

# Bianchi identities and weak gravitational lensing

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December 16, 2004

## **Abstract**

In 1919, the discovery of gravitational lensing from shifted star positions during a solar eclipse led to the acceptance of Einstein's general relativity. [2] For 70 years, poor resolution left gravitational lensing. [15] Recently, weak gravitational lensing, the distortion of shapes of light rays, has become a major research area in astrophysics, but non-relativistic approximations are used to analyze observations. [4] This paper formulates weak lensing from basic principles in general relativity. Assuming a weak field metric, the Ricci and Weyl curvature tensors are calculated. The components of these curvature tensors are recast in terms of complex scalar fields using Newman and Penrose's spin coefficient formalism. [10] The Bianchi identities of general relativity relate derivatives of these scalar fields and are the fundamental equations of weak lensing.

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# 1 Introduction

The phenomena of gravitational lensing occurs when a large object is in between an observer and light rays emitted from a distant star. According to general relativity, light is bent by massive objects. Weak lensing is when a massive object (the lens) is in between the source and the observer and the light from the source is bent in such a way as to make the images we would normally see more elliptical or sheared. Figure 1 is an example of weak lensing.

The projected mass density of the lens can be obtained from information about the amount of shearing that the source images undergo. The approach that astrophysicists use to reach this goal does not start from a fundamental equation in general relativity. But physicists like to categorize things; finding the fundamental equations in general relativity would make the theory more complete. In this thesis, we show that the Bianchi identities of general relativity are the fundamental equations of weak lensing.

We are speculating that integrating the Bianchi identities, a set of nine coupled partial differential equations which relate components of the Ricci curvature tensor to components of the Weyl curvature tensor, is the key to finding the mass density. To do this we will use the null tetrad and the spin coefficient formalism. The null tetrad is a way to track a light ray's path. This is done by identifying 4 vectors, 1 that is in the direction of the light ray, 1 that is perpendicular, and 2 complex ones that curl in opposite directions around the light ray. The spin coefficients are a set of 12 quantities that describe certain properties of the light rays in question. For example,  $\rho$  is the convergence of the light rays and  $\sigma$  is the shear of the light rays.

In section 2 we will explain basic relativity theory. Section 3 explains the math oriented approach to relativity. The Ricci and Weyl tensors are the curvature tensors that we hope to find and the topic in section 4. The null tetrad, explained in section 5, is the mathematical language we have employed to express weak lensing in, as well as the Ricci and Weyl tensors. The spin coefficient formalism follows from this and is also discussed in this section. The next section, 6, explains how to integrate the spin coefficients. Section 7 discusses asymptotically flat spacetimes and how they are important to isolated systems in general relativity. Section 8 explains how the first Bianchi identity could simplify down to the fundamental equation of weak lensing by good choice of tetrad, perturbing to first order in the weak perturbation of spacetime, and restricting the light rays to the lens plane. In

section 9, we discuss the relevance of our findings and future work with this project.

## 2 Intro to general relativity

In 1905, Einstein postulated the special theory of relativity in response to the failed Michelson-Morley experiment. The experiment concluded that light did not travel through ether and either classical mechanics or Maxwell's equations needed to be reconfigured. The 1905 paper radically changed the idea of classical mechanics.

The theory of special relativity begins with two postulates; the Principle of Relativity and the Principle of Constancy.

The Principle of Relativity:

*The test of a physical law by any experiment carried out in a uniformly moving frame of reference does not depend on the speed of that frame relative to any other frame moving uniformly with respect to it [1].*

This principle states that there is no way to tell if you are moving or not when you are in a frame that is not accelerating. So if you are on a train that is moving at a constant velocity of 60mph and all the windows are closed and there is an undetectable amount of jostling due to the friction between the wheels and the not so perfect rail, then you can not tell if your moving or not.

The Principle of Constancy:

*There exists a frame of reference  $S$  (call it the rest frame) with respect to which the speed of light is  $c$ . The speed is then also  $c$  in every other frame of reference moving uniformly with respect to  $S$ . This implies, as a corollary, that the speed of light is independent of the motion of the source [1].*

The speed of light,  $c$ , is  $c$  in a vacuum no matter how fast the source of the light is moving. Thus, if you had a car that moved the speed of light and then turned the headlights on, the light from the front of the car would move at the speed of light relative to the observer in the car and to another observer that is standing still or moving in a uniform way relative to the car.

Special relativity relates observers in uniform translation to each other. So another theory was needed to relate non-uniform translations from one observer to the next. This theory is named general relativity.

General relativity is a theory of gravity. On May 29, 1919 observations by Arthur Eddington of shifted star positions during a solar eclipse confirmed Einstein's theory of general relativity introduced in 1916. [?] According to general relativity, space and time are interwoven into one fabric called space-time. Massive objects, like stars for instance, "twist" or "warp" this space-time fabric around themselves. This physically implies that anything that is traveling in a straight line into a region of warped space-time will take a "warped" path. This occurs for light also as seen by the results of the 1919 solar eclipse observations. The fact that light bends due to gravity is a major reason that gravity could not be thought of as massive objects applying attractive forces to other massive objects and thus, move closer to each other. Since light is not massive and it is affected by gravity, this can not be the case.

In general relativity, there is a preferred observer. This is the one who is freely falling (the only forces on the observer are due to gravity). These observers are in their natural state of motion and we say that they are merely following the geodesic (shortest distance from one point to another) set for them by the curvature of the spacetime manifold that they exist in.

General relativity is based on two principles, the Weak Equivalence Principle and the Strong Equivalence Principle. The Weak Equivalence Principle states that the inertial mass is equal to the gravitational mass. This implies that "the motion of a neutral test body released at a given point in space-time is independent of it's composition." [3]. The Strong Equivalence Principle states that the results of all local experiments in a freely falling frame are independent of the motion and the results of said experiments are the same for all such frames at all places and all times. So physics is the same in all freely falling frames. This *does* violate special relativity. But this is because special relativity implies that the universe is covered by one manifold, a four dimensional object that embodies what the universe exists in, that can be expressed in one set of coordinates. Special relativity is thus a special case of general relativity in which the observer exists in a Minkowski manifold; at least in the local sense. The observer in the freely falling frame sees the world in a Minkowski (flat) manifold.

General relativity does not assume that all of the universe can be covered by a manifold that is expressed in one global set of coordinates. A patchwork

of manifolds must be laid out and “sewn together smoothly”. The reason for the difference is that different parts of the universe have different spacetime curvatures which are affected by mass. Einstein’s main accomplishment was to discover the relationship between curvature and mass.

### 3 Mathematical relativity

General relativity is a theory of gravity that is totally independent of the theories of the other 3 forces. General relativity is defined very precisely in analytical terms. Knowledge of how is imperative to understanding deeper theory and gives a better understanding of general relativity from a mathematical standpoint, so in this section we describe the building blocks.

An open ball is the set of all points within a ball of radius  $r$  excluding the shell. Open sets of  $R^n$  (the set of  $n$ -tuples of real numbers) can be represented by unions of open balls. A manifold is a collection of points that is the result of taking the union of open sets of  $R^n$ . These sets are allowed to overlap. A metric is a second degree equation that describes how the manifold curves at certain points. Putting these two ideas together creates a four dimensional manifold, a spacetime (with metric to describe curvature of the manifold due to gravity) in which three dimensions of space is identified along with one dimension of time.

The mathematical object, tensor, is used in general relativity because it is the only way to maintain covariance, or in other words, the only way that physical laws will maintain their integrity under coordinate transformations. A tensor  $T$ , of type  $(k, l)$  over  $V$  (vector space) is a multilinear map

$$T : \underbrace{V^* \times \dots \times V^*}_k \times \underbrace{V \times \dots \times V}_l \rightarrow \mathfrak{R}. \quad [16]$$

Since tensors are a collection of vectors it is convenient to work with them in matrix space. The relevant tensors will be of type  $(0, 2)$  and  $(0, 4)$ . In a four dimensional spacetime a type  $(0, 2)$  tensor takes the form

$$T_{ab} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix}.$$

A  $(0, 4)$  tensor,  $T_{abcd}$  can not be represented in matrix notation since it is a 4 dimensional lattice-like matrix.

