Lecture 8: Math 5 - metrics

Note Title 2/7/2006

In our previous lecture, we introduced a special tensor, type $\binom{0}{2}$ g_{ij}, which is symmetric ($g_{ij} = g_{ji}$) and non singular ($\det(g_{ij}) \neq 0$).

→ A manifold w/ such a tensor is called a metric space.

- In general, g_{ij} is a fnc of the coordinates, and transform according to

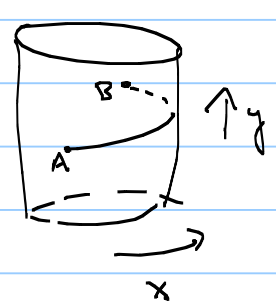
$$g'_{ab} = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} g_{cd}$$

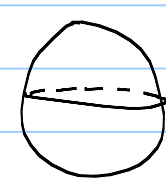
⟹ From this purely geometrical point of view, the g_{ij} can be determined arbitrarily.

But more physically, we want to use the metric as a rule for determining distances.

Kenyon p. 21-6

→ Consider 2 surfaces, a cylinder + a sphere





at least 2 coord systems

(different manifolds, R^2 + S^2)

B/c the cylinder is $\mathbb{R}^2$, the shortest distance b/w any two points is

$$s^2 = x^2 + y^2$$

For large distances b/w 2 points on a sphere of radius r, the distance b/w 2 points can not be given in this form.

cylinder: $ds^2 = dx^2 + dy^2$

sphere: $ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$

$\Big\} \longrightarrow$ or $ds^2 = g_{ij} \, dx^i dx^j$

$ds^2 = g_{11} (dx^1)^2 + g_{22} (dx^2)^2 + 2 g_{12} \, dx^1 dx^2$

happen to be diagonal g_{ij}

An important feature of any surface is that locally, one can always choose coordinates such that $g_{11} = g_{22} = 1$ and $g_{12} = 0$.

Let's do this! $\quad ds^2 = g_{11} \, dv^2 + 2 g_{12} \, dv \, dw + g_{22} \, dw^2$

New coord: $v = V(x,y) \qquad dv = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy = A\,dx + B\,dy$

$w = W(x,y) \qquad dw = \frac{\partial W}{\partial x} dx + \frac{\partial W}{\partial y} dy = C\,dx + D\,dy$

$$ds^2 = g_{11} \left(A^2 dx^2 + B^2 dy^2 + 2AB\,dx\,dy \right) + 2 g_{12} \left(AC\,dx^2 + BD\,dy^2 + (AD + BC)\,dx\,dy \right) + g_{22} \left(C^2 dx^2 + D^2 dy^2 + 2CD\,dx\,dy \right)$$

$$g'_{11}\,dx^2 + 2g'_{12}\,dx\,dy + g'_{22}\,dy^2 = dx^2\left(g_{11}A^2 + 2g_{12}AC + g_{22}C^2\right)$$
$$+ 2\,dx\,dy\left(g_{11}AB + g_{12}(AD+BC) + g_{22}CD\right)$$
$$+ dy^2\left(g_{11}B^2 + 2g_{12}BD + g_{22}D^2\right)$$

$$\left.\begin{aligned} g'_{11} &= 1 = g_{11}A^2 + 2g_{12}AC + g_{22}C^2 \\ g'_{12} &= 0 = g_{11}AB + g_{12}(AD+BC) + g_{22}CD \\ g'_{22} &= 1 = g_{11}B^2 + 2g_{12}BD + g_{22}D^2 \end{aligned}\right\} \rightarrow$$

3 eqns w/ 4 values A, B, C, D which we can pick.

So we can pick $V(x,y)$, $W(x,y)$ such that A, B, C, D make $g'_{11} = g'_{22} = 1$ + $g'_{12} = 0$!

