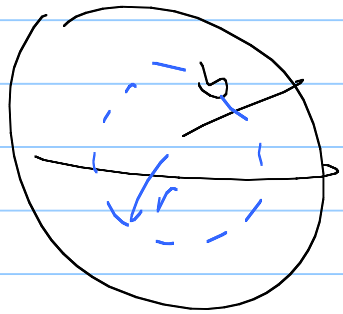
Midterm Solutions

Note Title 3/1/2011

1

$$\rho = kr \quad \text{for} \quad r < b$$



Inside $\oint \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \int \rho \, d\tau$

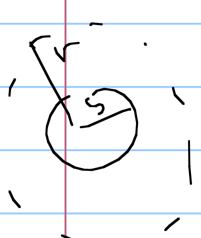
$$E(4\pi r^2) = \frac{4\pi}{\epsilon_0} \int_0^r (kr) \, r^2 dr$$

$$E(4\pi r^2) = \frac{4\pi}{\epsilon_0} k \left. \frac{r^4}{4} \right|_0^r = \frac{\pi k r^4}{\epsilon_0}$$

$$\boxed{E = \frac{k r^2}{4 \epsilon_0} \quad r < b}$$

Outside: $q_{enc} = \int_0^b (kr) \, r^2 \sin\theta \, dr \, d\theta \, d\phi$

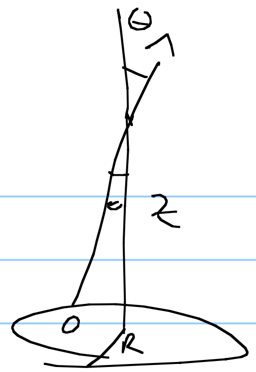
$$q_{enc} = 4\pi k \left. \frac{r^4}{4} \right|_0^b = \underline{\pi k b^4}$$



$$\oint \vec{E} \cdot d\vec{a} = 4\pi r^2 E = \frac{\pi k b^4}{\epsilon_0}$$

$$\boxed{E = \frac{k b^4}{4 \epsilon_0 r^2}}$$

(2)



$$E_z = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma \, da}{r^2}\left(\frac{z}{r}\right)$$

$$r = \sqrt{z^2+s^2} \qquad da = s\,ds\,d\phi$$

$$\sigma = ks$$

$$E_z = \frac{k}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{R} \frac{s\,z\,(s\,ds\,d\phi)}{(z^2+s^2)^{3/2}}$$

do ϕ integral

$$E_z = \frac{k(2\pi)z}{4\pi\epsilon_0}\int_0^R \frac{s^2\,ds}{(z^2+s^2)^{3/2}}$$

$$E_z = \frac{kz}{2\epsilon_0}\left[\frac{-s}{\sqrt{s^2+z^2}} + \ln\left(2\left(s+\sqrt{s^2+z^2}\right)\right)\right]_{s=0}^{s=R}$$

$$\boxed{E_z = \frac{kz}{2\epsilon_0}\left[\frac{-R}{\sqrt{R^2+z^2}} + \ln\left(\frac{R+\sqrt{R^2+z^2}}{z}\right)\right]}$$

(3) $$\vec{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 r^2}\,\hat{r} & r > R \\ 0 & r < R \end{cases}$$

$$V = -\int \vec{E}\cdot d\vec{l} = -\int_{\infty}^{0} \vec{E}\cdot d\vec{r}$$

$$V = -\int_{\infty}^{R} \frac{Q}{4\pi\epsilon_0 r^2}\,dr - \int_{R}^{0} \cancel{0}^{\,0}\,dr$$

$$\boxed{V = \frac{Q}{4\pi\epsilon_0}\,\frac{1}{r}\Big|_{\infty}^{R} = \frac{Q}{4\pi\epsilon_0 R}}$$

(4) $$P_2 = \frac{1}{2}(3x^2-1) \qquad P_3 = \frac{1}{2}(5x^3-3x)$$

$$(n+1)\,P_{n+1} = (2n+1)\,x\,P_n - n\,P_{n-1}$$

Take $n = 3$

$$4P_4 = 7x\,P_3 - 3P_2$$

$$P_4 = \frac{7}{4}x\left(\frac{1}{2}\right)(5x^3-3x) - \frac{3}{4}\left(\frac{1}{2}\right)(3x^2-1)$$

$$P_4 = \frac{35}{8}x^4 - \frac{21}{8}x^2 - \frac{9}{8}x^2 + \frac{3}{8}$$

$$\boxed{P_4 = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}}$$

at $r=R$

(5) $V_0(\theta) = 6\cos^2\theta - 2\cos\theta - 2 = 6x^2 - 2x - 2$

$P_2 = \frac{1}{2}(3x^2 - 1)$, $\boxed{4P_2 = 6x^2 - 2}$

$\boxed{-2P_1 = -2x}$

so $V_0(\theta) = 4P_2(\cos\theta) - 2P_1(\cos\theta)$

Inside $V = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ — need $l=2$ + $l=1$

at $r=R$ $V_0 = A_1 R P_1 + A_2 R^2 P_2 = 4P_2 - 2P_1$

$A_1 = -\frac{2}{R}$ $A_2 = \frac{4}{R^2}$

$$\boxed{V = \frac{-2}{R} r P_1 + \frac{4}{R^2} r^2 P_2 \qquad r < R}$$

Outside: $V = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

at $r=R$ $4P_2 - 2P_1 = \frac{B_2}{R^3} P_2 + \frac{B_1}{R^2} P_1$

$B_2 = 4R^3$, $B_1 = -2R^2$

$$\boxed{V = \frac{-2R^2}{r^2} P_1 + \frac{4R^3}{r^3} P_2 \qquad r > R}$$