## 4 Ricci and Weyl tensors

The Ricci tensor is one of the components of the decomposition of the Riemann tensor, the other is the (trace-free) Weyl tensor. Before we look at these tensors we must first introduce the Christoffel symbols defined as

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{2}\Sigma_{\sigma}g^{\rho\sigma} \left( \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right) \quad [16]. \quad (1)$$

It is the idea of covariance that is pivotal to general relativity. The idea of parallel transport of vectors is related to this. If one parallel transports a vector in flat space around a small closed loop with ordinary derivative operators, the vector will return to it's original position with its integrity intact. But this is not true in curved space or even for spheres in flat space. As we change coordinate systems we must use covariant derivatives to preserve the integrity of vectors. The Christoffel symbols mediate what the derivative operators are in different coordinate systems. The symbols are a map to the covariant derivatives for the particular coordinate system. The Riemann tensor is a  $(0, 4)$  curvature tensor that utilizes the fact that parallel transporting is path dependent in a curved space. It is defined in terms of the Christoffel symbols, with “, index” denoting partial derivative with respect to the index;

$$R_{\mu\nu\rho}{}^{\sigma} = \Gamma_{\mu\rho, \nu}^{\sigma} - \Gamma_{\nu\rho, \mu}^{\sigma} + \Sigma_{\alpha}(\Gamma_{\mu\rho}^{\alpha} \Gamma_{\alpha\nu}^{\sigma} - \Gamma_{\nu\rho}^{\alpha} \Gamma_{\alpha\mu}^{\sigma}) \quad [16]. \quad (2)$$

The Riemann tensor obeys 4 properties,

1.  $R_{abc}{}^d = -R_{bac}{}^d$ ,
2.  $R_{[abc]}{}^d = 0$ ,
3. For the derivative operator  $\nabla_a$  naturally associated with the metric,  $\nabla_a g_{bc} = 0$ , we have  $R_{abcd} = -R_{abdc}$ ,
4. The Bianchi identity holds:  $\nabla_{[a}R_{bc]d}{}^e = 0$ .

Contracting Eq. 2 yields the Ricci tensor,

$$R_{\mu\rho} = \Sigma_{\nu} R_{\mu\nu\rho}{}^{\nu} = \Sigma_{\nu} \Gamma_{\mu\rho, \nu}^{\nu} - \Sigma_{\nu} \Gamma_{\nu\rho, \mu}^{\nu} + \Sigma_{\alpha, \nu} (\Gamma_{\mu\rho}^{\alpha} \Gamma_{\alpha\nu}^{\nu} - \Gamma_{\nu\rho}^{\alpha} \Gamma_{\alpha\mu}^{\nu}) \quad [16]. \quad (3)$$

The Ricci tensor is the key needed to solve the Einstein equation. The Einstein equation, Einstein's greatest accomplishment, relates the curvature to the mass density and is a fundamental equation of classical physics. It is defined

$$R_{\mu\rho} - \frac{1}{2} g_{\mu\rho} R = 8\pi G T_{\mu\rho}. \quad [5] \quad (4)$$

The Weyl tensor is the trace-free, anti-symmetric part of the decomposition of the Riemann tensor. It is a product of commuting the metric and the Ricci tensor,

$$C_{abcd} = \frac{2}{n-2} (g_{a[c} R_{d]b} - g_{a[c} R_{d]a}) - \frac{2}{(n-1)(n-2)} R g_{a[c} g_{d]b} - R_{abcd} \quad [16], \quad (5)$$

where  $R$  is the scalar curvature, defined as the trace of the Ricci tensor. The properties that hold for the Riemann tensor also hold for the Weyl tensor.

## 5 Null tetrad and spin coefficient formalism

The null tetrad is a set of four spacetime, lightlike vectors and is used to track a light ray's path. The 4 vectors identified are in the direction of the light ray, perpendicular to it, and 2 complex ones that curl in opposite directions around the light ray. The Newman and Penrose spin coefficients are a set of 12 quantities that describe certain properties of the light rays in question. For example,  $\rho$  is the convergence of the light rays and  $\sigma$  is the shear of the light rays. [10]

Since we are using the null tetrad and spin coefficient formalism, the Ricci and Weyl tensors must be expressed in that way. We will compute the Ricci and the Weyl tensor in a weak field metric of a gravitational potential  $\varphi(x, y, z)$  satisfying  $\nabla^2 \varphi = 4\pi\rho$  where  $\rho(x, y, z)$  is the matter density.

We choose the tetrad to be

$$\begin{aligned}
\ell^a &= \frac{1}{\sqrt{2}}(1, 0, 0, 1) \\
n^a &= \frac{1}{\sqrt{2}}(1, 0, 0, -1) \\
m^a &= \frac{1}{\sqrt{2}}(0, 1, i, 0) \\
\bar{m}^a &= \frac{1}{\sqrt{2}}(0, 1, -i, 0).
\end{aligned} \tag{6}$$

It is convenient to define derivative operators as

$$D = \ell^a \partial_a = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial t} \right), \tag{7}$$

$$\Delta = n^a \partial_a = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial t} \right), \tag{8}$$

$$\delta = m^a \partial_a = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

$$\bar{\delta} = \bar{m}^a \partial_a = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \tag{9}$$

Our choice of tetrad makes the spin coefficients  $\kappa = \epsilon = \pi = 0$ . [11] The Weyl tensor calculated from this is

$$\begin{aligned}
\Psi_0 &= \Psi_4 = \frac{1}{2}(\varphi_{xx} - \varphi_{yy} + 2i\varphi_{xy}) \\
\Psi_1 &= -\bar{\Psi}_3 = \frac{1}{2}(-\varphi_{xz} - i\varphi_{yz}) \\
\Psi_2 &= \frac{1}{2}(\varphi_{zz} - \frac{\nabla^2 \varphi}{3}). \text{ see Appendix A.1}
\end{aligned} \tag{10}$$

We choose to perturbate to first order in  $\varphi$ . As a result, the Ricci tensor simplifies to

$$\Phi_{00} = \Phi_{22} = 2\Phi_{11} = \frac{\nabla^2 \varphi}{2}. \text{ see Appendix A.2} \tag{11}$$

In order to get non-trivial results from the Bianchi identities we must assume Minkowski (flat) space for the spin coefficients. This condition forces  $\tau = \bar{\alpha} + \beta$  and  $\rho = \bar{\rho}$ .

## 6 Integrating the spin coefficients

The Sachs equation is a differential equation that states how the light rays emitted from a star propagates to an observer. When a pencil of rays is emitted from a source, one ray can be identified. All of the rays around that ray are connected to that ray by a connecting vector,  $\mathbf{z}$ . The vector really has 4 components but we choose to give it 2 constraints so that the vector is contained in a plane. The Sachs equation [14] in Minkowski space is

$$D\mathbf{z} = -\mathbf{P}\mathbf{z} \quad (12)$$

where

$$\mathbf{P} = \begin{pmatrix} \rho & \sigma \\ \bar{\sigma} & \rho \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} \zeta \\ \bar{\zeta} \end{pmatrix}, \quad (13)$$

and  $\mathbf{z}$  is the distance on the z-axis. The spin coefficient that describes the divergence (the distance between the rays) of the rays is  $\rho$ . The shear of the rays is described by  $\sigma$ , a complex variable whose form is

$$\sigma = |\alpha| e^{i\phi}, \quad (14)$$

where  $\alpha$  is the elliptization constant and  $e^{i\phi}$  is the angle that the light rays rotate. The elliptization constant is the degree to which the bundle of light rays will shear for any change in  $\phi$ , the angle between the observer and the source image. To shear a circular bundle, or pencil, of light rays is to make the pencil more elliptical in shape or to rotate the ellipse. The greater the shear, the more eccentric the ellipse.

Taking another derivative of Eq. 12 yields

$$D^2\mathbf{z} = D(-\mathbf{P}\mathbf{z}) = -\mathbf{Q}\mathbf{z}, \quad (15)$$

where

$$\mathbf{Q} = \begin{pmatrix} \Phi_{00} & \Psi_0 \\ \bar{\Psi}_0 & \Phi_{00} \end{pmatrix}. \quad (16)$$

The curvature parameter,  $\Phi_{00}$ , is related to the Ricci tensor while  $\Psi_0$  is a curvature related to the Weyl tensor.

