Problem 1.39 part b

Note Title 9/11/2009

1.39 part b)

Check $\displaystyle\int_V \nabla\cdot\vec{v}\, d\tau = \oint_S \vec{v}\cdot d\vec{a}$

over hemisphere of radius R sitting on the x-y plane.

$$\nabla\cdot\vec{v} = 5\cos\theta - \sin\phi$$

$$\int \nabla\cdot\vec{v}\, d\tau = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} 5r^2 \sin\theta\cos\theta\, dr\, d\theta\, d\phi$$

$$+ \int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} -\sin\phi\, \sin\theta\, r^2\, dr\, d\theta\, d\phi$$

$1/2$

$$= \frac{5R^3}{3}(2\pi)\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta$$

$$- \frac{R^3}{3}\cancel{\int_0^{2\pi} \sin\phi\, d\phi}\; \int_0^{\pi/2} \sin\theta\, d\theta$$

0

$$\boxed{\int \nabla \cdot \vec{v}\, d\tau = \frac{5R^3\pi}{3}}$$

$$\oint \vec{v} \cdot d\vec{a} = \int_{Top} \vec{v} \cdot d\vec{a}_{Top} + \int_{bottom} \vec{v} \cdot d\vec{a}_{bot}$$

$$d\vec{a}_{top} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

$$d\vec{a}_{bott} = -\hat{z}\left(r \, dr \, d\phi\right) \qquad \text{(Think cylindrical)}$$

Top $\quad \vec{v} \cdot d\vec{a} = R^3 \sin\theta \cos\theta \, d\theta \, d\phi$

$$\int \vec{v} \cdot d\vec{a} = R^3 \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta \cos\theta = \pi R^3$$

<u>Bottom</u>: Look at eqn 1.64. At $\theta = \pi/2$ only

$\hat{\theta}$ pts in $\hat{z}$ dir,

$$\hat{\theta} = -\hat{z} \quad \text{at} \quad \theta = \pi/2$$

so $\vec{v} \cdot d\vec{a} = r\sin\theta \Big|_{\pi/2} \hat{\theta} \cdot (-\hat{z})\, r\,dr\,d\phi$

$= r^2\,dr\,d\phi$

$$\int_{\text{bott}} \vec{v} \cdot d\vec{a} = \int_0^{2\pi}\int_0^{R} r^2\,dr\,d\phi = \frac{2\pi R^3}{3}$$

$$\boxed{\oint \vec{v} \cdot d\vec{a} = \pi R^3 + \frac{2\pi R^3}{3} = \underline{\frac{5}{3}\pi R^3}}$$

which is the same as the previous result.