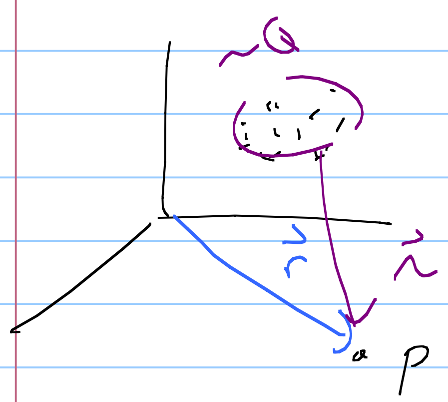
Lecture 10: Multipole expansion

Note Title 10/18/2009

Good HW: 3.28, 3.31

Suppose you have some charge distribution "near" the origin w/ a total charge Q.



Very far away, the field will look like a point charge... and if $\vec{r}$ is very big, $\vec{r}$ + $\vec{\mathscr{r}}$ will be about the same, so the potential will look like a pt Q at the origin.

But, what if Q is zero – no net charge?

Or, what if you need a more accurate description of V(r)?

This is the point of the Multipole expansion: to give approximate forms of V in powers of $\frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3}$, etc.

In general, we have $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}}$

$$\mathscr{r}^2 = r^2 + r'^2 - 2r'r\cos\theta'$$
$$\mathscr{r}^2 = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - \frac{2r'}{r}\cos\theta'\right]$$

$$(\dagger)$$

define $\quad \epsilon = \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\theta'\right)$, then

$$\mathscr{r} = r\sqrt{1+\epsilon}\,, \quad \text{and} \quad \frac{1}{\mathscr{r}} = \frac{1}{r\sqrt{1+\epsilon}}$$

The point is to assume ϵ is small; the charge is localized compared w/ the distance r.

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{\epsilon}{2} + \frac{3\epsilon^2}{8} - \frac{5}{16}\epsilon^3 + \dots\right)$$

Now plug in ϵ from $(\dagger)$, do algebra, and ...

$$\boxed{\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')}$$

which is remarkable!

Go back to $V(r)$ ...

$$V(r) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\vec{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\theta'\,\rho(\vec{r}')\,d\tau'\right.$$

$$+ \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\vec{r}')\,d\tau'$$

$$\left. + \dots \right]$$

Monopole term

Dipole term

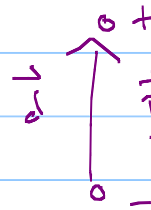
Quadrupole term

$$V_{mon} = \frac{1}{4\pi\epsilon_0 r}\int \rho \, d\tau = \frac{Q}{4\pi\epsilon_0 r} \; !$$

$$V_{dip} = \frac{1}{4\pi\epsilon_0 r^2}\int r' \cos\theta' \rho(r') \, d\tau'$$

$$r'\cos\theta' = \hat{r}\cdot\vec{r}', \quad \text{define } \underbrace{\vec{p} = \int \vec{r}'\rho(\vec{r}')\,d\tau'}_{\text{The dipole moment.}}$$

$$V_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$



$\underline{\vec{p} = q\vec{d}}$ for a physical dipole.

<u>Note</u>: if a pt charge is moved away from the origin, $\underline{V = \frac{1}{4\pi\epsilon_0}\frac{1}{\imath}}$

but $\frac{1}{\imath} \neq \frac{1}{r}$ ... there are dipole, other higher terms.

$\vec{E}$ of Dipole ... eqn $\underline{3.103}$

$$\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

.... can do work for pure dipole, get

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$