Chapter 2 HW, part 2

Note Title 9/14/2005

(1) Solving $\dot{N} = rN\left(1 - \frac{N}{K}\right)$ using $x = \frac{1}{N}$

$$\dot{x} = \frac{dx}{dN}\dot{N} = -\frac{1}{N^2}\dot{N}$$

$$\dot{x} = -x^2\left(r\left(\frac{1}{x}\right)\left(1 - \frac{1}{Kx}\right)\right)$$

$$\boxed{\dot{x} = -r\left(x - \frac{1}{K}\right)}$$

define $s = x - \frac{1}{K}$

$\dot{s} = \dot{x}$

$$\dot{s} = -rs \Rightarrow s = s_0 e^{-rt}$$

$$x - \frac{1}{K} = \left(x_0 - \frac{1}{K}\right)e^{-rt}$$

$$\boxed{x = \frac{1}{K} + \left(x_0 - \frac{1}{K}\right)e^{-rt}}$$

as $t \to \infty$,

$x \to \frac{1}{K} = \frac{1}{N_\infty}$

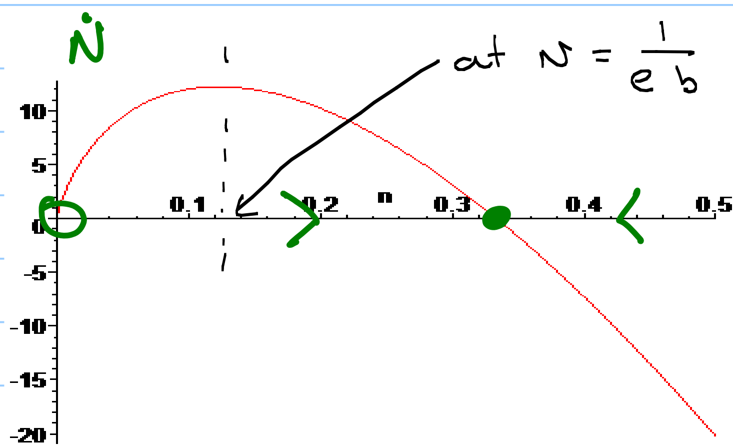
or $$\frac{1}{N} = \frac{1}{K} + \left(x_0 - \frac{1}{K}\right)e^{-rt}$$

$$\boxed{N = \frac{1}{\frac{1}{K} + \left(\frac{1}{N_0} - \frac{1}{K}\right)e^{-rt}}}$$

2. **2.3.3** The growth of cancerous tumors can be modeled by the Gompertz law $\dot{N} = -aN\ln(bN)$, where $N(t)$ is proportional to the number of cells in the tumor and $a, b > 0$ are parameters.

(a) Interpret a and b biologically.

(b) Sketch the vector field and graph $N(t)$ for various initial values.



at $N = \frac{1}{e\,b}$

for $a = 100$, $b = 3$

$\dot{N} = 0$ at $b = \frac{1}{N}$, so

a controls the max $\dot{N}$

$\frac{1}{b}$ is the carrying capacity.

③

(a) $\dot{x} = x(1-x)$

$f(x) = x - x^2$, $x^* = 0, 1$

$f' = 1 - 2x$, $f^*(0) = 1$ unstable

$f^*(1) = -1$ stable

(b) $\dot{x} = \tan x$ $\quad x^* = 0, \pi, -\pi, \ldots \quad x^* = n\pi$

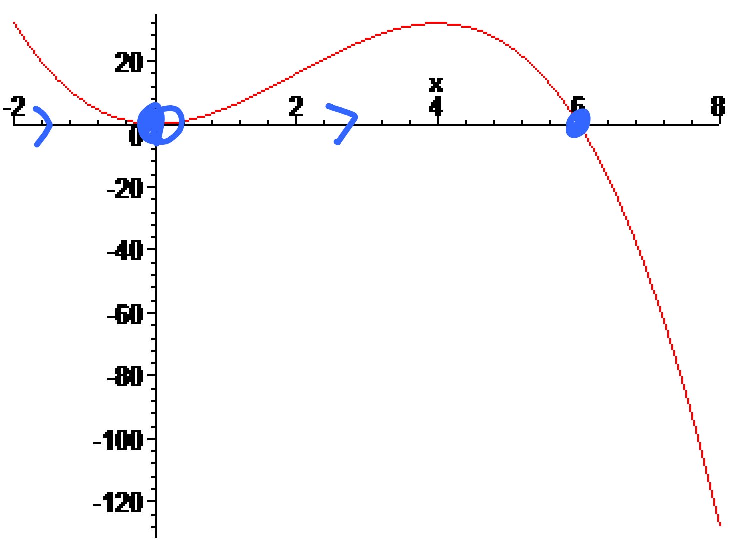
$f'(x) = \sec^2 x, \quad \sec^2 0 = 1$ – unstable.

