Lecture 2: Integral Calculus

Note Title 9/8/2009

HW – Chapter 1: 29, 31, 32, 33

There are 3 fundamental types of integrals in a 3-dim space.

(1) Line integrals

Final place

$$\int_{\vec{a}\ \mathcal{P}}^{\vec{b}} \vec{v} \cdot d\vec{l}$$

The extent to which $\vec{v}$ aligns to the path.

Initial place

along a particular path, divided into little parts $d\vec{l}$

A special case is when the integral's path is a closed loop.

$$\oint \vec{v} \cdot d\vec{l}$$ ← The "circulation" of $\vec{v}$ over the path.

Note: all ordinary, Calculus 1 integrals are line integrals

$$\int_{x_1}^{x_2} f(x)\, dx \longrightarrow \begin{array}{l} d\vec{l} \to dx\ \hat{x} \\ f(x) \to \vec{F} = (f, \underbrace{f_y, f_z}_{\text{not important.}}) \end{array}$$

(2) Surface integrals: flux of vector field through a surface

$$\int_S \vec{v} \cdot d\vec{a}$$

defined by the surface
- 2-D
- $\perp$ to surface

Special case: closed surface $\oint \vec{v} \cdot d\vec{a}$

Then $d\vec{a}$ pts outward

(3) Volume integrals: typically measure "stuff" inside.

$$\int_V T \, d\tau \qquad d\tau = dx \, dy \, dz$$

in cartesian coords.

$\longrightarrow$ Can do for a vector

$$\int_V \vec{S} \, d\tau = \hat{x} \int_V S_x \, d\tau + \hat{y} \int_V S_y \, d\tau + \hat{z} \int S_z \, d\tau$$

Usually the complicated issues are

(1) how to write $d\vec{l}$, $d\vec{a}$, or $d\tau$

(2) how to write the limits

(3) whether it is neccessary to replace one of the coordinates in the integration w/ a fnc of the other coordinates.

For example 1.28

$$\vec{v} = x^2\hat{x} + 2yz\,\hat{y} + y^2\hat{z}$$

→ integrate from origin $(0,0,0)$ to $(1,1,1)$

(a) Path 1 $\quad \underbrace{(0,0,0) \to (1,0,0)}_{d\vec{l}_1 = dx\,\hat{x}} \to \underbrace{(1,1,0)}_{d\vec{l}_2 = dy\,\hat{y}} \to \underbrace{(1,1,1)}_{d\vec{l}_3 = dz\,\hat{z}}$

$$I_a = \int_0^1 x^2\,dx + \int_0^1 2yz\,dy + \int_0^1 y^2\,dz$$

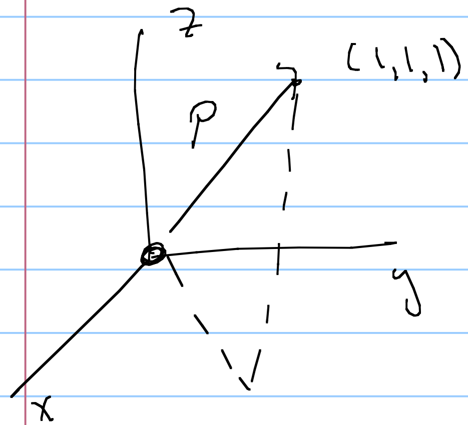
but here $z = 0$!

but here $y = 1$

$$I_a = \int_0^1 x^2\,dx + \int_0^1 dz$$

→ Consider part (c)

→ path is $(0,0,0) \rightarrow (1,1,1)$



Let $\vec{l} = x\hat{x} + y\hat{y} + z\hat{z}$

If $x = y = z = s$, then this is the path.

$$dx = dy = dz = ds$$

take $d\vec{l} = ds\,(\hat{x} + \hat{y} + \hat{z})$

$$\begin{aligned}\vec{v}\cdot d\vec{l} &= x^2 ds + 2yz\,ds + y^2 ds \\ &= s^2 ds + 2s^2 ds + s^2 ds = 4s^2 ds\end{aligned}$$

$$\underset{P}{\int_{(0,0,0)}^{(1,1,1)}} \vec{v}\cdot d\vec{l} = \int_0^1 4s^2\,ds = \frac{4}{3}\,!$$

Fundamental Theorem of Calculus.

$$\left.\begin{aligned}&\int_a^b F(x)\,dx = f(b) - f(a) \\ &\text{w/}\quad F(x) = \frac{df}{dx}\end{aligned}\right\}$$

Tells you how the integral of the derivative of $f(x)$ is related to $f(x)$.

or $$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

→ For <u>Gradients</u>

$$\int_{a\,\mathcal{P}}^{b} \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a)$$

How is the integral of the Gradient of T related to T

Really not that interesting.

→ If path is closed, a = b

$$\& \quad \oint \vec{\nabla} T \cdot d\vec{l} = 0$$

→ For Divergences

$$\int_V (\nabla \cdot \vec{v})\,d\tau = \oint_S \vec{v} \cdot d\vec{a}$$

How is the integral of the Divergence of $\vec{v}$ related to $\vec{v}$?

Green's Theorem or Divergence Theorem

$\oint \vec{v} \cdot d\vec{a}$ is flux through the bounding surface

$\int_V (\nabla \cdot \vec{v})\,d\tau$ is the total extent of divergence in bounding volume.

→ For curls

$$\int_S (\nabla \times \vec{v}) \cdot d\vec{a} = \oint_P \vec{v} \cdot d\vec{l}$$

How is the integral of the curl of $\vec{v}$ related to $\vec{v}$?

$\boxed{\text{Stoke's Theorem}}$

$\int_S (\nabla \times \vec{v}) \cdot d\vec{a}$ is the total amount of the curling of $\vec{v}$ in some area.

$\oint_P \vec{v} \cdot d\vec{l}$ is the amount $\vec{v}$ wraps around the boundary.