Using the product rule,

$$D(-\mathbf{P}\mathbf{z}) = -(D\mathbf{P})\mathbf{z} - \mathbf{P}(D\mathbf{z}) = (-D\mathbf{P} + \mathbf{P}^2)\mathbf{z}. \quad (17)$$

Combining Eq. 15 and Eq. 17 produces

$$D\mathbf{P} = \mathbf{P}^2 + \mathbf{Q}. \quad (18)$$

Minkowski space is also referred to as flat space. This is because there is no curvature in the manifold. Therefore, the components of the Riemann, Ricci, and Weyl tensors are all 0 in Minkowski space. Since  $\mathbf{Q}$  is a function of the Ricci and Weyl curvature tensors, it is 0 in flat space. Assuming flat space for Eq. 18, the equation becomes

$$D\mathbf{P} = \mathbf{P}^2. \quad (19)$$

Multiplying this equation on the right and the left by  $\mathbf{P}^{-1}$  (following the rules for m-space) reveals

$$\mathbf{I} = \mathbf{P}^{-1}(D\mathbf{P})\mathbf{P}^{-1}.$$

Differentiating and utilizing the product rule changes the equation to

$$D\mathbf{I} = 0 = D(\mathbf{P}^{-1}\mathbf{P}) = D(\mathbf{P}^{-1})\mathbf{P} + \mathbf{P}^{-1}D\mathbf{P}$$

Rearranging and substituting into Eq. 20 yields

$$\begin{aligned} -D(\mathbf{P}^{-1})\mathbf{P} &= \mathbf{P}^{-1}D\mathbf{P} \\ -\mathbf{I} &= D(\mathbf{P}^{-1})\mathbf{P}\mathbf{P}^{-1} = D(\mathbf{P}^{-1}). \end{aligned} \quad (20)$$

Since  $D = \frac{\partial}{\partial u}$ , integrating with respect to  $u$  from  $\rho_o$  to  $\rho$  and from  $\sigma_o$  to  $\sigma$  and then inverting solves for  $\mathbf{P}$ .

$$\begin{aligned} \mathbf{P}^{-1} &= \mathbf{A} - u\mathbf{I}, \\ \mathbf{P} &= (\mathbf{A} - u\mathbf{I})^{-1}. \end{aligned} \quad (21)$$

If

$$\mathbf{A} = \begin{pmatrix} \rho_o & \sigma_o \\ \bar{\sigma}_o & \rho \end{pmatrix}^{-1}, \quad (22)$$

then

$$\begin{aligned} \rho &= \frac{\rho_o - u(\rho_o^2 - \sigma_o \bar{\sigma}_o)}{1 - 2u\rho_o + u^2(\rho_o^2 - \sigma_o \bar{\sigma}_o)}, \\ \sigma &= \frac{\sigma_o}{1 - 2u\rho_o + u^2(\rho_o^2 - \sigma_o \bar{\sigma}_o)}. \quad (\text{see Appendix A.3}) \end{aligned} \quad (23)$$

Another spin coefficient,  $\tau$ , can be solved for using the Sachs equation. Including and solving for  $\tau$  is a little harder and requires a different method than did the solution for  $\rho$  and  $\sigma$ . Eq. 13 was a simplified version of the components of Eq. 12 referring only to abreast rays. Consider now not only a pencil of rays, but also rays that are not abreast to those rays being emitted from the source. Adding this height factor to the equation is shown by

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ \tau & \rho & \sigma \\ \bar{\tau} & \bar{\sigma} & \rho \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} h \\ \zeta \\ \bar{\zeta} \end{pmatrix}, \quad (24)$$

and

$$D\mathbf{P} = \mathbf{Q} = \begin{pmatrix} 0 & 0 & 0 \\ \Psi_1 + \Phi_{01} & \Phi_{00} & \Psi_0 \\ \bar{\Psi}_1 + \Phi_{10} & \bar{\Psi}_0 & \Phi_{00} \end{pmatrix}. \quad (25)$$

If we consider 3 independent solutions of Eq. 12 with the general definition of  $\mathbf{P}$ ,  $\mathbf{z}$ , and  $\mathbf{Q}$ ,

$$\mathbf{Z} = \begin{pmatrix} h_1 & h_2 & h_3 \\ \zeta_1 & \zeta_2 & \zeta_3 \\ \bar{\zeta}_1 & \bar{\zeta}_2 & \bar{\zeta}_3 \end{pmatrix}, \quad (26)$$

and recall Eq. 15,

$$D^2\mathbf{z} = D(-\mathbf{P}\mathbf{z}) = -\mathbf{Q}\mathbf{z},$$

then applying the derivative to the 3 independent solutions yields

$$D^2\mathbf{Z} = -\mathbf{Q}\mathbf{Z} = 0, \quad (27)$$

in Minkowski space.

Either  $\mathbf{Z} = 0$  or  $\mathbf{Z}$  is linear in  $u$  (or the 3 solutions are dependent which was already specified as not being). The former of the two is the trivial solution and is thus not very interesting. Assuming linearity,

$$\mathbf{Z} = \mathbf{Z}_1 u + \mathbf{Z}_2, \quad (28)$$

where  $\mathbf{Z}_1$  and  $\mathbf{Z}_2$  are constant. Substituting this into Eq. 12 finds that

$$\mathbf{Z} = -\mathbf{P}(\mathbf{Z}_1 u + \mathbf{Z}_2), \quad (29)$$

and so solving for  $\mathbf{P}$ ,

$$\mathbf{P} = -\mathbf{Z}_1(\mathbf{Z}_1 u + \mathbf{Z}_2)^{-1}. \quad (30)$$

The new  $\mathbf{P}$  has  $\rho$  and  $\sigma$  in it so our choice of  $\mathbf{Z}_1$  and  $\mathbf{Z}_2$  must still get the answers to  $\rho$  and  $\sigma$  that have been previously procured and must have the first row consist of zeroes. If we choose

$$\mathbf{Z}_1 = \begin{pmatrix} 0 & 0 & 0 \\ \tau_o & \rho_o & \sigma_o \\ \bar{\tau}_o & \bar{\sigma}_o & \rho_o \end{pmatrix}, \quad \mathbf{Z}_2 = -\mathbf{I}, \quad (31)$$

then these conditions are satisfied.

This leaves us with a solution for the general  $\mathbf{P}$ ,

$$\mathbf{P} = \mathbf{Z}_1(\mathbf{I} - u\mathbf{Z}_1)^{-1}. \quad (32)$$

The element  $\mathbf{P}_{10} = \tau$ , so

$$\tau = \tau_o + \frac{u[u\tau_o(\rho_o^2 + \sigma_o\bar{\sigma}_o) + \bar{\tau}_o\sigma_o - \tau_o\rho_o]}{1 - 2u\rho_o + u^2(\rho_o^2 - \sigma_o\bar{\sigma}_o)}. \quad (\text{see Appendix A.4}) \quad (33)$$

All of the other non-zero spin coefficients have been integrated in Minkowski (flat) space by a symbolic logic program. They are

$$\begin{aligned}
\alpha &= \lambda = \frac{c_1}{\sqrt{1 - 2u\rho_o + u^2\rho_o^2 - u^2\sigma_o\bar{\sigma}_o}} + \frac{c_2 \tanh^{-1}\left(\frac{\sqrt{\sigma_o\bar{\sigma}_o}(-\rho_o + u^2\rho_o - u\sigma_o\bar{\sigma}_o)}{\sigma_o c}\right)}{\sqrt{1 - 2u\rho_o + u^2\rho_o^2 - u^2\sigma_o\bar{\sigma}_o}}, \\
\beta &= \mu = \frac{c_1 + c_2 u}{1 + (\rho_o^2 - \sigma_o\bar{\sigma}_o)u^2 - 2\rho_o u}.
\end{aligned} \tag{34}$$

Two spin coefficients have been omitted for their large derivation lengths.

## 7 Asymptotically Flat Spacetimes

The subject of asymptotically flat spacetimes is particularly useful for studying isolated systems in general relativity. The main feature about such a spacetime is that it is compactifiable and thus, can define the spacetime's limits. In this way it is possible to set boundary conditions and avoid infinities that would otherwise show up.

The metric is an array (matrix) of values that fully describes a particular spacetime. If the metric can undergo a coordinate transformation that results in the new metric having no "points of infinity" then the original metric is considered compactified and as a result the spacetime is asymptotically flat. The relationship between the original metric and the transformed metric is  $ds'^2 = ds^2 \Omega^2$ .  $\Omega^2$  is the conformal factor that rescales the metric and allows the metric to reach a boundary in a uniform way. With this boundary of the spacetime defined we can argue that the space is compactified and thus asymptotically flat.

Let us look at an example of an asymptotically flat spacetime, Minkowski space, and prove both geometrically and mathematically that it is flat. This spacetime describes the motion of light far from significantly massive objects.

