Lecture 9: Gravity and metrics

Note Title 2/19/2011

- Hartle just tells us about metrics and gravity – see for example Table 6.1 on page 129 of sections 6.3, 6.5 + 6.6.

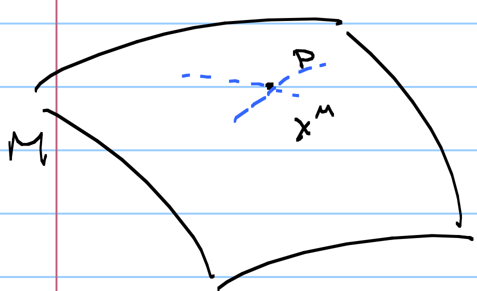
↳ Our goal is to understand this in more detail.

The equivalence principle tells us that acceleration and gravity are the same things.

<u>Fundamental Question</u>: How does comparing a freely falling observer's frame to the frame of a person who senses gravity lead to coding gravity into a metric theory?

- consider a 4-dim Manifold M, with coordinates

$$x^\mu = (t, x, y, z) = (x^0, x^1, x^2, x^3)$$



↳ Let these coordinates be <u>randomly</u> assigned.

<u>Important</u>: the x^μ coordinates are assigned arbitrarily.

Possibly the frame associated w/ x^μ is an observer who sees gravity, but we don't have to take this conclusion.

Now at P, there exists one freely falling frame,

↳ so $\exists$ a coordinate system ξ^α at P, and a change of coordinates b/w x^μ + ξ^α

$$x^\mu(\xi^\alpha) \quad \text{or} \quad \xi^\alpha(x^\mu)$$

Now define $e^\alpha{}_\mu = \left.\frac{\partial \xi^\alpha}{\partial x^\mu}\right|_{x=P}$ $\qquad e^\mu{}_\alpha = \left.\frac{\partial x^\mu}{\partial \xi^\alpha}\right|_{x=P}$ ← obviously inverses.

Two ways to think of $e^\alpha{}_\mu$: The Jacobian matrix

or A collection of 4 vectors

$(e^\mu{}_0, e^\mu{}_1, e^\mu{}_2, e^\mu{}_3)$ ← 4 different vectors which depend on P,

↑ 4 vector fields on the manifold: the tetrad field or vier-bein

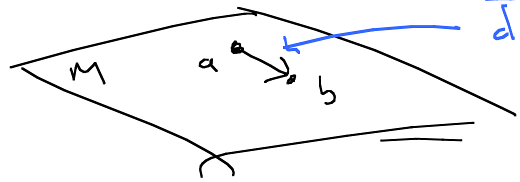
very important!

⟹ The tetrad vector field tells you how to transform a vector in the free fall frame to a vector in any given coordinate system:

$$V'^\mu(x) = e^\mu{}_\alpha V^\alpha$$ ← in free fall frame

or the other way

$$V^\alpha = e^\alpha{}_\mu V'^\mu$$



$\vec{d} = d\xi^a$ or dx^μ

- Now take two close pts (a, b) in manifold

$$\xi^\alpha_a, \quad \xi^\alpha_b, \quad x^\mu_a, \quad x^\mu_b$$

coordinates in free fall frame — coord. in any frame

↳ vector connecting a, b is $d\xi^\alpha = \xi^\alpha_a - \xi^\alpha_b = \frac{d\xi^\alpha}{dx^\mu} dx^\mu$

or $\boxed{d\xi^\alpha = e^\alpha_\mu \, dx^\mu}$

So the distance b/w $a + b$ is

$$ds^2 = \underbrace{\eta_{\alpha\beta} \, d\xi^\alpha d\xi^\beta}_{\text{person in freely falling frame uses Minkowski metric}} = \underbrace{\eta_{\alpha\beta} \, e^\alpha_\mu e^\beta_\nu}_{\text{re-arrange...}} dx^\mu dx^\nu$$

→ + we define $\boxed{g_{\mu\nu} = \eta_{\alpha\beta} \, e^\alpha_\mu e^\beta_\nu}$ so that

$$ds^2 = g_{\mu\nu} \, dx^\mu dx^\nu$$

Note: According to SEP, a freely falling observer will agree w/ special relativity physics, so the distance is $ds^2 = \eta_{\alpha\beta} \, d\xi^\alpha d\xi^\beta$

* For an arbitrary observer in a arbitrary coordinate system, the metric, $g_{\mu\nu}$, arises as a fnc of the x + is related to a transformation to the freely falling frame.

