Chapter 3 HW – Part 2

Note Title 10/13/2005

Sketch all the qualitatively different vector fields that occur as r is varied. Show that a pitchfork bifurcation occurs at a critical value of r to be determined and classify the bifurcation as supercritical or subcritical. Sketch the bifurcation diagram of x^* vs. r.

1. $\dot{x} = rx + 4x^3$

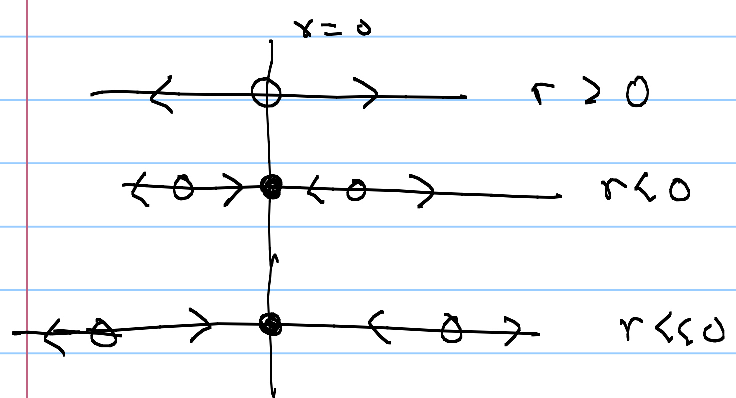
$$f(x) = rx + 4x^3 = x(r + 4x^2)$$

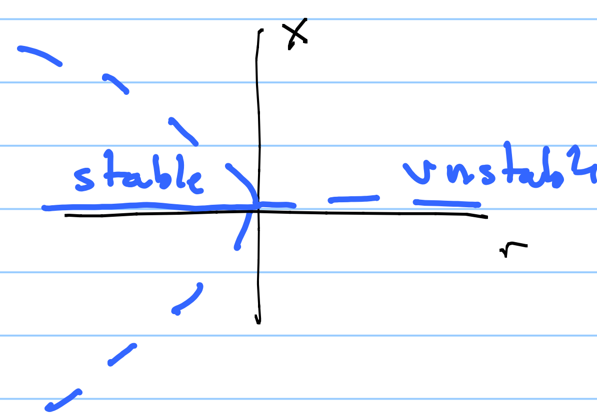
fixed pts are $x = 0$, $x = \pm\sqrt{\frac{-r}{4}}$

only for negative r! (pitchfork)

$$f' = r + 12x^2$$

so $f'(0) = r \rightarrow$ for $r > 0$, $x = 0$ is unstable

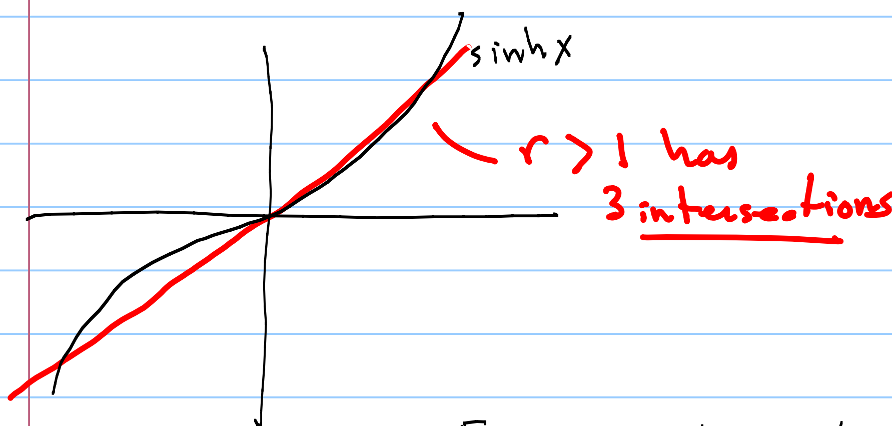




$\rightarrow$ system is subcritical pitchfork

2. $\dot{x} = rx - \sinh x$ Plot rx, $\sinh x$

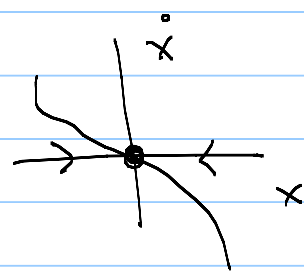
$rx^* = \sinh x^*$ defines fixed pts.



