# Lecture 7: Math 4 – Tensors and Mappings

Note Title 2/5/2006

Reading: Work on Chapter 7 + 21

- Return to idea of dual vectors. We can take a product of a vector $v^a$ with a dual $w_a$ and get a number – a scalar.

$$w_a v^a = f \in \mathbb{R}$$ ← does <u>not</u> depend on the coordinate system.

When all the indices are fully contracted, the result is a scalar independent of the coordinates – it is an <u>invariant</u>.

We can think of $w_a$ as an object that maps vectors, $v^a$, into the reals.

If $V$ is a vector space, & $V^*$ is the dual vector space w/ $v \in V$, $w \in V^*$, then $w^*(v) \equiv w_a v^a$ is a map from $V \to \mathbb{R}$.

Each element of $V^*$ provides a different mapping of every element in $V$ to the reals.

Other mappings: let $V, W$ be vector spaces.

$$F: V \Rightarrow W$$
$$F(\vec{v}) = \vec{\omega}$$

some map from one vector space to another.

$$\boxed{F^{\mu}{}_{i} v^{i} = w^{\mu}}$$

Index placement tells you what is happening.

note that $\dim(V)$ is not necc. equal to $\dim(W)$

→ If we restrict the action of $F$ to map from and to the same dimension, then we can identify the vector spaces $V$ & $W$, so that $F$ maps vectors in $V$ to different vectors in $V$.

$$F^{j}{}_{i} v^{i} = w^{j}$$

① $F^{i}{}_{j} v^{j} = w^{i}$ — $F^{i}{}_{j}$ maps a vector into a different vector

② $F_{ij} v^{j} = \omega_{i}$ — $F_{ij}$ maps a vector into a dual

③ $F_{ij} v^{i} w^{j} = f$ ⇐ $F_{ij}$ also maps <u>two</u> vectors into the reals.

Objects like these are called tensors if under complete contraction of indices, the resulting scalar is invariant under coordinate transformations.

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Case 2 interests us.

$\Longrightarrow$ We want a special $F_{ij} \Longrightarrow g_{ij}$ (the metric)
such that $g_{ij} = g_{ji}$ — symmetric
and $\det(g_{ij}) \neq 0$

We let $g^{ij}$ be the inverse of $g_{ij}$
$$g^{ij} g_{jk} = \delta^i{}_k.$$

$$\boxed{V_i = g_{ij} v^j \quad \& \quad v^j = g^{ji} v_i}$$

← We say that the metric raises & lowers indices.

With this special map, we can identify vectors + dual vectors.

$$v^i \Longleftrightarrow v_i$$

$\Rightarrow$ Also, scalar products of vectors or duals is taken w/ the metric.

$$\boxed{(\vec{v}, \vec{\omega}) = v_i\, w^i = g_{ij}\, v^i w^j}$$

We can now use $g_{ij}$ to define the length of vectors, or to define angles.

Other Tensors $\quad T^{ab}{}_{c} \leftarrow$ 2 ups, 1 down

(can have any number of up + down indices)

a $T^{\overbrace{a \ldots f}^{m}}{}_{\underbrace{g \ldots q}_{n}}$ is called a $\binom{m}{n}$ tensor.

Each index transforms like a vector or dual type index.

$V^a$ — a $\binom{1}{0}$ tensor $\qquad v'^a = \frac{\partial x'^a}{\partial x^b} v^b$

$\omega_a$ — a $\binom{0}{1}$ tensor $\qquad \omega'_a = \frac{\partial x^b}{\partial x'^a} \omega_b$

$g_{ab}$ – a $\binom{0}{2}$ tensor $\qquad g'_{ab} = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} g_{cd}$

