Lecture 8: Intro to Laplace's equation

Note Title 10/6/2009

HW: 3.6, 3.10

↳ Since $\vec{E} = -\nabla V$, if one can find V, then $\vec{E}$ is easy to compute.

↳ Mostly, want to solve for V around charges, in regions where $\rho = 0$.

So $\nabla^2 V = \frac{-\rho}{\epsilon_0} \Rightarrow \underline{\nabla^2 V = 0}$

Reading 3.1.2 – 3.1.4 is worthwhile, but the main point is

- V is the average of the values around it

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V \, da$$

divide by surface area

weighted average of V on surface

- This directly implies that V has no local minima or maxima.

$\nabla^2 V = 0$ is a Partial Differential Equation.

in Cartesian Coordinates: $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

for some fnc $V(x,y,z)$ in a region

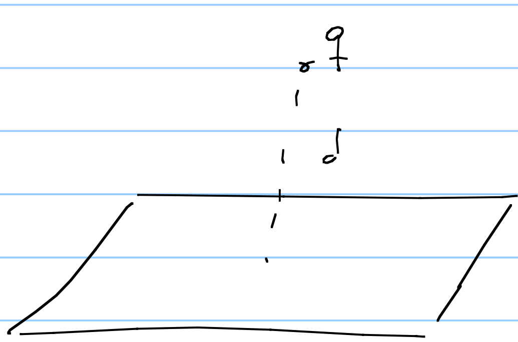
Two types of boundary conditions.

Dirichlet: specify V on the boundary

Neumann: specify the derivative of V on the boundary.

Our book focusses on Dirichlet boundary conditions, and insists that if you find a solution that fits the values of V on the boundary, then that solution is right.

Method of Images: useful if pt charges are held near a grounded conducting 2-D surface.

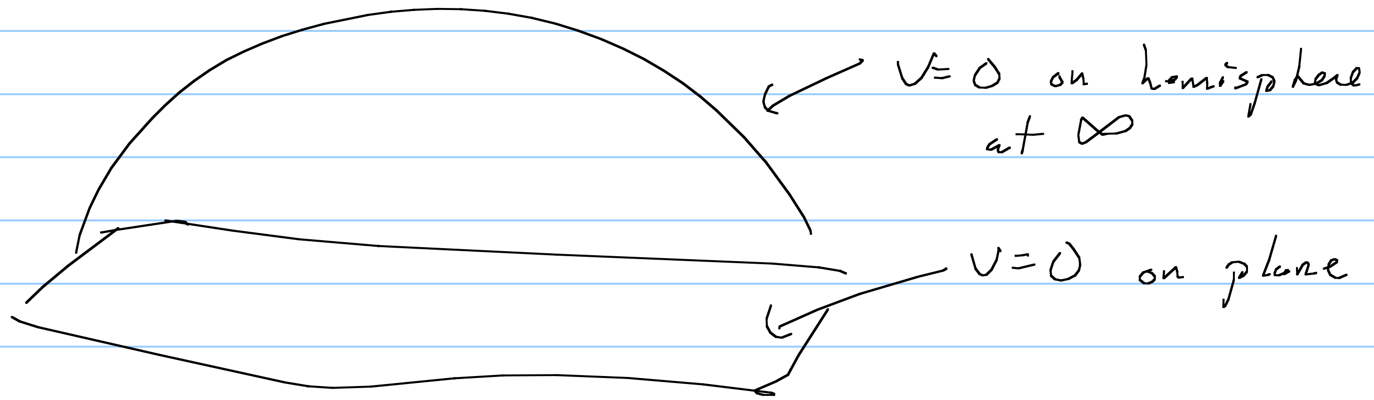


Classic problem: a charge q is held above an infinite conducting plane at $V=0$.

Note: grounded plane $\neq$ no charge on plane
↳ The charge q will induce opposite signed charge density σ to appear on the plane.

Otherwise, the plane would not have $V=0$; it would have V of the pt. charge.

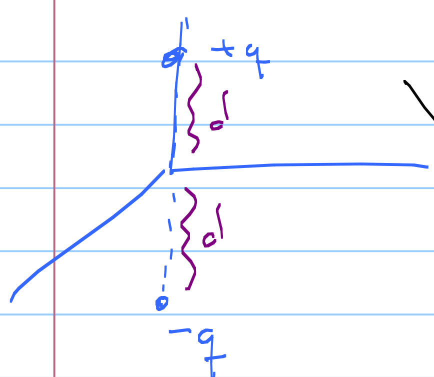
Note: Our b.c. are $V = 0$ on x-y plane and at infinity.



And we solve $\nabla^2 V = 0$ at every pt except the 1 pt where the charge is located.

→ The direct approach would involve figuring out what charge density σ is induced onto the x-y plane. This is hard.

→ So instead, consider the following 2 charges:



$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

Since $V=0$ at infinity and $V=0$ at $z=0$ (x-y plane)

This is the solution.

We can now find $\vec{E}$, or σ on the plane.

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} \quad \leftarrow \quad \frac{\partial V}{\partial n} \text{ is normal derivative on surface, so}$$

$$\frac{\partial V}{\partial n} = \frac{\partial V}{\partial z}\bigg|_{z=0}$$

Get $\sigma = \dfrac{-qd}{2\pi\,(x^2+y^2+d^2)^{3/2}}$ ... ok.

How much total charge is induced onto the surface?

$$Q = \int \sigma \, da = (\text{work it out}) = -q \; !$$

→ In image problems, you place a fake charge of opposite sign a distance on the other side of the conducting surface.

- Can also do w/ spheres – see Ex 3.2
- For problems like 3.10

extend $V=0$

and use enough images