# Lecture 8 Supplement: Curvature

Note Title 2/16/2006

- Go read Flatland QA699.A13 (Abbott)

Kenyon 22-33 — Suppose you are an ant living in Brazil. You live on a 2-D world along the surface of the earth.

→ Ant experiment: You and a friend both face due North, separated by 1 m. Upon walking north, you measure a decrease in your separation $\propto$ to your speed. There must be a force pulling you together!

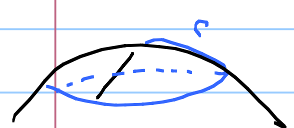
This is an effect of the curvature which the ants don't notice, but we, living in 3D think of as obvious.

Fundamental Question: How can we measure or infer the curvature without going to a "higher dimensional" space?

→ 1st way: put a peg down w/ a string of length r. Make a circle, and measure the circumference of that circle.

- If the world is flat, $C_F = 2\pi r$.
- If you live in or on a bowl, $C_P < 2\pi r$
- If you live on a saddle, $C_N > 2\pi r$

Positive curvature



$C_P < 2\pi r$

Negative Curvature

$C_N > 2\pi r$

$\boxed{2\pi r - C}$ ← fixes the sign of the curvature.

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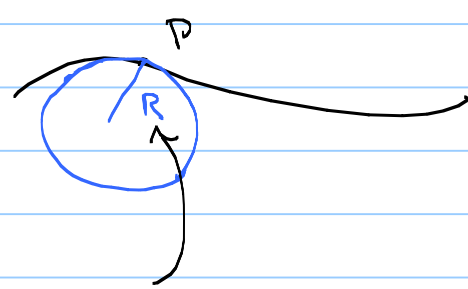
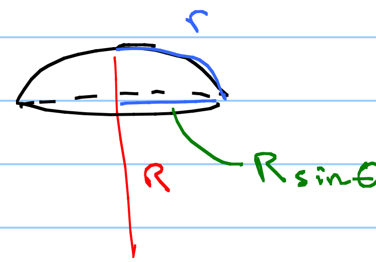
Radius of Curvature: Take a general smooth curve




At any pt P, there is a circle tangent to the curve. The radius R of that circle is the radius of curvature at P.

- Small R, very curved $\Longleftrightarrow$ $R = \infty$ – perfectly straight or flat

- Now take a sphere




$$\frac{r}{R} = \theta \implies C_p = 2\pi R \sin\theta$$

$$C_p = 2\pi R\left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{5!} - \dots\right) \qquad \text{(Taylor series)}$$

$$C_p = 2\pi r\left(1 - \frac{r^2}{6R^2} + \dots\right) \qquad \theta = \frac{r}{R}$$

$$C_p = 2\pi r - \frac{\pi r^3}{3R^2} \longrightarrow R^{-2} = \frac{3}{\pi}\left(\frac{2\pi r - C_p}{r^3}\right)$$

define $$\boxed{K = \lim_{r\to 0} R^{-2} = \frac{3}{\pi}\lim_{r\to 0}\left(\frac{2\pi r - C}{r^3}\right)}$$

For a sphere, $C = 2\pi r - \frac{\pi r^3}{3R^2} + \dots$

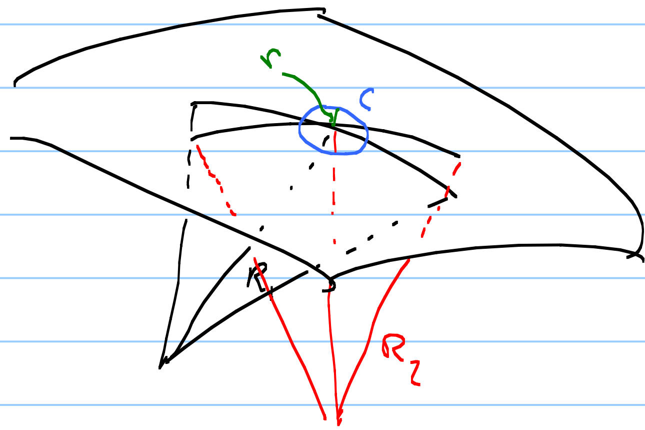
$+ \boxed{K = \frac{1}{R^2}}$

K is called the curvature $\qquad K \equiv \frac{3}{\pi}\lim_{r\to 0}\left(\frac{2\pi r - C}{r^3}\right)$

For a 2-D surface that is not a sphere,

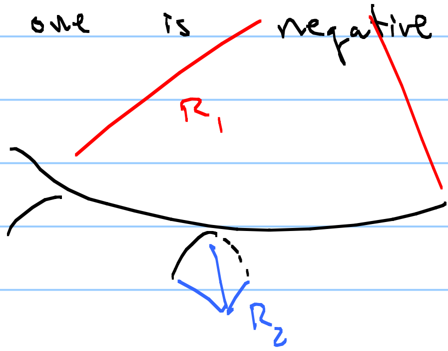
$$K = K_1 K_2 \quad w/ \quad K_1 = \frac{1}{R_1} \leftarrow \text{min rad. of curvature}$$

$$K_2 = \frac{1}{R_2} \leftarrow \text{max rad of curvatr}$$



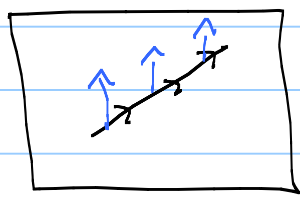

$$\boxed{K = \frac{1}{R_1}\frac{1}{R_2} = \frac{3}{\pi} \lim \left(\frac{2\pi r - C}{r^3}\right)}$$

<u>Note</u> for a hyperboloid, the 2 radii of curvature go in different directions, so one is positive + one is negative & $K = \frac{1}{R_1}\frac{1}{R_2} < 0$ (negative curvature)

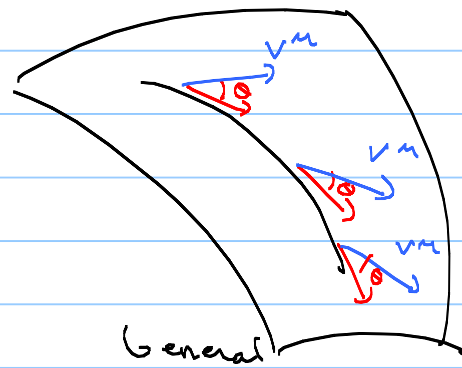


⇒ Another idea: Parallel transport (Very important)

Parallel transport of a vector means to carry a vector along some curve while preserving the angle b/w the tangent to the curve and the vector.



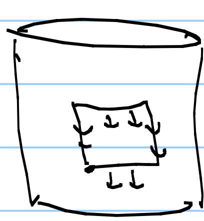
Flat space




General manifold

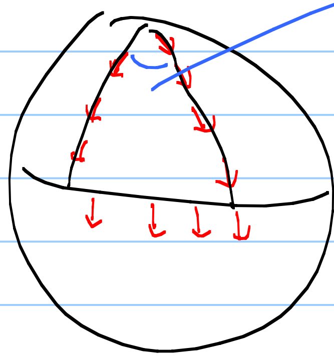
Note: we need a metric to do this — to define the angles.

Parallel transport around a closed curve on a surface gives the curvature of the surface.



vector does not rotate

cylinder is flat



$\phi$ — vector returns rotated by an angle $\phi$.

← Not flat!

$$\boxed{\phi = \kappa \,(\text{area enclosed})}$$

↑

Curvature!

We can (and will) define a tensor definition of parallel transport.