$g^{ab}$ – a $\binom{2}{0}$ tensor $\qquad g'^{ab} = \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} g^{cd}$

$$\Longrightarrow g'^{ab} g'_{bc} = \frac{\partial x'^a}{\partial x^e} \frac{\partial x'^b}{\partial x^f} g^{ef} \frac{\partial x^d}{\partial x'^b} \frac{\partial x^h}{\partial x'^c} g_{dh}$$

A sample calculation of tensor manipulation

$$= \frac{\partial x'^a}{\partial x^e} \underbrace{\frac{\partial x'^b}{\partial x^f} \frac{\partial x^d}{\partial x'^b}} \frac{\partial x^h}{\partial x'^c} g^{ef} g_{dh}$$

$\delta^d_f \rightarrow \delta^d_f g_{dh} = g_{fh}$

$$= \frac{\partial x'^a}{\partial x^e} \frac{\partial x^h}{\partial x'^c} \underbrace{g^{ef} g_{fh}}$$

$g^{ef} g_{fh} = \delta^e_h$ (inverses)

$$= \frac{\partial x'^a}{\partial x^e} \frac{\partial x^h}{\partial x'^c} \delta^e_h$$

$$= \frac{\partial x'^a}{\partial x^h} \frac{\partial x^h}{\partial x'^c} = \delta^a_c$$

$g'^{ab} g'_{bc} = \delta^a_c \leftarrow g'^{ab}$ is the inverse of $g'_{ab}$!

In general

$$T'^{a \dots f}{}_{g \dots q} = \frac{\partial x'^a}{\partial x^\mu} \cdots \frac{\partial x'^f}{\partial x^\nu} \frac{\partial x^\alpha}{\partial x'^g} \cdots \frac{\partial x^\beta}{\partial x'^q} T^{\mu \dots \nu}{}_{\alpha \dots \rho}$$

is the tensor transformation law.

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Example : Take $\mathbb{R}^2$, $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in $(x,y)$

1. What is $g'_{ij}(r,\theta)$ $\qquad g'_{ij} = \frac{\partial x^a}{\partial x'^i} \frac{\partial x^b}{\partial x'^j} g_{ab}$

$x = r\sin\theta$

$y = r\cos\theta$

$$g'_{11} = g'_{rr} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial r} g_{xx} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r} g_{yy}$$

$$g'_{11} = \cos^2\theta + \sin^2\theta = 1$$

$$g'_{12} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} g_{xx} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} g_{yy} = r\sin\theta\cos\theta\,(1) + \cos\theta(-r\sin\theta)(1)$$

$$g'_{12} = 0\,! = g'_{21} \quad \text{(symmetric)}$$

$$g'_{22} = g'_{\theta\theta} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} g_{xx} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} g_{yy} = r^2\sin^2\theta + r^2\cos^2\theta$$

$$g'_{22} = r^2$$

$$\boxed{g'_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}}$$

Note that if $ds^2 = g_{ij}\, dx^i dx^j$

$$ds^2 = g_{xx}\, dx^2 + g_{yy}\, dy^2 = dx^2 + dy^2$$
$$ds^2 = g_{rr}\, dr^2 + g_{\theta\theta}\, d\theta^2 = dr^2 + r^2 d\theta^2$$

② Let $v^a = (v^1, v^2)$ in $(x,y)$ coordinates. Find $v_a$ in $(x,y)$

$$v_a = g_{ab} v^b = (v^1, v^2) \rightarrow \begin{array}{l} v_1 = v^1 \\ v_2 = v^2 \end{array}$$

b/c $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

③ Let $v'^a(r,\theta) = (v'^1, v'^2)$ in $(r,\theta)$ coordinates. Find $v'_a(r,\theta)$

$$v'_1 = g'_{1j} v'^j = (1) v'^1 + 0 (v'^2) = v'^1$$

$$v'_2 = g'_{2j} v'^j = 0(v'^1) + r^2 v'^2 = r^2 v'^2$$

In polar coordinates, the components of the dual vector $v_a$ associated w/ the vector $v^a$ are not equal to the components of $v^a$!