- $g_{\mu\nu}$ is the gravitational field; it is a quantity telling you how to find the local inertial frame.

(The local inertial frame, i.e. the free fall frame, is the one in which $g_{\mu\nu} = \eta_{\mu\nu}$.)

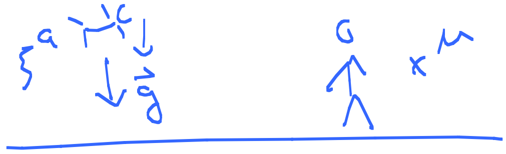
Note that $$g_{\mu\nu} = \eta_{\alpha\beta}\, e^{\alpha}{}_{\mu}\, e^{\beta}{}_{\nu} = \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} = g_{\nu\mu}$$

$g_{\mu\nu}$ is symmetric

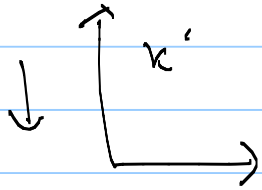
Let's do something overly simple... let our 2 observers be a freely falling observer in a <u>constant</u> grav. field and one at rest.

<u>Technical side note</u> — this works out very cleanly but is <u>not</u> equivalent to the general weak field metric. I think this is because the matter distribution needed to create a constant gravitational field is an infinite plane w/ constant mass density σ. Since the mass density extends to infinity, the resulting metric is "bad" in some fundamental ways and is not consistent with the usual weak field metric.

But it is still a nice example!



→ take a frame freely falling vertically in the earth's gravity



$\xi^\alpha = (t', x', y', z')$ & have $x^\mu = (t, x, y, z)$ on the earth

→ find $\xi^\alpha(x)$ to first order in v, g

→ b/c frames are accelerating, look at infinitesimal transformations

$$dt' = \gamma\,(dt - v\,dz) \qquad \text{w/ } v = -gt$$
$$dz' = \gamma\,(dz - v\,dt)$$

$$t' = t + gtz$$
$$z' = z + \tfrac{1}{2} g t^2$$

$$e^\alpha{}_\mu = \frac{\partial \xi^\alpha}{\partial x^\mu} = \begin{pmatrix} 1+gz & 0 & 0 & gt \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ gt & 0 & 0 & 1 \end{pmatrix}$$

$$g_{\mu\nu} = \eta_{\alpha\beta}\, e^\alpha{}_\mu\, e^\beta{}_\nu \qquad \eta_{\alpha\beta} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$g_{00} = \eta_{\alpha\beta}\, e^\alpha{}_0\, e^\beta{}_0 = \eta_{00}\, e^0{}_0\, e^0{}_0 + \eta_{33}\, e^3{}_0\, e^3{}_0$$

$$g_{00} = (-1)(1+gz)^2 + (+1)\cdot(gt)^2 \implies \boxed{g_{00} \ -(1+2gz)}$$

First order in g!

$$g_{33} = \eta_{\alpha\beta}\, e^{\alpha}_{3}\, e^{\beta}_{3} = \eta_{00}\, e^{0}_{3}\, e^{0}_{3} + \eta_{33}\, e^{3}_{3}\, e^{3}_{3}$$

$$= (-1)(gt)^2 + (+1)(1)^2$$

$$\boxed{g_{33} \approx 1}$$

$$g_{03} = \eta_{\alpha\beta}\, e^{\alpha}_{0}\, e^{\beta}_{3} = \eta_{00}\, e^{0}_{0}\, e^{0}_{3} + \eta_{33}\, e^{3}_{0}\, e^{3}_{3}$$

$$g_{03} = (-1)(1+gz)(gt) + (1)(gt)(1)$$

$$\boxed{g_{03} \approx gt - gt = 0.}$$

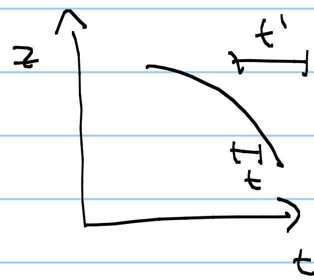
So to 1st order in g ... noting $\phi = gz$ is the gravitational potential

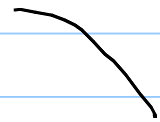
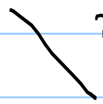
$$g_{\mu\nu} = \begin{pmatrix} -(1+gz) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -(1+2\phi) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

<u>Side note</u>: mass extending to ∞ prevented ϕ from appearing in spatial part of metric... but we still get time effects described in Hartle Ch. 6.