(c) $\dot{x} = x^2(6-x)$

$x^* = 0, 6 \qquad f' = 12x - 3x^2$

$f'(0) = 0$! – meta stable

$f'(6) < 0$ – stable



(d) $\dot{x} = x(1-x)(2-x)$ $\quad x^* = 0, 1, 2$

$f = x(2 - 3x + x^2)$

$f = 2x - 3x^2 + x^3$

$f' = 2 - 6x + 3x^2$

$f'(0) = 2$ – unstable

$f'(1) = -1$ – stable

$f'(2) = 2$ – unstable

(e) $\dot{x} = ax - x^3$ where a can be positive, negative, or zero. Discuss all the cases here.

$f(x) = ax - x^3 = 0 \rightarrow ax = x^3$

$x^* = 0$, if $x^* \neq 0$ $\quad a = x^2$

a changes the <u>number</u> of fixed pts.

$\begin{cases} \text{if } a > 0, & x^* = 0, \pm\sqrt{a} \\ \text{if } a \leq 0, & x^* = 0 \end{cases}$

$\underline{f' = a - 3x^2}$ $\quad$ if $\underline{a \geq 0}$

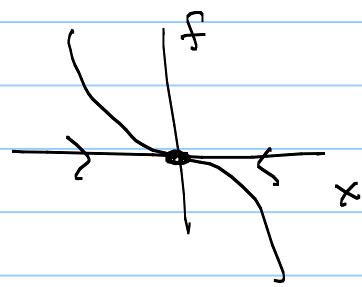
at $0 \leftarrow$ unstable

at $x^* = \pm\sqrt{a}$, stable

$f' = a - 3x^2$ if $a = 0$

$f' = 0$ at $x^* = 0$

$f = -x^3$



so 0 is stable

if $a < 0$

$f'(0) = a < 0$, so $x^* = 0$ is stable

Very important!

For $a \leq 0$, only fixed pt is $x^* = 0$ and this fixed point is stable.

For $a > 0$, this changes. $x^* = 0$ becomes unstable & 2 New stable fixed points appear

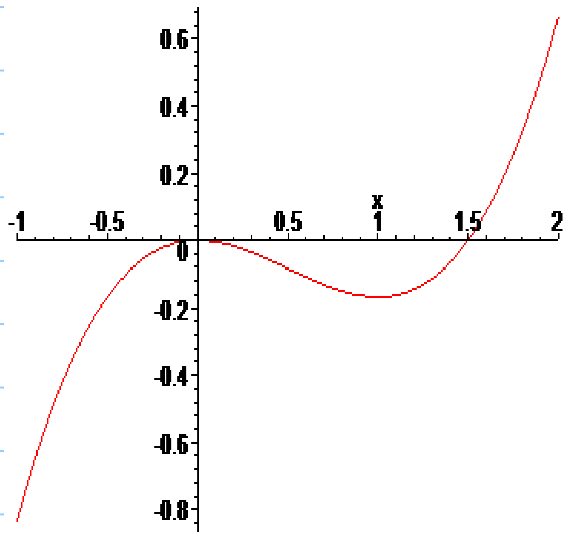
4. For each of the following vector fields, plot the potential $V(x)$ and identify all the fixed points and their stability.

(a) $\dot{x} = x(1-x)$

(b) $\dot{x} = \sin x$

(c) $\dot{x} = r + x - x^3$ for all values of r.

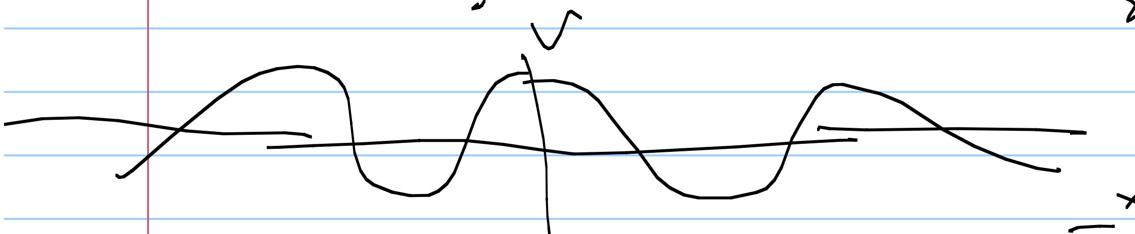
(a) $V = -\int dx\ (x - x^2) = -\dfrac{x^2}{2} + \dfrac{x^3}{3}$



