equation that holds at every point along the pencil of rays connecting our astronomer with the distant observed galaxy. To compare this relation with Eq. 23, we project into a lens plane by integrating out the z coordinate. Integrating $D\Psi_1$ in z from z_a to z_s effectively yields zero since $D \sim \partial_z$ and Ψ_1 evaluated far from the lens is assumed to be zero. Passing the z integration through the δ derivative operators, we have

$$\delta {}_L\Phi_{00} = \bar{\delta} {}_L\Psi_0. \quad (26)$$

We consider Eq. 26 as a field equation for ${}_L\Phi_{00}$ given a known ${}_L\Psi_0$. Equation 26 relates the fundamental quantities of weak gravitational lensing and is derived from first principles in general relativity.

VI. GREEN'S FUNCTION

In this section, we show that the relation between the Weyl and Ricci tensor derived from the Bianchi identity, Eq. 26, is the differential version of the integral relation used in standard weak lensing studies, Eq. 23. To do this, we employ a set of Green's functions first developed by Porter [15] and expanded in Ivancovich et al. [16], which we refer to as the Porter Green's functions.

The Porter Green's functions are designed to work with powers of the $\bar{\delta}$ and $\bar{\delta}$ differential operators. The $\bar{\delta}$ derivative operator acts on functions on the sphere of different spin weight in different ways and serves as a spin-weight raising and lowering operator [14]. Functions on the sphere of a spin weight s are expandable in a series of spin-weighted spherical harmonics ${}_sY_{lm}$. Because different powers of $\bar{\delta}$ and $\bar{\delta}$ annihilate the ${}_sY_{lm}$ with $|s| = l$, a set of Green's functions can be developed.

Our main result, Eq. 26, is a partial differential equation that holds in an (x, y) plane that we consider to be the lens plane. To make contact with the Porter Green's functions, we introduce the pair of complex coordinates $(\zeta, \bar{\zeta})$ defined by

$$\zeta = \frac{1}{\sqrt{2}}(x - iy). \quad (27)$$

In these coordinates, m^a points in the ζ direction and $\delta = \partial_\zeta$.

By stereographic projection of the sphere into a equatorial plane from the pole, the $(\zeta, \bar{\zeta})$ coordinates are complex coordinates for the sphere. With the exception of the point at infinity, where the projected curvature tensors ${}_L\Psi_0$ and ${}_L\Phi_{00}$ are assumed to be zero, introduction of the complex stereographic coordinates turns Eq. 26 into a partial differential equation on the sphere.

Since ${}_L\Phi_{00}$ is a spin-weight zero function, the application of $\bar{\delta}$ to it will take the form

$$\bar{\delta} {}_L\Phi_{00} = (1 + \zeta\bar{\zeta})\bar{\delta} {}_L\Phi_{00}. \quad (28)$$

For this reason, we multiply both sides of Eq. 26 by $(1 + \zeta\bar{\zeta})$ and our Bianchi identity takes the form

$$\bar{\delta} {}_L\Phi_{00} = (1 + \zeta\bar{\zeta})\bar{\delta} {}_L\Psi_0 = A_1(\zeta, \bar{\zeta}), \quad (29)$$

where $A_1(\zeta, \bar{\zeta})$ is a function of spin-weight 1.

The Porter Green's function for an equation of the form in Eq. 29 is

$${}_L\Phi_{00}(\zeta, \bar{\zeta}) = \int_{S^2} K_{0,-1}(\zeta, \bar{\zeta}; \eta, \bar{\eta}) (1 + \eta\bar{\eta}) \bar{\delta}_\eta {}_L\Psi_0 d\mu_\eta, \quad (30)$$

where the integral is taken over the sphere parameterized by complex coordinates $(\eta, \bar{\eta})$ with the area element

$$d\mu_\eta = \frac{2}{i} \frac{d\eta \wedge d\bar{\eta}}{(1 + \eta\bar{\eta})^2}.$$