⇒ Consequences of $g_{00} \neq 1$

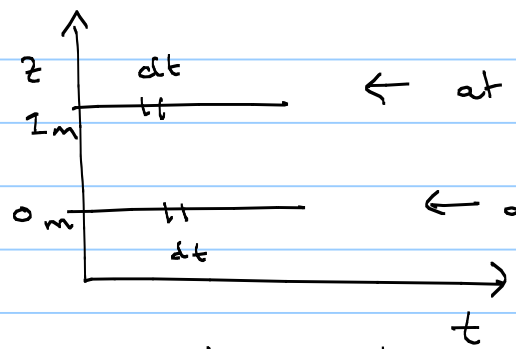
(1) In a fixed room, things accelerate



why  not  ?

- g_{00} changes w/ z
- more time elapses on a clock higher in the field
- physical path maximizes proper time, so spend as long as possible in spots higher in field.

(2) Physical clocks measure $\underline{ds}^2 = (1 + 2gz)\,dt^2 - 1z^2$



← at $z = 1$, $ds^2 = (1 + 2g(1))\,dt^2$

← at $z = 0$, $ds^2 = dt^2$

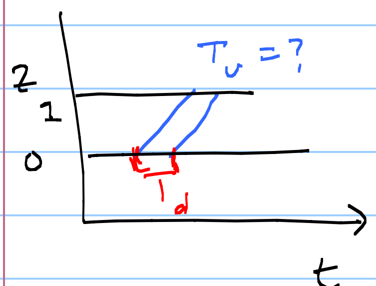
→ More time elapses on clocks higher up.

Time is <u>slow</u> deep in field.

Time is <u>fast</u> higher in field.

→ Suppose an event occurs at $t_d =$
and lasts for a time T

→ at $z=0$, clocks will say that T_d has elapsed, w/ $T_d = T$ 0, $z=0$



If light rays travel to an observer at $z=1$ at beginning & end of the event, a person at $z=1$ will measure T_u on his clocks

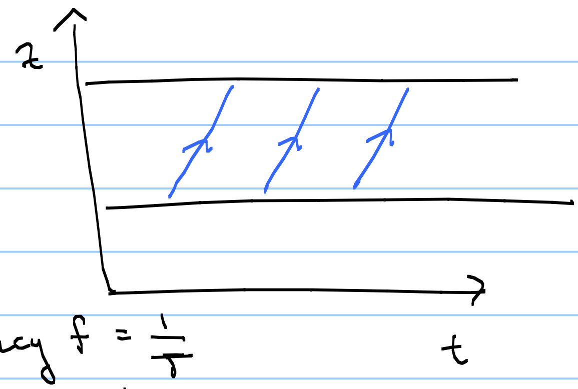
$$T_u = \Delta s = (1 + 2gz)^{1/2} dt \quad , \quad dt = T_d$$

$$\boxed{T_u = (1 + gz) T_d}$$

$$\frac{\Delta T}{T_d} = \frac{T_{up} - T_d}{T_d} = \frac{gz}{c^2} \sim \frac{10^{-16}}{} \quad \text{(small)}$$

makes units work

(3) Related idea



light rays sent up every T, or w/ frequency $f = \frac{1}{T}$ according to person at lower elevation.

- But $T_u > T_d$ ⇐ so there will be a frequency shift.

$$\frac{\Delta f}{f} = \left(\phi_{up} - \phi_{down}\right)$$ ← redshift at a higher potential

(longer time b/w wave crests at top, so frequency is lower, wavelength longer at top, i.e. redshift.)

All of this from $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$!