Lecture 14: Vector Properties of $\vec{B}$

Note Title 11/17/2009

Ampere's Law is $\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 i_{enc}}$

or w/ $i_{enc} = \int \vec{J}\cdot d\vec{a}$

$$\oint \vec{B}\cdot d\vec{l} = \int (\nabla\times\vec{B})\cdot d\vec{a} = \mu_0 \int \vec{J}\cdot d\vec{a}$$

so $\boxed{\nabla\times\vec{B} = \mu_0 \vec{J}}$

Derive it...

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\times\hat{\imath}}{\imath^2}\,d\tau'$$

$$\nabla\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \nabla\cdot\left(\frac{\vec{J}\times\hat{\imath}}{\imath^2}\right)d\tau'$$

taken w/ respect to $\vec{r}$ so it passes into integral

Front cover rule #6: $\nabla\cdot\left(\vec{J}\times\frac{\hat{\imath}}{\imath^2}\right) = \frac{\hat{\imath}}{\imath^2}\cdot\cancel{(\nabla\times\vec{J})}^{\,0} - \vec{J}\cdot\left(\nabla\times\frac{\hat{\imath}}{\imath^2}\right)$

$\vec{J}(\vec{r}\,')$

And $\nabla\times\frac{\hat{\imath}}{\imath^2} = 0$ ... so

$\boxed{\nabla\cdot\vec{B} = 0}$

Now for the curl...

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\vec{J} \times \frac{\hat{\imath}}{\imath^2} \right) d\tau' \qquad 4\pi \delta^3(\imath)$$

$$+ \quad \nabla \times \left(\vec{J} \times \frac{\hat{\imath}}{\imath^2} \right) = \vec{J} \left(\nabla \cdot \frac{\hat{\imath}}{\imath^2} \right) - \vec{J} \cdot \nabla \frac{\hat{\imath}}{\imath^2}$$

– int. to zero

w/ $\nabla \cdot \vec{J} = 0$ (last term in rule 8)

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}\,') \cdot 4\pi \cdot \delta^3(\vec{\imath})\, d\tau' = \mu_0 \vec{J} \, !$$

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J}}$$

Prove last term goes away...

$$-\vec{J} \cdot \nabla \left(\frac{\hat{\imath}}{\imath^2} \right) = \vec{J} \cdot \nabla' \left(\frac{\hat{\imath}}{\imath^2} \right)$$

↳ 3 components. For x:

$$\vec{J} \cdot \nabla' \left(\frac{x - x'}{\imath^3} \right) = \nabla' \cdot \left[\frac{x - x'}{\imath^3} \vec{J} \right] - \frac{(x - x')}{\imath^3} \cancel{\nabla' \cdot \vec{J}}$$

zero in magnetostatics

So

$$\left[-\vec{J} \cdot \nabla \frac{\hat{\imath}}{\imath^2} \right]_x = \nabla' \cdot \left[\frac{(x - x')}{\imath^3} \vec{J} \right]$$

put into integral

$$\int_{Vol} \nabla' \cdot \left[\frac{x - x'}{\imath^3} \vec{J} \right] d\tau' = \oint_S \frac{x - x'}{\imath^3} \vec{J} \cdot d\vec{a} = 0 \, !$$

This volume can be as big as you want. Take it where $\vec{J}\,' = 0$!

So we have Ampere's Law

$$\nabla \times \vec{B} = \mu_0 \vec{J} \quad \text{or} \quad \oint \vec{B} \cdot d\vec{l} = \mu_0 i_{enc}$$

also $\nabla \cdot \vec{B} = 0$

Useful for (a) 1 infinite straight wire
(b) infinite plane w/ current density $\vec{K}$
(c) infinite solenoids
(d) Toroids

And about nothing else! Go back & use Biot-Savart.

We have some geometrical pictures

$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ $\quad \nabla \times \vec{E} = 0$ $\quad$ $\vec{E}$ diverges & doesn't curl in statics

$\nabla \cdot \vec{B} = 0$ ← "no name" & no monopoles

$\nabla \times \vec{B} = \mu_0 \vec{J}$ ← $\vec{B}$ curls.

Since $\nabla \cdot \vec{B} = 0$, $\vec{B}$ is $\vec{B} = \nabla \times \vec{A}$

$\vec{A} \Rightarrow$ vector potential.

Now more than 1 $\vec{A}$ leads to the same $\vec{B}$. This is a gauge freedom.

$$\nabla \times \vec{B} = \nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}.$$

We can <u>choose</u> $\nabla \cdot \vec{A} = 0$. Why? You can always make it so.

- Suppose $\nabla \cdot A_0 \neq 0$ ...

write $\vec{A} = A_0 + \nabla \lambda$ for some fnc λ.

