Chapter 2 Homework, Part 2

Physics 459: Nonlinear Dynamics

You should complete this assignment and Chapter 2, Part 1 by Thursday, Feb 2. You should have made significant progress by Tuesday, Jan 31.

1. **2.3.1.(b)** Solve the logistic equation, $\dot{N} = rN(1 - N/K)$ exactly by substituting $x = 1/N$, and deriving and solving the differential equation for x.

2. **2.3.3** The growth of cancerous tumors can be modeled by the Gompertz law $\dot{N} = -aN\ln(bN)$, where $N(t)$ is proportional to the number of cells in the tumor and $a, b > 0$ are parameters.

   (a) Interpret a and b biologically.

   (b) Sketch the vector field and graph $N(t)$ for various initial values.

3. Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f(x^*) = 0$, use a graphical argument to determine the stability.

   (a) $\dot{x} = x(1 - x)$

   (b) $\dot{x} = \tan x$

   (c) $\dot{x} = x^2(6 - x)$

   (d) $\dot{x} = x(1 - x)(2 - x)$

   (e) $\dot{x} = ax - x^3$ where a can be positive, negative, or zero. Discuss all the cases here.

4. For each of the following vector fields, plot the potential $V(x)$ and identify all the fixed points and their stability.

   (a) $\dot{x} = x(1 - x)$

   (b) $\dot{x} = \sin x$

   (c) $\dot{x} = r + x - x^3$ for all values of r.

5. **2.5.6** (The leaky bucket) The following example (Hubbard and West 1991, p. 159) shows that in some physical situations, non-uniquesness is natural and obvious, not pathological.

Consider a water bucket with a hole in the bottom. If you see and empty bucket with a puddle beneath it, can you figure out when the bucket was full? No, of course not! It could have finished emptying a minute ago, ten minutes ago, or whatever. The solution of the corresponding differential equation must be non-unique when integrated backwards in time.

Here's a crude model of the situation. Let $h(t)$ = height of the water remaining in the bucket at time t; a = area of the hole; A = cross sectional area of the bucket (assumed constant); $v(t)$ = velocity of the water passing through the hole.

(a) Show $av(t) = A\dot{h}(t)$. What physical law are you invoking?

(b) To derive an additional equation, use conservation of energy. First, find the change in the potential energy of the system, using that the height of the water in the bucket decreases by an amount Δh and that the water has density ρ. Then find the kinetic energy transported out of the bucket by the escaping water. Finally, assuming all the potential energy is converted into kinetic energy, derive the (obvious equation) $v^2 = 2gh$. (Note that this is just Bernoulli's law applied to the system. – tpk)

(c) Combining (a) and (b) show $\dot{h} = -C\sqrt{h}$, where $C = \sqrt{2g}\frac{a}{A}$.

(d) Given $h(0) = 0$ (bucket empty at $t = 0$), show that the solution for $h(t)$ is non-unique *backwards* in time, i.e., for $t < 0$.