Actually, we can do more! A, B, C, D - 4 free values

⑤ $\dfrac{\partial A}{\partial x} = \dfrac{\partial^2 V}{\partial x^2}$ ⑥ $\dfrac{\partial B}{\partial y} = \dfrac{\partial^2 V}{\partial y^2}$ $\dfrac{\partial B}{\partial x} = \dfrac{\partial^2 V}{\partial x \partial y}$ ⑦

⑧ $\dfrac{\partial C}{\partial x} = \dfrac{\partial^2 W}{\partial x^2}$ $\dfrac{\partial C}{\partial y} = \dfrac{\partial^2 W}{\partial y \partial x}$ ⑨ $\dfrac{\partial D}{\partial y} = \dfrac{\partial^2 W}{\partial y^2}$ ⑩

→ <u>10</u> free values; we can pick

Let's choose those 10 values so that

$$g'_{11} = 1 \qquad \frac{\partial g'_{11}}{\partial x} = 0 \qquad \frac{\partial g'_{22}}{\partial x} = 0 \qquad \frac{\partial g'_{12}}{\partial x} = 0$$

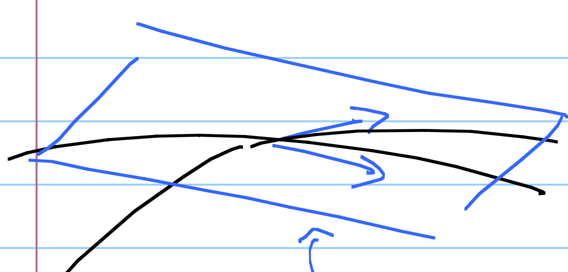
$$g'_{22} = 1 \qquad \frac{\partial g_{11}}{\partial y} = 0 \qquad \frac{\partial g'_{22}}{\partial y} = 0 \qquad \frac{\partial g'_{12}}{\partial y} = 0$$

$$g'_{12} = 0$$

9 conditions, w/ 10 free fncs

This means that A, B, C, D can be chosen at P such that the area close to P looks flat, and the metric is

$$ds^2 = dx^2 + dy^2 \quad \text{locally}$$



Note that you can not arrange to make the 2nd der. vanish, so the plane eventually pulls away from the surface.

Can always lay a flat tangent plane tangent to a surface which is locally flat.

Important point:

Write metric as $ds^2 = \left(g_{11}^{1/2}\,du + \frac{g_{12}}{g_{11}^{1/2}}\,dw\right)^2 + \left(g_{22} - \frac{g_{12}^2}{g_{11}}\right)dw^2$

substitute $dx = g_{11}^{1/2}\,du + \frac{g_{12}}{g_{11}^{1/2}}\,dw$

$dy = \left(g_{22} - \frac{g_{12}^2}{g_{11}}\right)^{1/2} dw$

+ will get $ds^2 = dx^2 + dy^2$ iff $g_{22} - \frac{g_{12}^2}{g_{11}} \geq 0$

If $g_{11}g_{22} - g_{12}^2 < 0$, you get $ds^2 = dx^2 - dy^2$

Psuedo-Riemannian – locally flat, but different.

Psuedo-Riemannian spaces are very important.

→ Spacetime is psuedo-Riemannian

SR metric: Minkowski Spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$

$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$

Save $\eta_{\mu\nu}$ for Minkowski metric

See Kenyon 38-41

In general, spacetime will be described by a metric $g_{\mu\nu}(x)$ which depends on the coordinates, + is psuedo-Riemannian w/ signature $(+,-,-,-)$

Note that some books use $(-,+,+,+)$ — a different convention — just as good....

Metrics w/ signature $(+,-,-,-)$

Foster & Nightingale p. 45-6

(1) inner product of vectors $g_{ab}\lambda^a\lambda^b$ can be positive, negative, or zero.

(2) Length of vector $|g_{ab}\lambda^a\lambda^b|^{1/2} = |\lambda_a\lambda^a|^{1/2}$

if $\lambda_a\lambda^a = 0$, the vector is called a null vector, and describes a light ray.

(3) Angle b/w 2 non-null vectors λ^a, μ^a

$$\cos\theta = \frac{g_{ab}\lambda^a\mu^b}{|\lambda_a\lambda^a|^{1/2}\,|\mu_b\mu^b|^{1/2}}$$

(4) Timelike, spacelike, null vectors

$$g_{ab}\lambda^a\lambda^b \begin{cases} > 0 & \text{timelike} \\ = 0 & \text{null or lightlike} \\ < 0 & \text{space like} \end{cases}$$

(5) If γ is a curve described by $x^a(\tau)$, $v^a = \frac{dx^a}{d\tau}$ is tangent.
The length of γ b/w A & B is

$$L = \int_A^B |g_{ab}v^a v^b|^{1/2}\,d\tau$$