The kernel of the Green's function for an equation of the type in Eq. 29 is

$$K_{0,-1}(\zeta, \bar{\zeta}; \eta, \bar{\eta}) = \frac{1}{4\pi} \frac{1 + \eta\bar{\eta}}{\bar{\zeta} - \bar{\eta}}. \quad (31)$$

Putting all this together, we have

$${}^L\Phi_{00}(\zeta) = \frac{1}{2i\pi} \int_{S^2} \frac{\bar{\delta}_\eta {}^L\Psi_0(\eta)}{\bar{\zeta} - \bar{\eta}} d\eta \wedge d\bar{\eta}. \quad (32)$$

It is convenient to multiply by 1 in the form $1 = (\zeta - \eta)/(\zeta - \eta)$ to obtain

$${}^L\Phi_{00}(\zeta) = \frac{1}{2i\pi} \int_{S^2} \frac{\zeta - \eta}{(\bar{\zeta} - \bar{\eta})(\zeta - \eta)} \bar{\delta}_\eta {}^L\Psi_0(\eta) d\eta \wedge d\bar{\eta}. \quad (33)$$

Because we want to show that Eq. 33 is equivalent to Eq. 23, we reintroduce cartesian coordinates

$$\zeta = \frac{1}{\sqrt{2}}(x - iy) \quad \eta = \frac{1}{\sqrt{2}}(x' - iy'). \quad (34)$$

In these coordinates, Eq. 33 becomes

$${}^L\Phi_{00}(x, y) = \frac{1}{2\pi} \int \frac{[(x - x') - i(y - y')]}{(x - x')^2 + (y - y')^2} (\partial_{x'} - i\partial_{y'}) {}^L\Psi_0 dx' dy'. \quad (35)$$

Comparing Eq. 35 with Eq. 23, we see that we need to flip the $\bar{\delta}_\eta$ derivative operator that is inside the integral by integrating by parts. Using the definition of the angle η as the angle the vector $\vec{r} - \vec{r}'$ makes with the $+\hat{x}$ axis and integrating Eq. 35 by parts yields

$${}^L\Phi_{00}(x, y) = - \int d\vec{r}' \frac{{}^L\Psi_0}{\pi} \frac{e^{-2i\eta}}{|\vec{r} - \vec{r}'|^2} \quad (36)$$

plus the surface term

$$\frac{1}{2\pi} \int dx' dy' (\partial_{x'} - i\partial_{y'}) \left\{ \frac{[(x - x') - i(y - y')]}{|\vec{r} - \vec{r}'|^2} {}^L\Psi_0 \right\}. \quad (37)$$

One can see that the surface term will vanish by breaking the integral into two parts for the $\partial_{x'}$ and $\partial_{y'}$ derivatives. Each part vanishes as the integrand is evaluated at either $x' = \infty$ or $y' = \infty$.

Therefore, we have shown that the application of the Porter Green's function to the Bianchi identity, Eq. 26, results in the integral relation used in weak gravitational lensing studies.

VII. AXIAL SYMMETRY

In this section, we consider those mass distributions which are axially symmetric around the line of sight. In practice, this is a very important simplifying assumption used in gravitational lensing studies. Under this assumption, Eq. 26 is easily integrated directly, without using the Porter Green's function.

First, we change coordinates in the lens plane from the cartesian (x, y) coordinates to standard axial coordinates (r, ϕ) . In the axial coordinate system, the δ directional derivative operator in Eq. 12 is

$$\delta = \frac{e^{i\phi}}{\sqrt{2}} \left(\partial_r + \frac{i}{r} \partial_\phi \right). \quad (38)$$

As an ansatz for an axially symmetric lens, we assume that the Ricci tensor is a function of r only,

$${}_{L}\Phi_{00} = {}_{L}\Phi_{00}(r), \quad (39)$$

and that the Weyl tensor can be written as

$${}_{L}\Psi_0 = \Upsilon(r) e^{2i\phi}. \quad (40)$$

This second assumption is motivated by the concept of spin weight in the NP formalism [14].