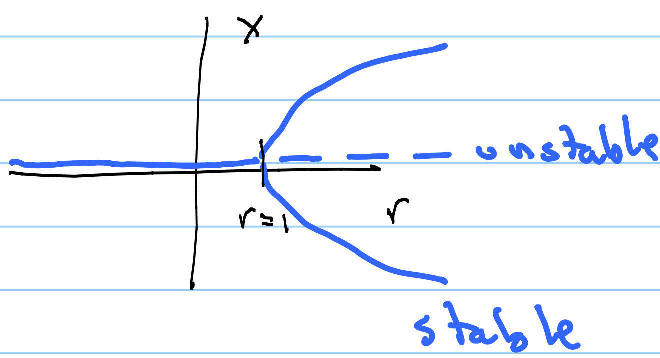
$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$

For $r < 1$, only 1 intersection for a fixed point at $x = 0$.

For $r = 0$,



so the $x = 0$ fixed point is stable for $r < 1$.



$\boxed{\text{Supercritical Pitchfork bifurcation}}$

3. $\cancel{\dot{x} = r - x + \frac{rx}{1+x^2}}$

This problem was supposed to be

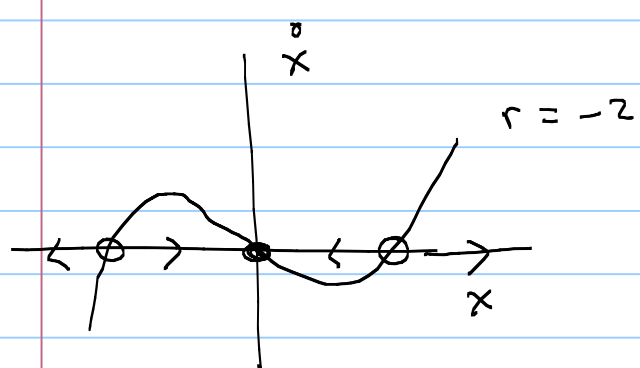
$$\dot{x} = x + \frac{rx}{1+x^2}$$

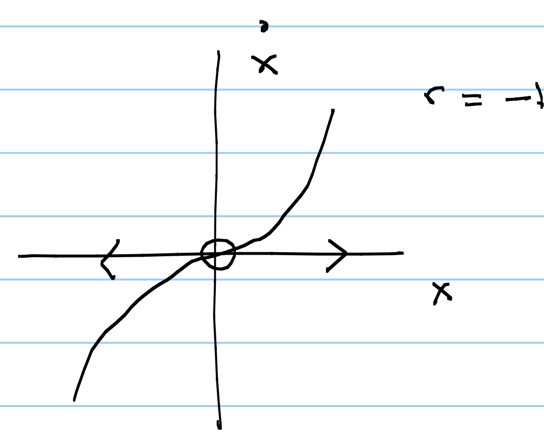
<u>Note</u>: always a fixed pt at $x=0$.

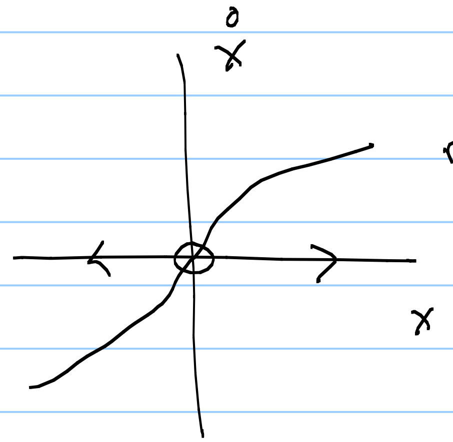
$$f = x + \frac{rx}{1+x^2} = \frac{(1+x^2)x + rx}{1+x^2}$$

$$f = \frac{x(r+1+x^2)}{1+x^2} \qquad \begin{array}{l} x = 0 \\ x = \pm\sqrt{-r-1} \end{array}$$

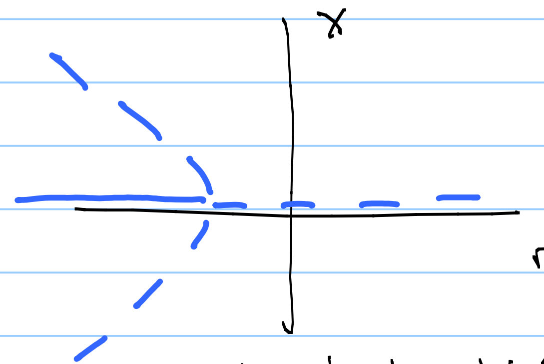
for $\underline{r < -1}$







$r > -1$



Subcritical pitchfork bifurcation.

