Lecture 3: Mathematical Aspects

Note Title 9/12/2009

Chapter 1 HW, part 3: 1.43, 1.45, 1.46, 1.47, 1.49, 1.52

Dirac Delta Fnc:
(a distribution)

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

$$\& \quad \int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

In general:

$$\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a)$$

limits can be anything as long as $\underline{a}$ is inside

In 3-d:

$$\int_{\substack{\text{all}\\\text{space}}} f(\vec{r})\, \delta^3(\vec{r}-\vec{a})\, d\tau = f(\vec{a})$$

Now:

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\, \delta^3(\vec{r})$$

or

$$\nabla \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\, \delta^3(\vec{\mathscr{r}})$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$$

We use delta functions to place charges & charge densities in particular places.

Problem 1.46

(a) What is $\rho(\vec{r})$ for a pt charge q at $\vec{r}\,'$?

$$\rho(\vec{r}) = q\,\delta^3(\vec{r}-\vec{r}\,')$$

$$\Rightarrow \int_{\text{all space}} \rho(\vec{r})\,d\tau = q!$$

(b) $\rho(\vec{r})$ for $-q$ at origin + $+q$ at $\vec{a}$:

$$\rho(\vec{r}) = -q\,\delta^3(\vec{r}) + q\,\delta^3(\vec{r}-\vec{a})$$

(c) Thin spherical shell radius R, total charge Q

$$\rho = \frac{Q}{4\pi R^2}\,\delta(r-R)$$

Test it (in spherical coordinates)

$$I = \int_{\text{all space}} \frac{Q}{4\pi R^2}\,\delta(r-R)\,r^2 \sin\theta\,dr\,d\theta\,d\phi$$

from θ, ϕ integrals

$$I = \frac{4\pi\, Q}{4\pi R^2} \int r^2\, \delta(r-R)\, dr = \frac{4\pi\, Q}{4\pi R^2} R^2 = Q$$

<u>The Helmholz Theorem</u>: if a field goes to zero at infinity, then

$$\nabla \cdot \vec{F} = \vec{D} \qquad \nabla \times \vec{F} = \vec{C}$$

uniquely determines $\vec{F}$.

(Or, given the boundary condition that $\lim_{\vec{r}\to\infty} F(\vec{r}) = 0$, a field is determined by its divergence and curl.)

$$\begin{tikzcd}
\nabla \times \vec{F} = 0 \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & \oint \vec{F}\cdot d\vec{\ell} = 0 \arrow[d, Leftrightarrow] \\
\boxed{\vec{F} = -\nabla V} \arrow[r, Leftrightarrow] & \int_{\vec{a}}^{\vec{b}} \vec{F}\cdot d\vec{\ell} \text{ is path independent for any } \vec{a}, \vec{b}
\end{tikzcd}$$

All of these are equivalent.

$$\begin{tikzcd}
\nabla \cdot \vec{F} = 0 \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & \oint \vec{F} \cdot d\vec{a} = 0 \arrow[d, Leftrightarrow] \\
\boxed{\vec{F} = \nabla \times A} \arrow[r, Leftrightarrow] & \int \vec{F} \cdot d\vec{a} \text{ is independent of the surface for a given boundary line}
\end{tikzcd}$$

All of these are equivalent.

In Nature we find no magnetic monopoles (which itself leads to an interesting question) so

$$\nabla \cdot \vec{B} = 0$$

So we can introduce a Vector Potential $\vec{A}$, so that

$$\vec{B} = \nabla \times \vec{A}$$

In electrostatics $\left(\frac{d\vec{B}}{dt} = 0 \right)$ and $\nabla \times \vec{E} = 0$, so

$$\vec{E} = -\nabla V,$$ where V is the scalar electric potential