Applying Eq. 38 to our ansatz for the solution to Eq. 26, the ϕ dependence drops out, and one is left with

$$\partial_r {}_{L}\Phi_{00}(r) = \partial_r \Upsilon(r) + \frac{2}{r} \Upsilon(r), \quad (41)$$

which is now a one dimensional equation.

Recall that we have shown that ${}_{L}\Psi_0$ or $\Upsilon(r)$ is a known, measurable quantity and that ${}_{L}\Phi_{00}$ is essentially the projected matter density. This means that after integrating over r , Eq. 41 gives us the functional form of the projected matter distribution:

$${}_{L}\Phi_{00}(r) = \Upsilon(r) + \int \frac{2}{r} \Upsilon dr + C. \quad (42)$$

The constant of integration represents the mass sheet degeneracy present in gravitational lensing and can be taken as zero.

VIII. TWO AXIALLY SYMMETRIC EXAMPLES

In this section, we examine two axially symmetric mass distributions in the context of weak lensing as expressed by Eq. 42. The models we consider include a singular, Schwarzschild-like point lens and a singular isothermal sphere (SIS) model. For each model, we will determine the gravitational potential, $\psi(r)$, then compute the projected Weyl and Ricci tensor components and show that they obey Eq. 42.

A. Point lens

As our first example, we consider a Schwarzschild point lens of mass M at the origin. For this mass distribution, the gravitational potential ψ is given by

$$\psi = 2M \ln(r) = 2M \ln \left(\sqrt{x^2 + y^2} \right). \quad (43)$$

One can show by direct computation that

$${}_{L}\Psi_0 = -\frac{2M}{r^2} e^{2i\phi}, \quad (44)$$

which matches Eq. 40 yielding

$$\Upsilon = -\frac{2M}{r^2}. \quad (45)$$

Inserting this functional form for Υ into the right hand side of Eq. 42 gives

$$\frac{-2M}{r^2} + \int dr \frac{-4M}{r^3} = 0. \quad (46)$$

Equation 46 indicates, through the Bianchi identity Eq. 42, that the projected Ricci tensor is zero at all radii $r > 0$, which is the expected result for a Schwarzschild, point-like lens.

B. SIS model

The SIS model is a one parameter model given by

$$\Sigma(r) = \frac{\sigma_v^2}{2r}, \quad (47)$$

where σ_v is the velocity dispersion. Although unphysical due to the singularity at the origin and infinite total mass, SIS models do explain the flat rotation curves of galaxies and are widely used as models in gravitational lensing studies.

For our purposes, the simplest way to integrate the projected gravitational potential for the SIS model is to follow the argument of Schneider et al. [13] in section 8.1, where it is shown that one can write the projected gravitational potential as

$$\psi(r) = 8\pi \int_0^r r' dr' \Sigma(r') \ln\left(\frac{r}{r'}\right). \quad (48)$$

It is then relatively simple to show that

$$\psi = 4\pi \sigma_v^2 r, \quad (49)$$

so that the projected Weyl tensor is

$${}^L\Psi_0 = -\frac{2\pi\sigma_v^2}{r} e^{2i\phi}. \quad (50)$$

From Eq. 40, we have $\Upsilon = -2\pi\sigma_v^2/r$ and from the Bianchi identity

$$\Upsilon + \int dr \frac{2\Upsilon}{r} = \frac{2\pi\sigma_v^2}{r}. \quad (51)$$

By direct computation,

$${}^L\Phi_{00} = \frac{1}{2}(\psi_{xx} + \psi_{yy}) = \frac{2\pi\sigma_v^2}{r}, \quad (52)$$

so that the Bianchi identity, Eq. 42 is preserved.