The metric for Minkowski spacetime in rectangular coordinates is

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{35}$$

Changing to spherical coordinates using rules for transforming the metric,

$$g'_{\mu\nu} = \frac{\partial x^\gamma}{\partial x'^\mu} \frac{\partial x^\delta}{\partial x'^\nu} g_{\gamma\delta}, \tag{36}$$

with coordinates defined

$$\begin{aligned}
t &= t, \\
x &= r \sin \theta \cos \phi, \\
y &= r \sin \theta \sin \phi, \\
z &= r \cos \theta,
\end{aligned} \tag{37}$$

yields

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}. \quad (\text{see Appendix B.1}) \tag{38}$$

We transform Eq. 38 into double null, or lightlike, coordinates using the same rules for metric transformation where

$$\begin{aligned}
u &= t + r, \\
v &= t - r, \\
\theta &= \theta, \\
\phi &= \phi.
\end{aligned} \tag{39}$$

When plugged into Eq. 38 it turns into

$$\eta_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -(\frac{1}{2}v - \frac{1}{2}u)^2 & 0 \\ 0 & 0 & 0 & -(\frac{1}{2}v - \frac{1}{2}u)^2 \sin^2 \theta \end{pmatrix}. \quad (\text{see Appendix B.2}) \tag{40}$$

At this point we allow  $\Omega = \frac{2}{\sqrt{(1+u^2)(1+v^2)}}$ , then

$$ds'^2 = \Omega^2 ds^2 = \frac{4dudv}{(1+u^2)(1+v^2)} - \frac{(v-u)^2(d\theta^2 + \sin^2 \theta d\phi^2)}{(1+u^2)(1+v^2)}. \quad (41)$$

Now we will assign our “points at infinity” finite coordinates by substituting  $u = \tan p$  and  $v = \tan q$  so that the metric becomes

$$ds'^2 = \Omega^2 ds^2 = 4dpdq - \sin^2(q-p)(d\theta^2 + \sin^2 \theta d\phi^2). \quad (\text{see Appendix B.3}) \quad (42)$$

We can see from fig. 1 that there are limits on the new coordinates of the Minkowski metric.

Looking at fig. 1, the vertical line on the left corresponds to  $q - p = 0$  and the dotted line on the left corresponds to  $q - p = \frac{\pi}{2}$ . The values of  $q - p$  between 0 and  $\pi$  will produce the two steadily sloped lines that meet at  $i^0$  (spatial infinity). These slopes correspond to past null infinity ( $\mathcal{I}^-$ ) and future null infinity ( $\mathcal{I}^+$ ). In other words it is the limit of where light could have come from and could go from/to the point  $i^0$ , spatial infinity. The point curls back onto itself and conforms to the cylinder. Spatial infinity is thus, point-like, and would suggest that no matter which direction one moves in Minkowski space the point is the “edge of space”. The point at the top corresponds to future timelike infinity ( $i^+$ ) while the point at the bottom corresponds to past timelike infinity ( $i^-$ ).

Since  $-\sin^2(q-p)$  is periodic and has an effective domain  $0 \leq q-p \leq \pi$ , fig. 1 represents all of Minkowski space. This space can be “wrapped up” into a larger spacetime. This would compactify Minkowski space. One more coordinate transformation must be done where

$$\begin{aligned} \tau &= p + q, \\ \rho &= q - p. \end{aligned} \quad (43)$$

Using basic calculus and algebra,

$$ds'^2 = \Omega^2 ds^2 = d\tau^2 - d\rho^2 + \sin^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2). \quad (\text{see Appendix B.4}) \quad (44)$$

After the coordinate transformation the spacetime looks like figure 2

Looking at figure 2, it is easy to see geometrically that Minkowski space can be compactified or bound within a larger manifold. This is commonly referred to as the Einstein cylinder for the Minkowski example. However, a manifold is only asymptotically flat if it follows specific mathematical criteria. The above cylinder is key to understanding the analytical criteria involved in the technical definition of an asymptotically flat spacetime. Wald gives the technical definition as:

*A vacuum spacetime  $(M, g_{ab})$  is called to be asymptotically flat at null and spatial infinity if there exists a spacetime  $(M', g'_{ab})$ —with  $g'_{ab}$   $C^\infty$  everywhere except possibly at a point  $i^0$  where it is  $C^{>0}$ —and a conformal isometry  $\psi : M \rightarrow \psi[M] \subset M'$  with conformal factor  $\Omega$  (so that  $g'_{ab} = \Omega^2 \psi^* g_{ab}$  in  $\psi[M]$ ) satisfying the following conditions: [16]*

A vacuum spacetime complete with manifold and companion metric that describes curvature in the manifold is one that is devoid of all matter in it. That  $g'_{ab}$  is  $C^\infty$  is a statement that the curvature of the larger manifold  $M'$  is infinitely differentiable.  $C^{>0}$  at  $i^0$  means that the metric is differentiable at  $i^0$  a finite number of times (but it is not important exactly how many times). An isometry is a bijective (two way) map between two metric spaces that preserves physical distances. Two geometric figures such as the manifolds connected by an isometry are geometrically congruent. Thus, if one of the manifolds is shown to be compactifiable, then by transitivity, the other manifold is also compactifiable.  $\psi : M \rightarrow \psi[M] \subset M'$  denotes a map from the manifold to a function of the image of the manifold which is a subset of the larger manifold. The image of the manifold is a function that represents a subset of the manifold.

This definition of asymptotic flatness must satisfy 5 conditions:

1.  $\overline{(J^+(i^0))} \cup \overline{(J^-(i^0))} = M' - M$ . [16]  
 $\overline{(J^+(i^0))} \cup \overline{(J^-(i^0))} = M' - M$  is the larger manifold minus the compactifiable manifold (Minkowski space in this instance).  $(J^+(i^0))$  is called the causal future of  $i^0$ . Basically,  $i^0$  could have affected  $i^+$ . Similarly,  $(J^-(i^0))$  is called the causal past. The causal past is anything in  $i^0$ 's past light cone all the way up to  $i^-$ . This states that  $i^-$  could have affected  $i^0$ . The closure of a set is the smallest set that contains all of the set. The closure of the causal future is the surface of the cylinder up to its upper limit in figure 2 from  $i^-$  to  $i^0$ . The closure of the causal

past is the surface of the cylinder up to its upper limit in figure 2 from  $i^0$  to  $i^+$ . The union of the two sets in question can now be seen as the area between the larger manifold and the smaller one.

2. There exists an open neighborhood with a radius existing as a subset of the manifold space, of the boundary of the compactifiable manifold. In other words, The boundary of the manifold, which is the set of all points that exist in the closure of the manifold and in the closure of the complement of the manifold (the edges of Minkowski space on the cylinder), can extend an infinitesimally small radius on either side of  $i^0 \cup \mathcal{I}^+ \cup \mathcal{I}^-$  (another way to say the boundary of the manifold) and will still remain in the boundary of  $M$ . This implies that the subset of the manifold space is why the edges of Minkowski space look the way they do on the cylinder.
3. The conformal factor can increase its magnitude such that it affects all of the cylinder. One can make the magnitude of the cylinder bigger or smaller to fit  $M$  in it. The function that results is differentiable a maximum of two times at  $i^0$  and infinitely differentiable everywhere else. This is true for the conformal factor used where  $\Omega = \frac{2}{\sqrt{(1+u^2)(1+v^2)}}$ . The conformal factor can be expressed as a trigonometric function, all of which, are infinitely differentiable (see Appendix B.5).
4. On  $\mathcal{I}^+$  to  $\mathcal{I}^-$ ,  $\Omega$  is 0 and the  $\nabla'_a$ (covariant derivative, an operator associated with the metric) of  $\Omega$  is different from 0. As a point gets really close to (mathematically, the limit)  $i^0$ , the  $\nabla'_a$  of  $\Omega$  approaches 0. Also as a point approaches  $i^0$ ,  $\nabla'_a \nabla'_b$  (corresponding to two lattices of the metric) of  $\Omega$  go to  $2g'_{ab}(i^0) = 0$ .
5. “The map of null directions at  $i^0$  into the space of integral curves of  $n^a = g'_{ab} \nabla'_a \Omega$  on  $\mathcal{I}^+$  to  $\mathcal{I}^-$  is a diffeomorphism”. [16] Integral curves are the space of the solutions of differential equations when the constant of integration is varied.  $n^a = g'_{ab} \nabla'_a \Omega$  is the set of curves in the metric. Let these curves vary by a constant  $C$ . A tensor, call it  $T$ , is a map from null directions at  $i^0$  to curves of the metric on the interval  $\zeta^+$  to  $\zeta^-$ .  $T_b^a(x^b(i^0)) = n^a$  is a diffeomorphism or map between manifolds that is differentiable and has a differentiable inverse. For any smooth function  $\omega$  on the cylinder except at  $i^0$ , “ $\omega > 0$  on  $M \cup \mathcal{I}^+ \cup \mathcal{I}^-$ ” which satisfies  $\nabla'_a(\omega^4 n^a) = 0$  on  $\mathcal{I}^+$  to  $\mathcal{I}^-$ , the vector field  $\omega^{-1} n^a$  is complete on  $\mathcal{I}^+$

to  $\mathcal{I}^-$ .” [16] A complete vector field is a subspace that can locate any vector on the limits.