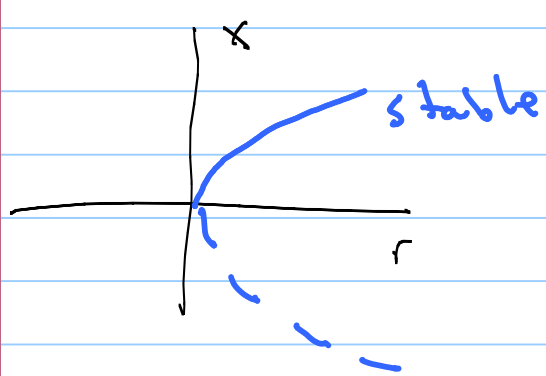
The systems in the next exercises undergo some sort of bifurcation. Determine the value of r where the bifurcations occur, sketch the bifurcation diagrams, and classify the bifurcations.

1. $\dot{x} = r - 3x^2$

$f = r - 3x^2$, $\quad x^* = \pm\sqrt{\frac{r}{3}}$, $\; r > 0$

(Saddle-node!)

$f' = -6x \rightarrow f'\left(+\sqrt{\frac{r}{3}}\right) < 0$ - stable



2. $\dot{x} = rx - \frac{x}{1+x^2}$

$$f = \frac{rx(1+x^2) - x}{1+x^2}$$

$$f = \frac{x\,(r(1+x^2)-1)}{1+x^2}$$

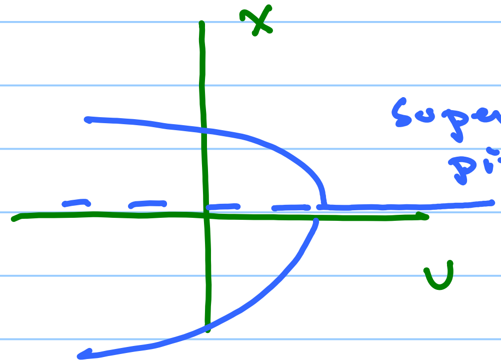
$$x^* = 0$$
$$x^* = \pm\sqrt{\frac{1}{r} - 1}$$

$1 > r > 0$

define $u = \frac{1}{r}$

$$\dot{x} = \frac{x}{u} - \frac{x}{1+x^2} \rightarrow x^* = 0$$
$$x^* = \pm\sqrt{u-1}$$

$u > 1$



supercritical pitchfork.

$$f' = r - \frac{1}{1+x^2} + \frac{2x^2}{1+x^2}$$

$$f'(0) = r - 1$$

$r = 0$, $x^* = 0$ is stable

$f'(0) = \frac{1}{u} - 1$, $u > 1$ stable

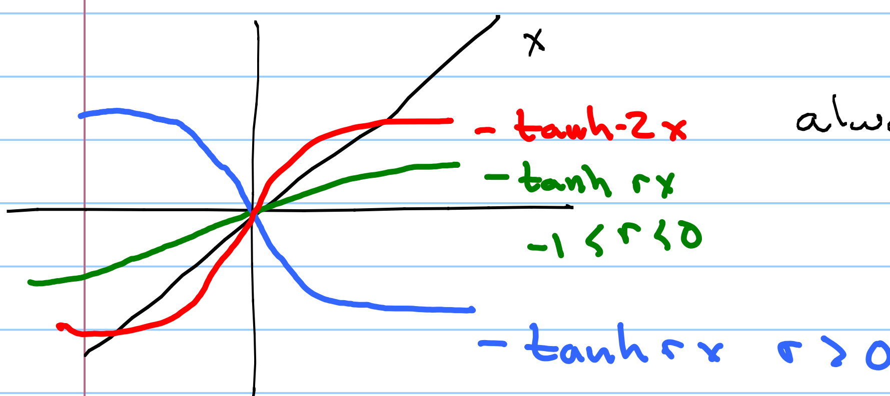
3. $\dot{x} = x + \tanh(rx)$

$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

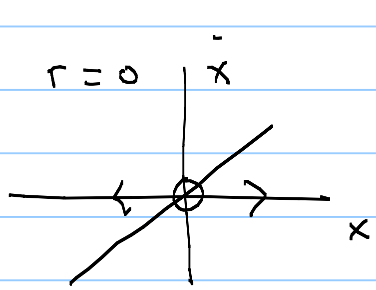
$$\tanh(0) = 0$$

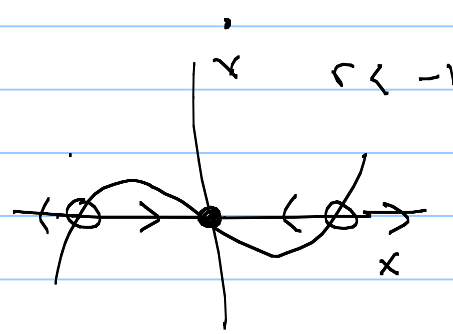
$$\lim_{x \to \pm\infty} \tanh x = 1$$

at $x = 0$, slope of $\tanh rx = \underline{r}$!