IX. DISCUSSION

The main results of this paper are that the fundamental quantities of weak gravitational lensing are projected components of the Ricci and Weyl tensors and that the Bianchi identity provides a field equation for the projected matter density derived from first principles. We explicitly show that the common integral equation used in weak gravitational is an integral version of the Bianchi identity. To our knowledge, these results are known in the lensing or relativity communities.

From this perspective, this paper extends the work presented in a series of recent papers that has helped reunite applied gravitational lensing with its roots in general relativity. The underlying perspective of these papers has been that any observed lensing phenomena must be encoded into the past light cone of the observer.

A primary purpose of these papers has been to identify the equation from general relativity that is approximated in the thin lens treatment used by practicing astrophysicists. Frittelli & Newman [8] first pointed out that the lens should be coded into a space-time metric, and then solving the null geodesic equations would simultaneously give the “time of flight” equation and lens mapping. In a two-paper set, Frittelli et al. [9] related general relativity’s optical scalars to the “shears” and “convergences” cited in the thin-lens literature. These papers also wrote out integral relations that showed how image distortion grew continuously along the pencil of rays connected a source and observer. A generalization of the Fermat principle of least time was present in Frittelli et al. [17].

Because Eq. 23 is kinematically derived, it does not indicate a true connection between observational weak lensing and relativistic first principles. The derivation of Eq. 23 relies only on the definitions of Φ_{00} and Ψ_0 .

However, we show in this paper that the projected Bianchi identity, Eq. 26 is the differential version of Eq. 23. Thus, we have a first principles derivation of the basic equation of weak gravitational lensing for the first time and a potentially useful new PDE approach to the subject.

Most directly, Eq. 26 provides a new PDE approach for determining the projected mass density in weak lensing studies. While numerical approaches to solving PDEs with real data tend to be less stable than integral approaches, they also tend to provide higher accuracy, so it is not unreasonable to pursue approaches to data analysis based directly on these results.

Another potential application of this work might be in the cumulative analysis of thick lenses or repeated lensing by multiple clusters along the line of sight. In this case, one would not apply a thin-lens assumption, but would use Eq. 25 as a field equation along the line of sight for a full metric that encodes a more realistic cosmological model.

Acknowledgments

The authors wish to acknowledge Simonetta Frittelli, Ted Newman and Al Janis for helpful suggestions and comments. BK would like to thank the Adrian Tinsley Program for Undergraduate Research of BSC for making his involvement possible.

APPENDIX A: WEAK FIELD RICCI AND WEYL TENSORS

In this appendix, we give the usual component form of the Ricci and Weyl tensors considered in the text, along with their definitions in the Newman & Penrose [1] formalism. In Eqs. A1-A3, the usual Einstein summation conventions are not employed.

For the weak field metric in Eq. 3, to first order in the perturbation, φ , the non-zero Christoffel symbols are

$$\begin{aligned} \Gamma^0_{0i} &= \varphi_i & \Gamma^i_{ik} &= -\varphi_k \\ \Gamma^i_{00} &= \varphi_i & \Gamma^i_{kk} &= \varphi_i \quad (i \neq k) \end{aligned} \quad (\text{A1})$$

where $\varphi_i = \partial_i \varphi$, 0 denotes time and i, j, k denote spatial components. The non-zero, first order Ricci tensor components are

$$R_{00} = -\nabla^2 \varphi \quad R_{ii} = -\nabla^2 \varphi, \quad (\text{A2})$$

with $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$, while the non-zero, first order Weyl tensor components are

$$\begin{aligned} C_{0i0i} &= \frac{1}{3} (-3\varphi_{ii} + \nabla^2 \varphi) & C_{0i0j} &= -\varphi_{ij} \quad i \neq j \\ C_{ijij} &= \frac{1}{3} (3\varphi_{kk} - \nabla^2 \varphi) \quad i \neq j \neq k & C_{ijik} &= -\varphi_{jk} \quad i \neq j \neq k. \end{aligned} \quad (\text{A3})$$