## 8 1st Bianchi identity: the fundamental equation of weak lensing?

At this point we can create a differential relation between the Ricci and Weyl tensors by using the results of the previous sections and a thin lens approximation. We believe that the first Bianchi identity may simplify down to the fundamental equations of weak lensing. The Bianchi identities are the result of taking covariant derivatives of the commuted Riemann tensor and contracting it. They are defined as

$$\nabla_{[a}R_{bc]d}{}^e = 0. \quad (45)$$

In the spin coefficient version, the first Bianchi identity is

$$\begin{aligned} \bar{\delta}\Psi_0 &- D\Psi_1 + D\Phi_{01} - \delta\Phi_{00} = (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2 \\ &+ (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} + 2(\epsilon + \bar{\rho})\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02}, \end{aligned} \quad (46)$$

where  $D$ ,  $\bar{\delta}$ , and  $\delta$  are derivative operators associated with the null tetrad.

We will assume that an astronomer is observing the pencil of light rays through a modern telescope which sees a very small portion of the sky [8]. This implies that the incoming light rays hit the lens at such a small difference in angle that the rays are all approximately parallel the observer’s line of sight making

$$\ell_{ast}^a = \frac{1}{\sqrt{2}}(1, 0, 0, 1). \quad (47)$$

The pencil was lensed at some point before entering the lens and the deflection is proportional to the two-dimensional gradient of  $\varphi$ . The first tetrad vector may be written as

$$\ell^a = \ell_{ast}^a + \epsilon^a, \quad (48)$$

where  $\epsilon^a$  is first order in  $\varphi$ . Figure ?? illustrates this model. We wish to work to first order in  $\varphi$ . Since we will be contracting the tetrad vectors with

the Ricci and Weyl tensors, we will ignore  $\epsilon^a$  in order to keep our results first order in  $\varphi$ . The tetrad simplifies to

$$\ell^a = \frac{1}{\sqrt{2}} (1, 0, 0, 1). \quad (49)$$

The main idea that this result conveys is that the Newman-Penrose spin coefficients are 0 for this pencil. The first Bianchi identity after the simplifications discussed are

$$\bar{\delta}\Psi_0 - D\Psi_1 - \delta\Phi_{00} = 0. \quad (50)$$

Assume that the Weyl tensor has support in the lens plane and define  $\Phi_{00} =_L \Phi_{00}\delta(z-L)$  (the lens is thin) and  $\Psi_i =_L \Psi_i\delta(z-L)$  where  $\delta(z-L) = \begin{cases} 1, & z=L \\ 0, & z \neq L \end{cases}$  and  $\int \delta(z)dz = 1$ . Integrate each side over all space with respect to  $z$ . Then

$$\begin{aligned} \int_{-\infty}^{\infty} \bar{\delta}\Psi_0 dz &= \int_{-\infty}^{\infty} \bar{\delta}_L\Psi_0 \delta(z-L)dz = \bar{\delta}_L\Psi_0, \\ \int_{-\infty}^{\infty} \delta\Phi_{00} dz &= \int_{-\infty}^{\infty} \delta_L\Phi_{00} \delta(z-L)dz = \delta_L\Phi_{00}. \end{aligned} \quad (51)$$

We see that  $\Psi_1$  takes the form of  $\varphi_{x^i z}$  where  $x^i$  is restricted to  $x$  or  $y$ . We can conclude that it is an odd function since  $\varphi$  takes the form  $\frac{-M}{R} = \frac{-M}{\sqrt{x^2+y^2+z^2}}$  and two mixed derivatives yields a rational function  $\frac{-3Mx^i z}{\sqrt{x^2+y^2+z^2}}$ . We also know that the integral over symmetric space of an odd function is 0. Therefore

$$\int_{-\infty}^{\infty} D\Psi_1 dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} \frac{\partial}{\partial z} dz \Psi_1 = \Psi_1|_{-\infty}^{\infty} = 0. \quad (52)$$

Now we can write the final result as

$$\delta_L\Phi_{00} = \bar{\delta}_L\Psi_0, \quad (53)$$

with the mathematical consistency shown in Appendix C.1. Furthermore, The Weyl tensor is a measurable quantity so the mass density can be calculated from this.

## 9 Discussion

An integral relationship between the Ricci and the Weyl tensors has already been found where the Weyl tensor is some kernel of the Ricci tensor and the setting is the lens plane,

$${}_L\Psi_{00}(\vec{r}) = \int (d\vec{r}') \frac{{}_L\Psi_0\vec{r}}{\pi} \frac{e^{-2i\eta}}{|\vec{r} - \vec{r}'|^2}, [9] \quad (54)$$

where  $\vec{r}$  is the coordinate distance to the lens plane  $\vec{r}'$  is the coordinate distance from the source to the lens plane and  $\eta$  is the angle between  $\vec{r}$  and  $\vec{r}'$ .

We have found a differential relation between the two. These two different formats for weak lensing lends to each other's relevance. The integral relation is stable, smooths out noise and is less precise than the differential version which is at times unstable, but more exact.

A very important aspect about the differential version is that it was found by starting with first principles. If this equation is relevant, then it lends to the foundation and the strength of general relativity as a theory, as well as being another approach to weak lensing. Also it could serve as a first principle to other physicists who would like to find an equation for something starting with first principles in general relativity.

To prove the relevance of this equation of weak lensing we must first plug in values for a simple mock case. If the numbers come out correct there, then we will try it out with real data. This involves writing code that will map the values of the spin coefficients and Weyl tensor at every point in the lens plane to the Ricci tensor that we expect.

## A Spin coefficient formalism calculations

### A.1 Weyl tensor

The Weyl tensor components ( $C_{abcd}$ ) in spin coefficient formalism ( $\Psi_A$ ) are computed by contracting over a combination of 4 vectors from the null tetrad chosen,

$$\begin{aligned}
 \ell^a &= \frac{1}{\sqrt{2}}(1, 0, 0, 1), \\
 n^a &= \frac{1}{\sqrt{2}}(1, 0, 0, -1), \\
 m^a &= \frac{1}{\sqrt{2}}(0, 1, i, 0), \\
 \bar{m}^a &= \frac{1}{\sqrt{2}}(0, 1, -i, 0).
 \end{aligned} \tag{55}$$

The spin coefficient version components of the Weyl tensor are defined:

$$\begin{aligned}
 \Psi_0 &= -C_{0202} = -C_{abcd}\ell^a m^b \ell^c m^d, \\
 \Psi_1 &= -C_{0102} = -C_{abcd}\ell^a n^b \ell^c m^d, \\
 \Psi_2 &= -\frac{1}{2}(C_{0101} - C_{0123}) = -\frac{1}{2}(C_{abcd}\ell^a n^b \ell^c n^d - C_{abcd}\ell^a n^b m^c \bar{m}^d), \\
 \Psi_3 &= C_{0113} = C_{abcd}\ell^a n^b n^c \bar{m}^d, \\
 \Psi_4 &= -C_{1313} = -C_{abcd}n^a \bar{m}^b n^c \bar{m}^d.
 \end{aligned} \tag{56}$$

A symbolic logic program calculated the components of the Weyl tensor to be

$$\begin{aligned}
 C_{0i0i} &= \frac{1}{3}(-3\varphi_{ii} + \nabla^2\varphi), \\
 C_{0i0j} &= -\varphi_{ij}, \quad i \neq j \\
 C_{ijij} &= \frac{1}{3}(3\varphi_{kk} - \nabla^2\varphi), \quad i \neq j \neq k \\
 C_{ijik} &= -\varphi_{jk}, \quad i \neq j \neq k
 \end{aligned} \tag{57}$$

where  $i$ ,  $j$  and  $k$  can represent either  $x$ ,  $y$  or  $z$ .