Try to make $\nabla \cdot \vec{A} = 0$ ...

$$\nabla \cdot \vec{A} = 0 = \nabla \cdot \vec{A}_0 + \nabla^2 \lambda$$

or $\boxed{\nabla^2 \lambda = - \nabla \cdot \vec{A}_0}$ Poisson's eqn.

Solution is $\lambda = \dfrac{1}{4\pi} \displaystyle\int \dfrac{\nabla \cdot \vec{A}_0}{r} \, d\tau'$

But the point is that $\vec{A}$ & $\vec{A}_0$ w/ $\vec{A} = A_0 + \nabla \lambda$ have the same $\vec{B}$!

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \vec{A}_0 + \cancelto{0}{\nabla \times \nabla \lambda} = \vec{B}_0 \,!$$

So $\underline{\text{Assume}}$ $\nabla \cdot \vec{A} = 0$ +

$$\nabla \times \vec{B} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} \Rightarrow \boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

$\underline{\text{Three}}$ Poisson's eqns!

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}} \, d\tau'}$$

So to find $\vec{B}$, you can try to find $\vec{A}$. (Note: this is not as easy as $\vec{E}$ + V!)

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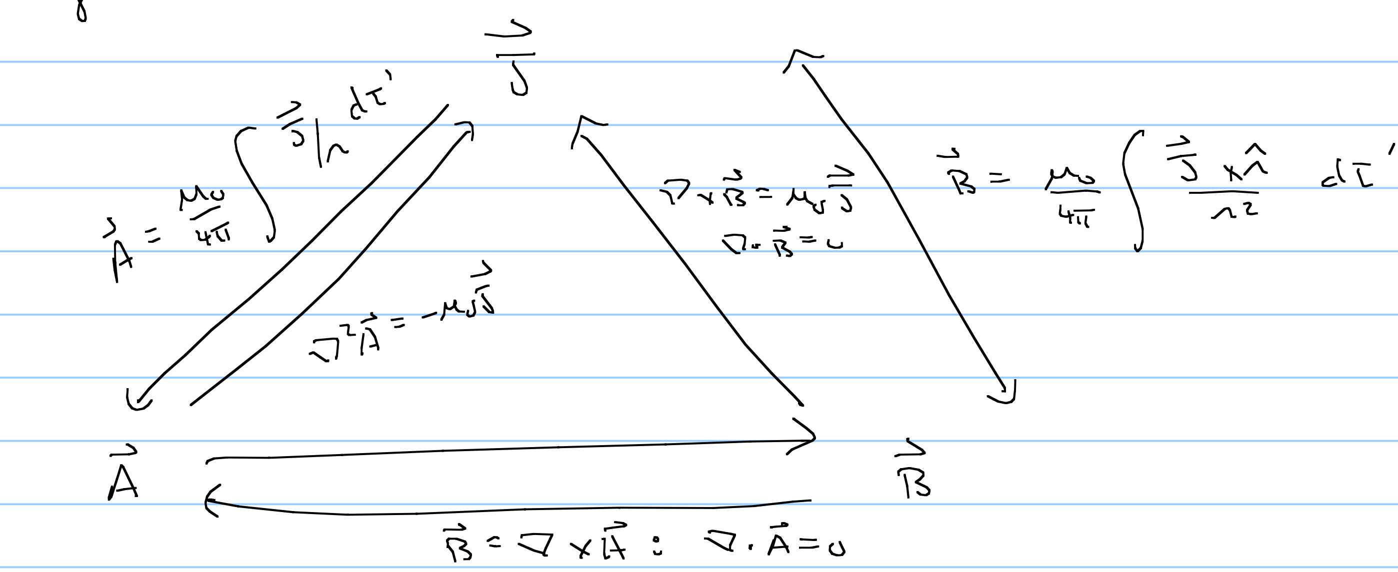

Similar to the boundary conditions on $\vec{E}$, we can get b.c. for $\vec{B}$ around a 2-D, current carrying sheet w/ density $\vec{K}$

$$\vec{B}_{above} - \vec{B}_{below} = \mu_0 (\vec{K} \times \hat{n})$$

$$\text{so} \quad B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K$$

$$B^{\perp}_{above} = B^{\perp}_{below}$$

$\vec{A}$ is continuous (like V), w/ a discontinuity in the derivative (like V).

$$\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}$$

Since $\nabla^2 \vec{A} = -\mu_0 \vec{J}$, many of the tricks of Chap 2 work again. In particular there is a multipole expansion - starting w/ the dipole of course.

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} \qquad \vec{m} = I \int d\vec{a} = I\vec{a}$$

HW: Ch 5 Problems: 19, 20, 21, 22, 23, 34