Lecture 1: Differential Calculus

Note Title 8/31/2009

<u>HW</u> – Chapter <u>1</u>: 3, 5, 11, 12, 13, 15, 18, 25, 26, 27, 39, 42

We know about vectors. $\vec{A} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}$

or $\vec{A} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$

(1) Vectors <u>Add</u>: $a(\vec{A} + \vec{B}) = a\vec{A} + a\vec{B}$

(2) Dot Product: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} = |A||B| \cos\theta$

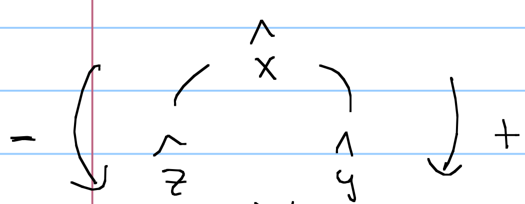
$= a_x b_x + a_y b_y + a_z b_z$

Note $|A| = \sqrt{\vec{A} \cdot \vec{A}}$ is the magnitude of $\vec{A}$

(3) Cross Product: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} = |A||B| \sin\theta \, \hat{n}$

Note $\hat{n}$ is the unit normal (in this case in the dir. $\perp$ to $\vec{A} + \vec{B}$ following the RHR (bottom of page 3).)

Also $\vec{A} \times \vec{A} = 0$



$\hat{x} \times \hat{y} = +\hat{z}$

$\hat{y} \times \hat{x} = -\hat{z}$

etc

or $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

(4) There are triple product rules - see front cover - ugly to prove, hard to remember. Just look them up.

Important: Separation vectors.

Let $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$r = |r| = \sqrt{x^2+y^2+z^2}$

$\hat{r} = \frac{\vec{r}}{r}$ ← unit vector pointing from origin towards $\vec{r}$.

Separation vector:

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$$

$$\mathscr{r} = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}}$$

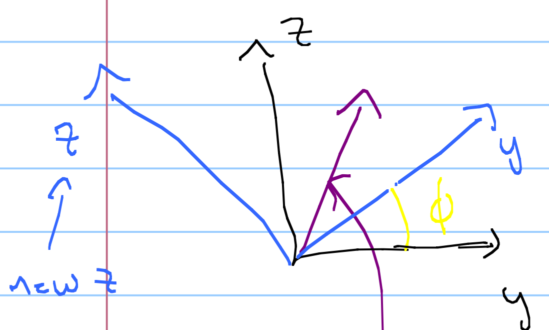
Infinitesimal displacement ← will be important

$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

Question: What makes a vector a vector?

Answer: How it transforms under coordinate changes.

Suppose you want to rotate coordinates about the x axis. The same vector will have new y & z components



(x is out of page)

vector doesn't change, but the numbers do

Use a rotation matrix $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$

$$\boxed{\vec{A}' = R\vec{A}}$$

This generalizes... if

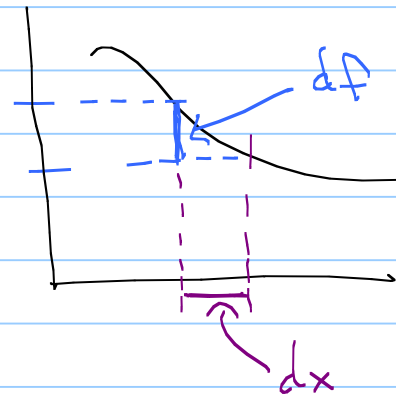
$$\left.\begin{aligned} x' &= X'(x,y,z) \\ y' &= Y'(x,y,z) \\ z' &= Z'(x,y,z) \end{aligned}\right\}$$ 3 functions expressing how coords change

then $\quad \underbrace{J}_{\text{Jacobian Matrix of the transformation}} = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{pmatrix}$

and $\quad \underline{\vec{v}' = J\,\vec{v}}$

Calculus: $\quad$ 1-D

$$df = \left(\frac{df}{dx}\right) dx$$



3-D: have $T(x,y,z)$

$$dT = \left(\underbrace{\frac{\partial T}{\partial x}}_{\text{The change in } \hat{x} \text{ dir}}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)\cdot d\vec{l}$$

$$dT = \boxed{\vec{\nabla} T} \cdot d\vec{l}$$

given definition of gradient.

$$\nabla T = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)$$

$\Rightarrow$ Direction = maximum change

$\Rightarrow$ Magnitude = rate of change in that direction

Pull off the operator: $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$

Get also <u>Divergence</u>: $\nabla \cdot \vec{A}$ – how much does $\vec{A}$ "spread out"

<u>Curl</u>: $\nabla \times \vec{A}$ – how much does $\vec{A}$ "wrap around"

All sorts of product rules w/ ∇ ... I can't remember them, but they are on page 21.

You must do lots of HW w/ these!

<u>Second derivatives</u>:

(1) $\nabla \cdot (\nabla T) = \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$

The Laplacian! (Super important)

(2) $\nabla \times (\nabla T) = 0$! (Always, so remember!)

(3) $\nabla(\nabla \cdot \vec{v}) = ?$ (who knows! Not important)

(4) $\nabla \cdot (\nabla \times \vec{v}) = 0$! (Always, so remember!)

(5) $\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla \cdot \vec{v}) - \nabla^2 \vec{v}$

(Ugly – turns out it is needed much later.)

All this is for cartesian coordinates. What about spherical (r, θ, ϕ) or polar (s, ϕ, z)?

See the front cover!

There is theory for these formulas. It comes from adding little distances in front of derivatives w/ respect to angles.

$\frac{\partial}{\partial x} \leftarrow$ units $= \frac{1}{\text{meters}}$

$\nabla \leftarrow$ units $= \frac{1}{\text{meters}}$

$\Longrightarrow$ You could learn that theory, but in practice you don't need to.

Act like a trained monkey + use the front cover!