always a f.p. at $x=0$.
for $r < -1$, 3 f.p.





subcritical pitchfork

4. $\dot{x} = rx + \frac{x^3}{1+x^2}$

$$f = \frac{rx(1+x^2) + x^3}{1+x^2}$$

$$x\left(r(1+x^2) + x^2\right) = 0 \quad \text{for fixed pts}$$

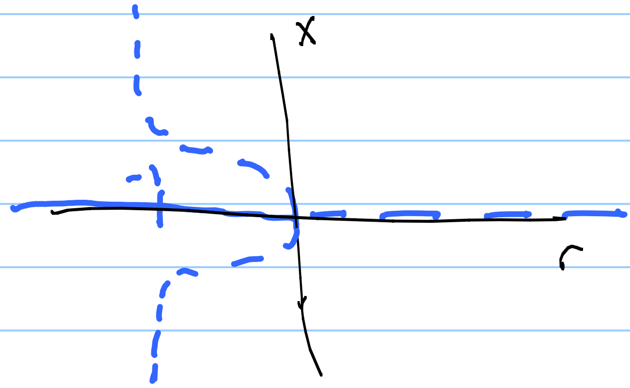
$$x^* = 0$$

$$r + (r+1)x^2 = 0 \Rightarrow x^* = \pm\sqrt{\frac{-r}{r+1}}$$

$-1 < r < 0$

$$f' = r + \frac{3x^2}{1+x^2} - \frac{6x^4}{(1+x^2)^2}$$

$f'(0) = r$ — stable for $r < 0$



unusual subcritical pitchfork

Consider the system $\dot{x} = rx + x^3 - x^5$, which exhibits a subcritical pitchfork bifurcation.

1. Find algebraic expressions for all the fixed points as r varies.
2. Sketch the vector fields as r varies. Be sure to indicate all the fixed points and their stability.
3. Calculate r_s, the parameter value at which nonzero fixed points are born in a saddle-node bifurcation.

$$f = x(r + x^2 - x^4) \qquad x^* = 0$$

Let $u = x^2$

$$-u^2 + u + r = 0$$

$$u = \frac{-1 \pm \sqrt{1+4r}}{-2}$$

$$u = \frac{1}{2} \mp \frac{1}{2}\sqrt{1+4r}$$

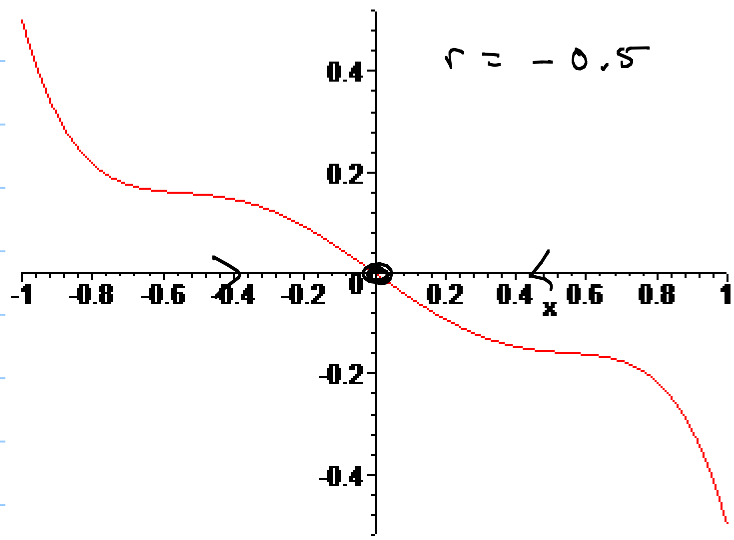
$$x^* = +\sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1+4r}}, \quad -\sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1+4r}}$$

$$x^* = +\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1+4r}}, \quad -\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1+4r}}$$