$x = 0$ – unstable

$x = 1$, stable

(b) $V = -\int dx\ \sin x = \cos x$

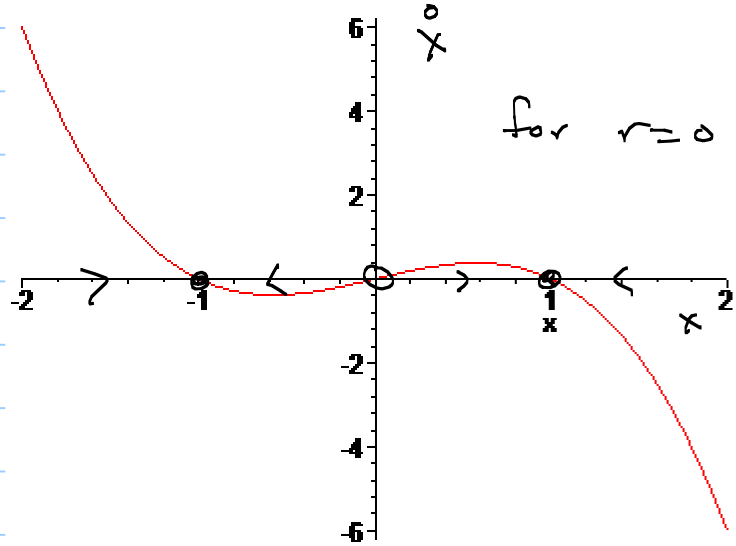


$x = 2\pi n$ – unstable

$x = \pi + 2\pi n$ – stable

(c) $\dot{x} = r + x - x^3$

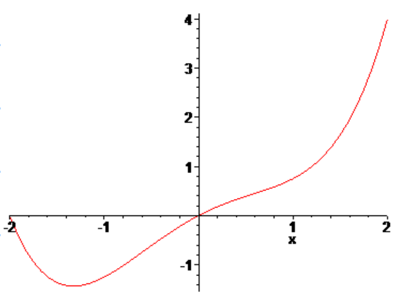
$V = -rx - \frac{x^2}{2} + \frac{x^4}{4}$



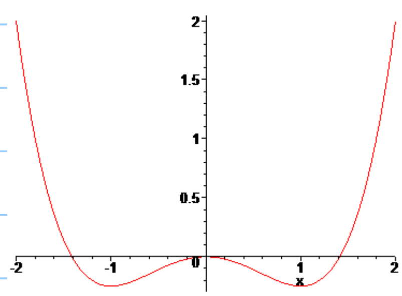
As r changes, the curve moves up and down.

For $r=0$, there are 3 fixed pts.

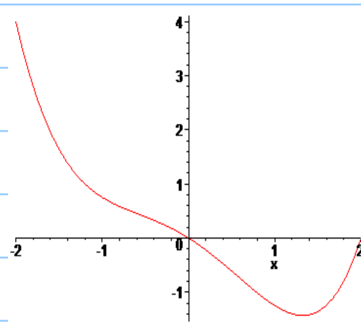
For $r \neq 0$, there may be only one fixed pt.



$r = -1$

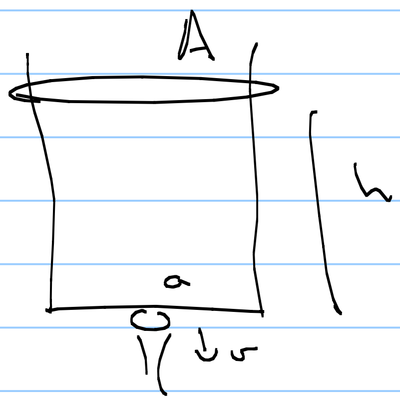


$r = 0$



$r = +1$

(5) Leaky bucket



(a) The rate equation says

$$A_1 v_1 = A_2 v_2$$

so $\boxed{A\dot{h} = a v}$

(This is conservation of mass or water.)

(b) if the water level decreases a height Δh, then

$\rho A \Delta h$ is the mass of water lost in pot. energy

and $\Delta U = (\rho A \Delta h) g h$

This water will have $\Delta KE = \frac{1}{2}(\rho A \Delta h) v^2$

$\Delta KE + \Delta U = 0$ implies

$$\cancel{\rho}\cancel{A}\cancel{\Delta h} g h + \frac{1}{2}\cancel{\rho}\cancel{A}\cancel{\Delta h} v^2 = 0$$

or $\underline{v^2 = 2gh} \Rightarrow v = -\sqrt{2gh}$

(c) $A\dot{h} = av \Rightarrow \dot{h} = -\frac{a}{A}\sqrt{2gh}$

$\boxed{\dot{h} = -C\sqrt{h}}$ $\quad C = \frac{a}{A}\sqrt{2g}$

our flow

$f(h) = -C\sqrt{h}$, note $f' = \frac{-C}{2\sqrt{h}}$

and $f'(0)$ is undefined.

(d) for $h(0) = 0$, one solution is

$h = 0$ for all t! (The bucket never had any water.)

Note $\dot{h} < 0$, so as time moves forward, $h \downarrow$.

define $\tau = -t$, so $\tau \uparrow$ is $t \downarrow$

$$\frac{dh}{d\tau} = \frac{dh}{dt}\frac{dt}{d\tau} = -\dot{h}$$

$$\frac{dh}{d\tau} = c\sqrt{h}$$

$$\int_0^h \frac{dh}{\sqrt{h}} = \int_0^\tau c\, d\tau$$

can change to any τ_0, so bucket has $h=0$ at <u>any</u> τ_0.

$$2\sqrt{h} = c\tau$$

$$\underline{h = \left(\frac{c\tau}{2}\right)^2 = \left(\frac{-ct}{2}\right)^2}$$

another solution.