In the NP formalism, the components of the Ricci and Weyl tensors are contracted with the null tetrad to give named components. The Weyl tensor components are defined by

$$\Psi_0 = -C_{abcd} \ell^a m^b \ell^c m^d, \quad \Psi_1 = -C_{abcd} \ell^a n^b \ell^c m^d,$$

$$\Psi_2 = -\frac{1}{2} (C_{abcd}\ell^a n^b \ell^c n^d - C_{abcd}\ell^a n^b m^c \bar{m}^d),$$

$$\Psi_3 = C_{abcd}\ell^a n^b n^c \bar{m}^d, \quad \Psi_4 = -C_{abcd}n^a \bar{m}^b n^c \bar{m}^d, \quad (\text{A4})$$

and the Ricci tensor components are

$$\begin{aligned} \Phi_{00} &= -\frac{1}{2}R_{ab}\ell^a\ell^b, & \Phi_{10} &= -\frac{1}{2}R_{ab}\ell^a\bar{m}^b, & \Phi_{20} &= -\frac{1}{2}R_{ab}\bar{m}^a\bar{m}^b, \\ \Phi_{01} &= -\frac{1}{2}R_{ab}\ell^am^b, & \Phi_{11} &= -\frac{1}{4}(R_{ab}\ell^an^b + R_{ab}m^a\bar{m}^b), & \Phi_{21} &= -\frac{1}{2}R_{ab}n^a\bar{m}^b, \\ \Phi_{02} &= -\frac{1}{2}R_{ab}m^am^b, & \Phi_{12} &= -\frac{1}{2}R_{ab}n^am^b, & \Phi_{22} &= -\frac{1}{2}R_{ab}n^an^b, \\ & & \Lambda &= \frac{1}{24}R. \end{aligned} \quad (\text{A5})$$

-
- [1] Newman, E.T. & Penrose, R., *J. Math Phys.* **3**, 566-78 (1962)
 - [2] Miralda-Escude, J., *IAU Symp.* **173**, 131 (1996)
 - [3] Hawking, S.W. & Ellis, G.F.R., *The Large Scale Structure of Space-time*, (Cambridge University Press, Cambridge, 1973)
 - [4] Miralda-Escude, J., *Astrophys. J.* , **370**, 1 (1991)
 - [5] Kaiser, N., & Squires, G., *Astrophys. J.* **404**, 441 (1993)
 - [6] Mellier, Y., *Ann. Rev. A&A* **37**, 127 (1999)
 - [7] Schneider, P., & Bartelmann, M., *Phys. Rept.* **340**, 291 (2001)
 - [8] Frittelli, S., & Newman, E.T., *Phys. Rev. D* , **59**, 124001 (1999)
 - [9] Frittelli, S., Kling, T.P., & Newman, E.T., *Phys. Rev. D* , **63**, 023006 (2001)
 - [10] Frittelli, S., Kling, T.P., & Newman, E.T., *Phys. Rev. D* , **63**, 023007 (2001)
 - [11] Perlick, V., *Living Rev. Relativity* 7, 9. <http://www.livingreviews.org/lrr-2004-9> cited on June 27, 2005
 - [12] Tyson, J.A. et al., *ASPC*, **237**, 417T (2001)
 - [13] Schneider, P., Ehlers, J. & Falco, E.E., *Gravitational Lenses*, (Springer-Verlag, Berlin, Heidelberg, 1992)
 - [14] Newman, E.T. & Tod, K.P., *General Relativity and Gravitation: One Hundred Years after Einstein, vol 2*, (Perseus Publishing, New York, 1980)
 - [15] Porter, J.R., *GRG*, **13**, 531 (1981)
 - [16] Ivancovich et al., *J. Math Phys.* **30**, 45 (1989)
 - [17] Frittelli, S., Kling, T.P., & Newman, E.T., *Phys. Rev. D* , **65**, 123007 (2002)