need $\quad 1+4r \geq 0 \longrightarrow r > -\frac{1}{4}$

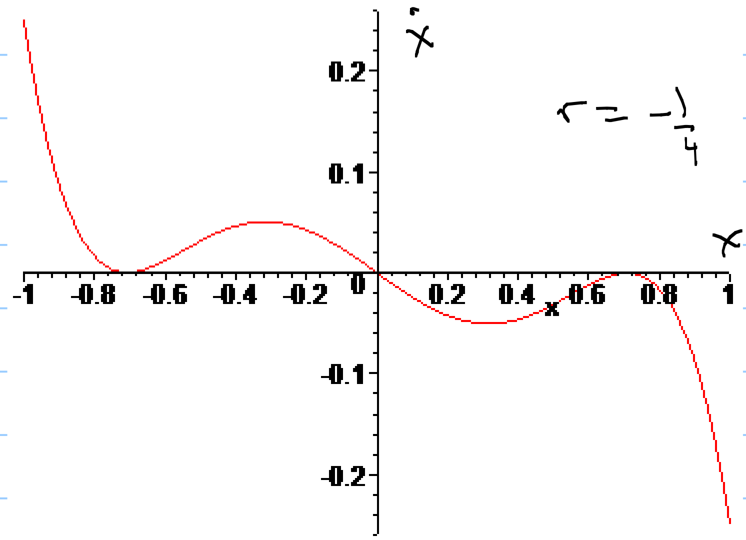
$\frac{1}{2} - \frac{1}{2}\sqrt{1+4r} \geq 0 \longrightarrow r < 0$

For $\quad r < -\frac{1}{4}$ — only f.p. is $x = 0$, and is stable

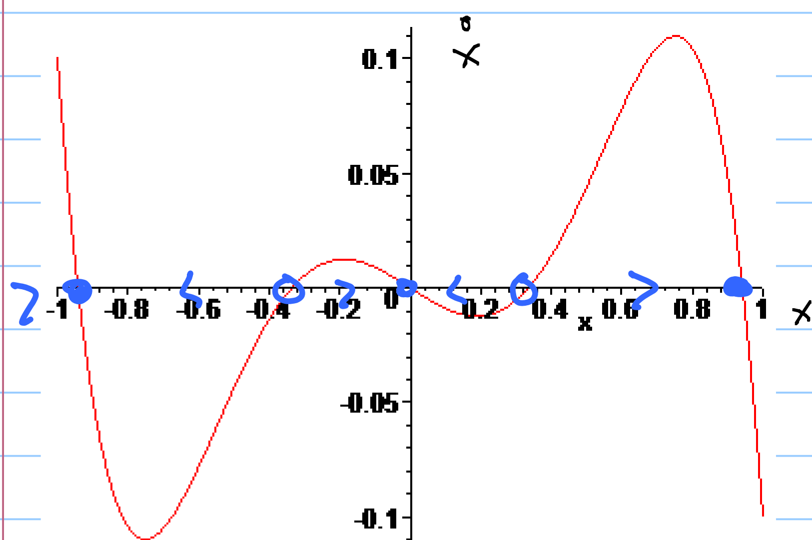


at $r = -\frac{1}{4}$, we have a saddle-node bifurcation

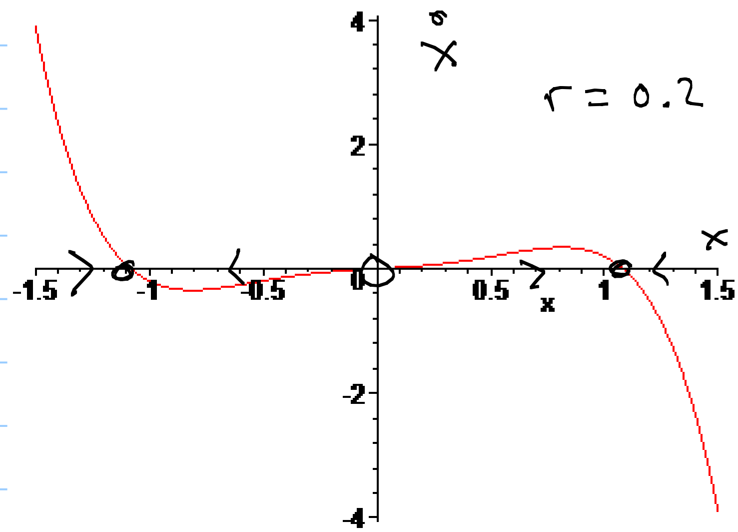
$r_s = -\frac{1}{4}$



For $-\frac{1}{4} < r < 0$, get 5 f.p.



For $r > 0$, have only 3 f.p.



Bifurcation diagram

at $r = -\frac{1}{4}$

$x^* = \pm\sqrt{\frac{1}{2}}$

value of r_s