The non-zero calculations omitting the specific vector contractions are

$$\begin{aligned}
\Psi_0 &= \Psi_4 = -C_{0202} = -(C_{obod} + C_{3b3d}) = \frac{1}{2}(\varphi_{xx} - \varphi_{yy} + 2i\varphi_{xy}), \\
\Psi_1 &= -C_{0102} = -(C_{03cd} + C_{30cd}) = -\frac{1}{2}(\varphi_{xz} + i\varphi_{yz}), \\
\Psi_2 &= -\frac{1}{2}(C_{0101} - C_{0123}) = -2(C_{03cd} + C_{30cd}) = \frac{1}{2}(\varphi_{zz} - \frac{\nabla^2\varphi}{3}), \\
\Psi_3 &= C_{0113} = C_{30cd} + C_{03cd} = \frac{1}{2}(\varphi_{xz} - i\varphi_{yz}). \tag{58}
\end{aligned}$$

## A.2 Ricci tensor

The Ricci tensor components ( $R_{ab}$ ) in spin coefficient formalism ( $\Phi_{AA}$ ) are computed by contracting over a combination of 2 vectors from the null tetrad chosen,

$$\begin{aligned}
\ell^a &= \frac{1}{\sqrt{2}}(1, 0, 0, 1), \\
n^a &= \frac{1}{\sqrt{2}}(1, 0, 0, -1), \\
m^a &= \frac{1}{\sqrt{2}}(0, 1, i, 0), \\
\bar{m}^a &= \frac{1}{\sqrt{2}}(0, 1, -i, 0). \tag{59}
\end{aligned}$$

The contractions that produce non-trivial results are defined:

$$\begin{aligned}
\Phi_{00} &= -\frac{1}{2}R_{00}, \\
\Phi_{11} &= -\frac{1}{4}(R_{01} + R_{23}), \\
\Phi_{22} &= -\frac{1}{2}R_{11}. \tag{60}
\end{aligned}$$

The subscripts in the component denotes which vector to contract with. For instance,  $R_{01} = R_{ab}\ell^a n^b$  while  $R_{23} = R_{ab}m^a \bar{m}^b$ . When the vectors are

chosen only the contractions over the same part of respective vectors are eligible to be non-zero since  $R_{aa} = \nabla^2\varphi$ .

$$\begin{aligned}
R_{00} &= R_{ab}\ell^a\ell^b = \left(\frac{1}{2} + \frac{1}{2}\right)\nabla^2\varphi = \nabla^2\varphi, \\
R_{01} &= R_{ab}\ell^an^b = 0, \\
R_{23} &= R_{ab}m^a\bar{m}^b = \left(\frac{1}{2} - i^2\frac{1}{2}\right)\nabla^2\varphi = \nabla^2\varphi, \\
R_{11} &= R_{ab}n^an^b = \left(\frac{1}{2} + \frac{1}{2}\right)\nabla^2\varphi = \nabla^2\varphi.
\end{aligned} \tag{61}$$

Plugging these results into the definition of the spin coefficient version of the Ricci tensor yields

$$\Phi_{00} = \Phi_{22} = 2\Phi_{11} = -\frac{\nabla^2\varphi}{2}. \tag{62}$$

### A.3 Solving P for abreast light rays

Begin with Eq(22)

$$\mathbf{A} = \begin{pmatrix} \rho_o & \sigma_o \\ \bar{\sigma}_o & \rho_o \end{pmatrix}^{-1}.$$

Inverting it yields

$$\mathbf{A} = \frac{1}{\rho_o^2 - \sigma_o\bar{\sigma}_o} \begin{pmatrix} \rho_o & -\sigma_o \\ -\bar{\sigma}_o & \rho_o \end{pmatrix}. \tag{63}$$

Next expand  $uI$  into matrix form where  $I$  is the identity matrix,

$$u\mathbf{I} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}. \tag{64}$$

$$\mathbf{A} - u\mathbf{I} = \begin{pmatrix} \frac{\rho_o - u(\rho_o^2 - \sigma_o\bar{\sigma}_o)}{\rho_o^2 - \sigma_o\bar{\sigma}_o} & \frac{-\sigma_o}{\rho_o^2 - \sigma_o\bar{\sigma}_o} \\ \frac{-\bar{\sigma}_o}{\rho_o^2 - \sigma_o\bar{\sigma}_o} & \frac{\rho_o - u(\rho_o^2 - \sigma_o\bar{\sigma}_o)}{\rho_o^2 - \sigma_o\bar{\sigma}_o} \end{pmatrix}, \tag{65}$$

Recall Eq(21),

$$\mathbf{P} = (\mathbf{A} - u\mathbf{I})^{-1} = \begin{pmatrix} \frac{\rho_o - u(\rho_o^2 - \sigma_o\bar{\sigma}_o)}{\rho_o^2 - \sigma_o\bar{\sigma}_o} & \frac{-\sigma_o}{\rho_o^2 - \sigma_o\bar{\sigma}_o} \\ \frac{-\bar{\sigma}_o}{\rho_o^2 - \sigma_o\bar{\sigma}_o} & \frac{\rho_o - u(\rho_o^2 - \sigma_o\bar{\sigma}_o)}{\rho_o^2 - \sigma_o\bar{\sigma}_o} \end{pmatrix}^{-1}. \quad (66)$$

For ease let  $\beta = (\rho_o^2 - \sigma_o\bar{\sigma}_o)$ . Inverting the right hand side solves for  $P$ .

$$\begin{aligned} \mathbf{P} &= \left( \frac{(\rho_o - u\beta)(\rho_o - u\beta) - \sigma_o\bar{\sigma}_o}{\beta^2} \right)^{-1} \begin{pmatrix} \frac{\rho_o - u\beta}{\beta} & \frac{\sigma_o}{\beta} \\ \frac{\bar{\sigma}_o}{\beta} & \frac{\rho_o - u\beta}{\beta} \end{pmatrix}, \\ &= \begin{pmatrix} \frac{\rho_o - u\beta}{1 - 2u\rho_o + u^2\beta} & \frac{\sigma_o}{1 - 2u\rho_o + u^2\beta} \\ \frac{\bar{\sigma}_o}{1 - 2u\rho_o + u^2\beta} & \frac{\rho_o - u\beta}{1 - 2u\rho_o + u^2\beta} \end{pmatrix}. \end{aligned} \quad (67)$$

Since  $P_{oo} = \rho$  and  $P_{01} = \sigma$

$$\begin{aligned} \rho &= \frac{\rho_o - u(\rho_o^2 - \sigma_o\bar{\sigma}_o)}{1 - 2u\rho_o + u^2(\rho_o^2 - \sigma_o\bar{\sigma}_o)}, \\ \sigma &= \frac{\sigma_o}{1 - 2u\rho_o + u^2(\rho_o^2 - \sigma_o\bar{\sigma}_o)}. \end{aligned} \quad (68)$$

#### A.4 Solving $\mathbf{P}$ for nonabreast light rays

Beginning with Eq(32) expanded,

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ \tau_o & \rho_o & \sigma_o \\ \bar{\tau}_o & \bar{\sigma}_o & \rho_o \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\tau_o u & (1 - \rho_o)u & -\sigma_o u \\ -\bar{\tau}_o & -\bar{\sigma}_o & (1 - \rho_o)u \end{pmatrix}^{-1}. \quad (69)$$

Inverting the second matrix yields

$$\frac{1}{1 - 2u\rho_o + u^2(\rho_o^2 - \sigma_o\bar{\sigma}_o)} \begin{pmatrix} 1 - 2u\rho_o + u^2(\rho_o^2 - \sigma_o\bar{\sigma}_o) & 0 & 0 \\ \bar{\tau}_o\sigma_o u^2 - \tau_o u(1 - \rho_o u) & 1 - \rho_o u & \sigma_o u \\ \tau_o\bar{\sigma}_o u^2 + \bar{\tau}_o u(1 - \rho_o u) & \bar{\sigma}_o u & 1 - \rho_o u \end{pmatrix}$$

Multiplying row 2 of the first matrix and column 1 of the second matrix yields  $\mathbf{P}_{10}$ , or in other words,  $\tau$ .

$$\begin{aligned}
\tau &= \frac{\tau_o[1 - 2u\rho_o + u^2(\rho_o^2 - \sigma_o\bar{\sigma}_o)] + \rho_o[\bar{\tau}_o\sigma_o u^2 - \tau_o u(1 - \rho_o u)] + \sigma_o[\tau_o\bar{\sigma}_o u^2 + \bar{\tau}_o u(1 - \rho_o u)]}{1 - 2u\rho_o + u^2(\rho_o^2 - \sigma_o\bar{\sigma}_o)} \\
&= \frac{2u^2\rho_o^2 + u(\bar{\tau}_o\sigma_o - 3\tau_o\rho_o + \tau_o)}{1 - 2u\rho_o + u^2(\rho_o^2 - \sigma_o\bar{\sigma}_o)}. \tag{70}
\end{aligned}$$

## B Compactification

### B.1 Minkowski metric in spherical coordinates

Start with the Minkowski metric,

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{71}$$

and change to spherical with coordinates defined

$$\begin{aligned}
x^0 &= x'^0, \\
x^1 &= x'^1 \sin x'^2 \cos x'^3, \\
x^2 &= x'^1 \sin x'^2 \sin x'^3, \\
x^3 &= x'^1 \cos x'^2. \tag{72}
\end{aligned}$$

The equation for transforming the metric is

$$\eta'_{\mu\nu} = \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x^\mu}{\partial x'^\nu} \eta_{\mu\nu}. \tag{73}$$

