

Lecture 3: Mathematical Aspects

Note Title

9/12/2009

Chapter 1 HW, part 3: 1.43, 1.45, 1.46, 1.47, 1.49, 1.52

Dirac Delta Fnc:
(a distribution)

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

$$+ \int_{-\infty}^{\infty} \delta(x) dx = 1$$

In general:

$$\int_{-\infty}^{\infty} f(x) \delta(x-a) dx = f(a)$$



limits can be anything
as long as a is inside

In 3-d:

$$\int_{\text{all space}} f(\vec{r}) \delta^3(\vec{r}-\vec{a}) d\tau = f(\vec{a})$$

Now:

$$\nabla \cdot \left(\frac{\hat{r}}{r^2} \right) = 4\pi \delta^3(\vec{r})$$

$$\text{or } \nabla \cdot \left(\frac{\hat{r}}{r^2} \right) = 4\pi \delta^3(\vec{r})$$

$$\vec{r} = \vec{r} - \vec{r}'$$

We use delta functions to place charges & charge densities in particular places.

Problem 1.46

(a) What is $\rho(\vec{r})$ for a pt charge q at \vec{r}' ?

$$\rho(\vec{r}) = q \delta^3(\vec{r} - \vec{r}')$$

$$\Rightarrow \int_{\text{all space}} \rho(\vec{r}) d\vec{r} = q!$$

(b) $\rho(\vec{r})$ for $-q$ at origin + $+q$ at \vec{a} :

$$\rho(\vec{r}) = -q \delta^3(\vec{r}) + q \delta^3(\vec{r} - \vec{a})$$

(c) Thin spherical shell radius R , total charge Q

$$\rho = \frac{Q}{4\pi R^2} \delta(r - R)$$

Test it (in spherical coordinates)

$$I = \int_{\text{all space}} \frac{Q}{4\pi R^2} \delta(r - R) r^2 \sin\theta dr d\theta d\phi$$

$$I = \frac{4\pi Q}{4\pi R^2} \int r^2 \delta(r-R) dr = \frac{4\pi Q}{4\pi R^2} R^2 = Q$$

from θ, ϕ integrals

The Helmholtz Theorem: if a field goes to zero at infinity, then

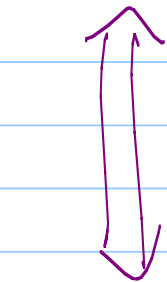
$$\nabla \cdot \vec{F} = \vec{D} \quad \nabla \times \vec{F} = \vec{C}$$

uniquely determines \vec{F} .

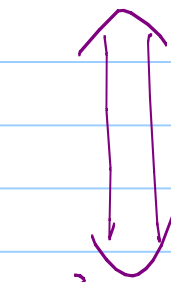
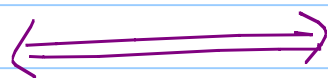
(Or, given the boundary condition that $\lim_{\vec{r} \rightarrow \infty} F(\vec{r}) = 0$,

a field is determined by its divergence and curl.)

$$\nabla \times \vec{F} = 0 \quad \iff \quad \oint \vec{F} \cdot d\vec{\ell} = 0$$



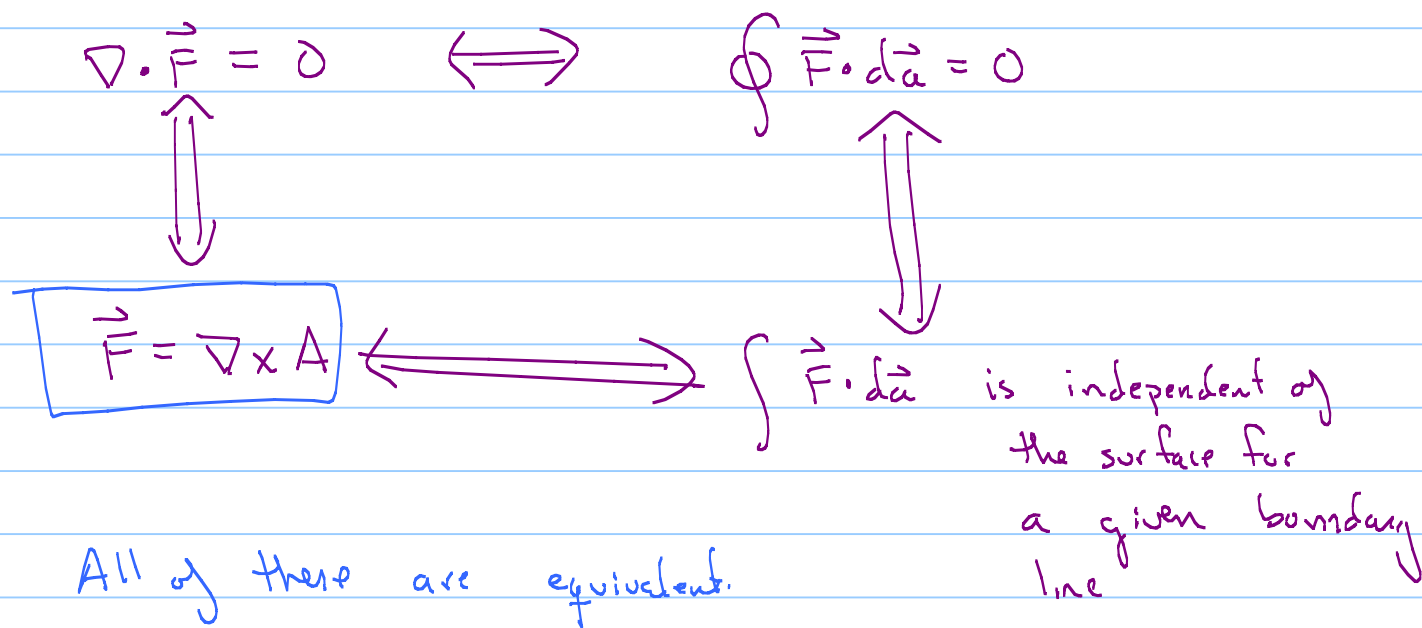
$$\vec{F} = -\nabla V$$



$$\int_{\vec{a}}^{\vec{b}} \vec{F} \cdot d\vec{\ell}$$

is path independent for any \vec{a}, \vec{b}

All of these are equivalent.



In Nature we find no magnetic monopoles
 (which itself leads to an interesting question)
 so

$$\nabla \cdot \vec{B} = 0$$

So we can introduce a Vector Potential \vec{A} , so
 that

$$\vec{B} = \nabla \times \vec{A}$$

In electrostatics ($\frac{d\vec{B}}{dt} = 0$) and $\nabla \times \vec{E} = 0$, so

$$\vec{E} = -\nabla V, \text{ where } V \text{ is the scalar electric Potential}$$