The non-trivial components of the transformation yield

$$\begin{aligned}
\eta'_{00} &= \frac{\partial x^0}{\partial x'^0} \frac{\partial x^0}{\partial x'^0} \eta_{00} = 1, \\
\eta'_{11} &= \frac{\partial x^1}{\partial x'^1} \frac{\partial x^1}{\partial x'^1} \eta_{11} + \frac{\partial x^2}{\partial x'^1} \frac{\partial x^2}{\partial x'^1} \eta_{22} + \frac{\partial x^3}{\partial x'^1} \frac{\partial x^3}{\partial x'^1} \eta_{33}, \\
&= \sin^2 x'^2 \cos^2 x'^3 (-1) + \sin^2 x'^2 \sin^2 x'^3 (-1) + \cos^2 x'^2 (-1), \\
&= -\sin^2 x'^2 (\cos^2 x'^3 + \sin^2 x'^3) - \cos^2 x'^2 = -1,
\end{aligned}$$

$$\begin{aligned}
\eta'_{21} &= \frac{\partial x^1}{\partial x'^2} \frac{\partial x^1}{\partial x'^1} \eta_{11} + \frac{\partial x^2}{\partial x'^2} \frac{\partial x^2}{\partial x'^1} \eta_{22} + \frac{\partial x^3}{\partial x'^2} \frac{\partial x^3}{\partial x'^1} \eta_{33}, \\
&= (x'^1 \cos x'^2 \cos x'^3)(\sin x'^2 \cos x'^2)(-1) + (x'^1 \cos x'^2 \sin x'^3)(\sin x'^2 \sin x'^3)(-1), \\
&\quad + (-x'^1 \cos x'^2 \sin x'^2)(-1) = x'^1 \sin x'^2 \cos x'^2(1 - 1) = 0, \\
\eta'_{31} &= \frac{\partial x^1}{\partial x'^3} \frac{\partial x^1}{\partial x'^1} \eta_{11} + \frac{\partial x^2}{\partial x'^3} \frac{\partial x^2}{\partial x'^1} \eta_{22}, \\
&= x'^1 \sin^2 x'^2 \sin x'^3 \cos x'^3 - x'^1 \sin^2 x'^2 \sin x'^3 \cos x'^3 = 0 \\
\eta'_{32} &= \frac{\partial x^1}{\partial x'^3} \frac{\partial x^1}{\partial x'^2} \eta_{11} + \frac{\partial x^2}{\partial x'^3} \frac{\partial x^2}{\partial x'^2} \eta_{22}, \\
&= (x'^1)^2 \sin^2 x'^2 \sin x'^3 \cos x'^2 \cos x'^3 - (x'^1)^2 \sin^2 x'^2 \sin x'^3 \cos x'^2 \cos x'^3 = 0 \\
\eta'_{22} &= \frac{\partial x^1}{\partial x'^2} \frac{\partial x^1}{\partial x'^2} \eta_{11} + \frac{\partial x^2}{\partial x'^2} \frac{\partial x^2}{\partial x'^2} \eta_{22} + \frac{\partial x^3}{\partial x'^2} \frac{\partial x^3}{\partial x'^2} \eta_{33}, \\
&= (x'^1)^2 (\cos^2 x'^2 \cos^2 x'^3 + \cos^2 x'^2 \sin^2 x'^3 + \sin^2 x'^2) \\
&= -(x'^1)^2 [\cos^2 x'^2 (\cos^2 x'^3 + \sin^2 x'^3) + \sin^2 x'^2] = (\cos^2 x'^2 + \sin^2 x'^2) = -(x'^1)^2, \\
\eta'_{33} &= \frac{\partial x^1}{\partial x'^3} \frac{\partial x^1}{\partial x'^3} \eta_{11} + \frac{\partial x^2}{\partial x'^3} \frac{\partial x^2}{\partial x'^3} \eta_{22} = -(x'^1)^2 (\sin^2 x'^2) (\sin^2 x'^2 + \cos^2 x'^3) \\
&= -(x'^1)^2 \sin^2 x'^2. \tag{74}
\end{aligned}$$

Since  $\eta_{\mu\nu} = \eta_{\nu\mu}$ , only seven elements need to be calculated. The result is

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}, \tag{75}$$

the Minkowski metric in spherical coordinates.

## B.2 Minkowski metric double null coordinates

Start with the Minkowski metric in spherical coordinates,

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}, \tag{76}$$

and change to double null with coordinates defined

$$\begin{aligned}
x^0 &= \frac{1}{2}(x'^0 + x'^1), \\
x^1 &= \frac{1}{2}(x'^0 - x'^1), \\
x^2 &= x'^2, \\
x^3 &= x'^3.
\end{aligned} \tag{77}$$

Employing the equation for transforming the metric and picking out the non-trivial solution yields,

$$\begin{aligned}
\eta'_{00} &= \frac{\partial x^0}{\partial x'^0} \frac{\partial x^0}{\partial x'^0} \eta_{00} + \frac{\partial x^1}{\partial x'^0} \frac{\partial x^1}{\partial x'^0} \eta_{11} = (.5)(.5) + .5(-.5) = 0, \\
\eta'_{10} &= \frac{\partial x^0}{\partial x'^1} \frac{\partial x^0}{\partial x'^0} \eta_{00} + \frac{\partial x^1}{\partial x'^1} \frac{\partial x^1}{\partial x'^0} \eta_{11} = (.5)(.5) + (.5)(.5) = .5, \\
\eta'_{11} &= \frac{\partial x^0}{\partial x'^1} \frac{\partial x^0}{\partial x'^1} \eta_{00} + \frac{\partial x^1}{\partial x'^1} \frac{\partial x^1}{\partial x'^1} \eta_{11} = (.5)(.5) + (.5)(-.5) = 0, \\
\eta'_{22} &= \frac{\partial x^2}{\partial x'^2} \frac{\partial x^2}{\partial x'^2} \eta_{22} = -\left(\frac{1}{2}v - \frac{1}{2}u\right)^2, \\
\eta'_{33} &= \frac{\partial x^3}{\partial x'^3} \frac{\partial x^3}{\partial x'^3} \eta_{33} = -\left(\frac{1}{2}v - \frac{1}{2}u\right)^2 \sin^2 \theta.
\end{aligned} \tag{78}$$

Since  $\eta_{\alpha\beta} = \eta_{\beta\alpha}$ , only five elements need to be calculated. The result is

$$\eta_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\left(\frac{1}{2}v - \frac{1}{2}u\right)^2 & 0 \\ 0 & 0 & 0 & -\left(\frac{1}{2}v - \frac{1}{2}u\right)^2 \sin^2 \theta \end{pmatrix}, \tag{79}$$

The Minkowski metric in double null coordinates.

### B.3 Minkowski metric transformed

We start for with the expression for the rescaled metric,

$$ds' = \Omega^2 = \frac{4dudv}{(1+u^2)(1+v^2)} - \frac{(v-u)^2(d\theta^2 + \sin^2 \theta d\phi^2)}{(1+u^2)(1+v^2)}, \tag{80}$$

and change to  $(\tau, \rho, \theta, \phi)$  with coordinates defined  $u = \tan p$ ,  $v = \tan q$ . Take the derivative of  $u$  with respect to  $p$  and take the derivative of  $v$  with respect to  $q$  to get

$$\begin{aligned} du &= \sec^2 p \, dp, \\ dv &= \sec^2 q \, dq. \end{aligned} \tag{81}$$

Then,

$$\frac{4dudv}{(1+u^2)(1+v^2)} = \frac{4 \sec^2 p \, dp \sec^2 q \, dq}{(1+\tan^2 p)(1+\tan^2 q)} = 4dpdq, \tag{82}$$

and

$$ds' = \Omega^2 ds = 4dpdq - \frac{(v-u)^2}{(1+u^2)(1+v^2)}. \tag{83}$$

Focusing on the second part of the right-hand side of the previous equation,

$$\begin{aligned} \frac{(v-u)^2}{(1+u^2)(1+v^2)} &= \frac{(\tan q - \tan p)^2}{(\sec^2 q)(\sec^2 p)} = \frac{(\tan^2 q - 2 \tan q \tan p + \tan^2 p)}{(\sec^2 p)(\sec^2 q)}, \\ &= \sin^2 q \cos^2 p - 2 \sin q \sin p \cos q \cos p + \sin^2 p \cos^2 q, \end{aligned}$$

and using the trigonometric powers identities,

$$\begin{aligned} &= \left(\frac{1}{2} - \frac{1}{2} \cos(2q)\right)\left(\frac{1}{2} + \frac{1}{2} \cos(2p)\right)\left(\frac{1}{2} - \frac{1}{2} \cos(2p)\right)\left(\frac{1}{2} + \frac{1}{2} \cos(2q)\right), \\ &= \frac{1}{4} - \frac{1}{4} \cos(2q) + \frac{1}{4} \cos(2p) - \frac{1}{4} \cos(2q) \cos(2p) + \frac{1}{4} + \frac{1}{4} \cos(2q) - \frac{1}{4} \cos(2p) \\ &\quad - \frac{1}{4} \cos(2q) \cos(2p), \\ &= \frac{1}{2} - \frac{1}{2} \cos(2q) \cos(2p) = 2 \sin p \sin q \cos p \cos q = \frac{1}{2} \sin(2q) \sin(2p), \\ &= \frac{1}{2} - \frac{1}{2} (-\sin(2p) \sin(2q) + \cos(2p) \cos(2q)), \\ &= \frac{1}{2} - \frac{1}{2} \cos(2q - 2p) = \frac{1}{2} - \frac{1}{2} \cos(2(q-p)), \\ &= \sin^2(q-p). \end{aligned} \tag{84}$$

Combining equations 83 and 84 results in

$$ds' = \Omega^2 ds = 4dpdq - \sin^2(q - p)(d\theta^2 + \sin^2 \theta d\phi^2). \quad (85)$$

#### B.4 Minkowski metric transformed again

Starting with the rescaled conformal factor,

$$ds' = \Omega^2 ds = 4dpdq - \sin^2(q - p)(d\theta^2 + \sin^2 \theta d\phi^2), \quad (86)$$

perform a coordinate transformation with coordinates defined

$$\begin{aligned} \tau &= p + q, \\ \rho &= q - p. \end{aligned} \quad (87)$$

Taking a full derivative of  $\tau$  yields

$$d\tau = dp + dq.$$

Squaring  $d\tau$  turns it onto

$$d\tau^2 = dp^2 + 2dpdq + dq^2.$$

Taking a full derivative of  $\rho$  yields

$$d\rho = dq - dp.$$

Squaring  $d\rho$  turns it into

$$d\rho^2 = dp^2 - 2dpdq + dq^2.$$

Combining the equations results in

$$d\tau^2 - d\rho^2 = 4dpdq.$$

Thus,

$$ds' = d\tau^2 - d\rho^2 \sin^2(\rho)(d\theta^2 + \sin^2 \theta d\phi^2). \quad (88)$$

## B.5 The Conformal factor

Take the square root of the conformal factor to obtain

$$\Omega = \frac{2}{\sqrt{(1+u^2)^2(1+v^2)^2}}. \quad (89)$$

According to a transformation done in B.3 it is correct to let

$$\begin{aligned} u &= \tan p, \\ v &= \tan q. \end{aligned} \quad (90)$$

Substituting yields

$$\Omega = \frac{2}{\sqrt{(1+\tan^2 p)(1+\tan^2 q)}}.$$

Using the pythagorean trigonometric identity,

$$\Omega = \frac{2}{\sqrt{(\sec^2 p)(\sec^2 q)}} = 2(\cos p)(\cos q). \quad (91)$$

The cosine function can be replaced by an infinite sum of polynomials by a Taylor series expansion. The product of these two functions are a subset of  $P^\infty$  (the set of all polynomials of infinite order).  $p^\infty$  is infinitely differentiable, therefore,  $\Omega = \frac{2}{\sqrt{(1+u^2)^2(1+v^2)^2}}$  is also infinitely differentiable.

## C Bianchi identities

### C.1 Consistency of the final result

In this section we would like to prove the mathematical consistency of

$$\delta_L \Phi_{00} = \bar{\delta}_L \Psi_0. \quad (92)$$

From the definitions laid out in section 5, we remember that the derivative operators in this final result are

$$\begin{aligned}\delta &= m^a \partial_a = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \\ \bar{\delta} &= \bar{m}^a \partial_a = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right),\end{aligned}$$

and the Ricci and Weyl tensors in the final result are

$$\begin{aligned}\Psi_0 &= \frac{1}{2}(\varphi_{xx} - \varphi_{yy} + 2i\varphi_{xy}), \\ \Phi_{00} &= \frac{\nabla^2 \varphi}{2}.\end{aligned}$$

We multiply  $\bar{\delta}\Psi_0$  by  $2\sqrt{2}$  for convenience and expand it to yield

$$\begin{aligned}2\sqrt{2} \bar{\delta}\Psi_0 &= \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (\varphi_{xx} - \varphi_{yy} + 2i\varphi_{xy}) \\ &= \varphi_{xxx} - \varphi_{xyy} + 2i\varphi_{xxy} - i\varphi_{xxy} + i\varphi_{yyy} + 2\varphi_{xyy} \\ &= \varphi_{xxx} + \varphi_{yyx} + i(\varphi_{xxy} + \varphi_{yyy}).\end{aligned}\tag{93}$$

We choose to rearrange the equation by factoring derivatives out. The factored equation is

$$\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \nabla^2 \varphi = 2\sqrt{2} \delta\Phi_{00}.\tag{94}$$

We have  $2\sqrt{2} \bar{\delta}\Psi_0 = 2\sqrt{2} \delta\Phi_{00}$  and it is easy to see that

$$\bar{\delta}\Psi_0 = \delta\Phi_{00}.\tag{95}$$

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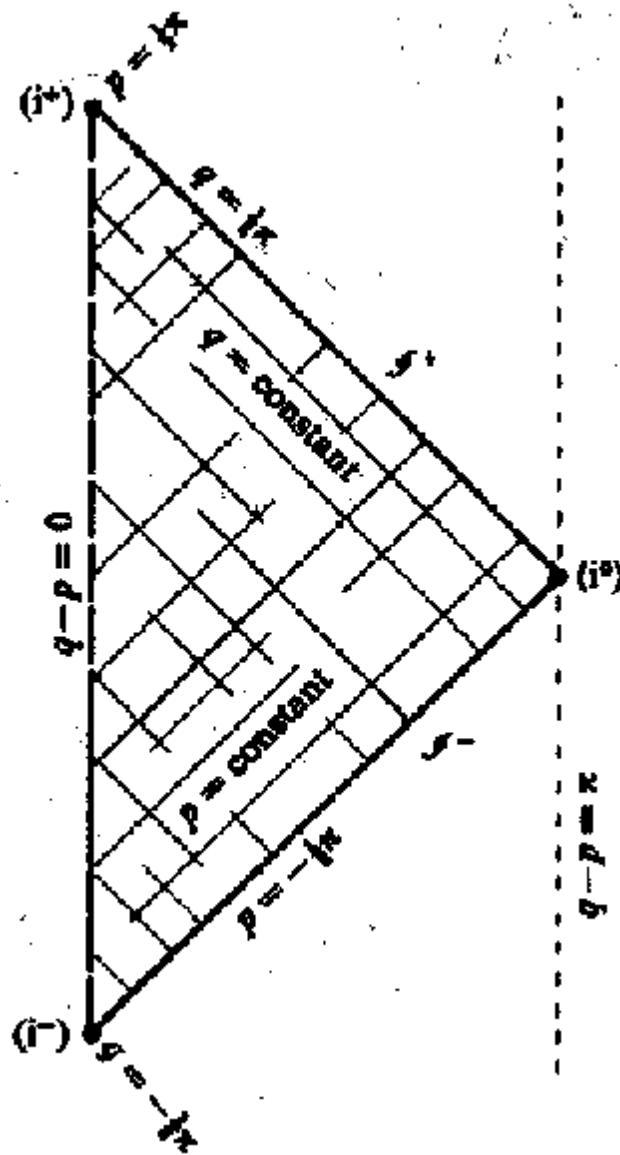


Figure 1: The shaded area represents all of Minkowski space. The borders of Minkowski, the interior of the diamond, space are characterized by light rays. The line that constitutes the base of the triangle and the dotted line that is tangent to the triangle represent the periodicity of the sine function.

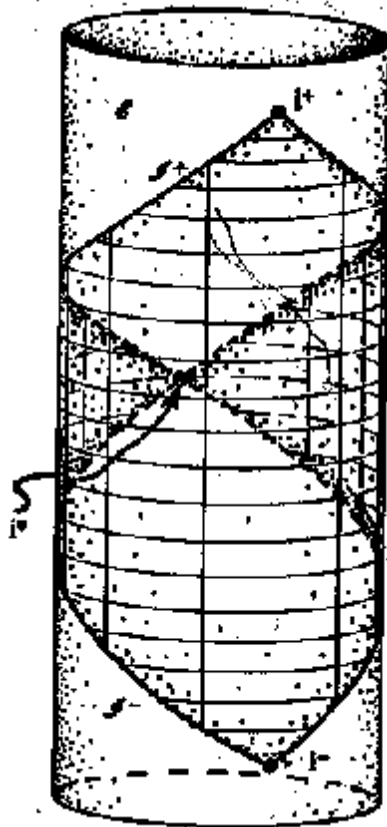


Figure 2: The Einstein cylinder is a larger manifold that encompasses all of the smaller manifold, Minkowski space. This geometrically “compactifies” Minkowski space.

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