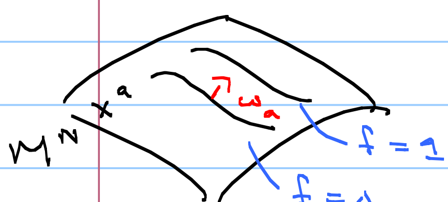
Lecture 6 – Math 3: Dual Vectors

Note Title 1/31/2006

⇒ On a general manifold, vectors V^a arise naturally as tangents to curves. Under coordinate trans., the components of vectors transform according to

$$V'^a = \frac{\partial x'^a}{\partial x^b} V^b .$$

→ There is another natural way to get vectors.



Suppose there is a fnc $f(x^a)$ defined on the manifold.

Then the gradients $\frac{\partial f}{\partial x^a} \equiv \omega_a$ is a vector-like thing.

But since the index is <u>downstairs</u>, ω_a is not a vector like V^a. We call ω_a a dual vector, or covector, or covariant vector, or just a dual.

(V^a is called a contravariant vector, or just a vector.)

Dual vectors have different transformation properties than vectors.

Suppose $x^a = x^a(x'^a)$, $f(x^a) = f(x^a(x'^b))$

$$\boxed{\omega'_a = \frac{\partial f}{\partial x^b}\frac{\partial x^b}{\partial x'^a} = \omega_b \frac{\partial x^b}{\partial x'^a}}$$

What can you do w/ dual vectors?

$\Rightarrow$ With no additional structure, you can define a product of a dual vector w/ a vector at any pt.

$$\omega_a v^a = \# = \omega_1 v^1 + \omega_2 v^2 + \dots + \omega_N v^N$$

We know how to

(1) handle coordinate transformations on a manifold

(2) transform components of vectors and dual vectors

(3) take scalar products b/w a dual + a vector at a given pt.

Class Problem:

Foster + Nightingale p. 11-12

Consider coordinates $\underbrace{(x,y,z)}^{x^a}$ on $\mathbb{R}^3$ + $\underbrace{(u,v,w)}^{y^a}$ given by

$$x = u + v, \quad y = u - v, \quad z = 2uv + w$$

① Find the inverse coord. transformation: $u(x,y,z)$
$v(x,y,z)$
$w(x,y,z)$

② Find the basis vectors $\vec{e}_u, \vec{e}_v, \vec{e}_w$ in terms of $\vec{e}_x, \vec{e}_y, \vec{e}_z$

③ Are the new coordinates orthogonal?

④ Find the components of the vector $t^a = (2, 1, 5)$ at a general pt x^a in the y^a basis. (Find $t'^a(u,v,w)$.)

⑤ Find the components of the dual $q_a = (-1, 0, 6)$ at a general pt x^a in the y^a basis. (Find $q'_a(u,v,w)$.)

⑥ What is $t^a q_a$? Is this equal to $t'^a q'_a$?

Solution

① $u = \frac{1}{2}(x+y) \qquad v = \frac{1}{2}(x-y) \qquad w = z - \frac{1}{2}(x^2 - y^2)$

② $\vec{e}_u = \frac{d\vec{r}}{du} = \frac{d}{du}\left(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z\right)$

$$\begin{aligned}
\vec{e}_u &= \vec{e}_x + \vec{e}_y + 2v\,\vec{e}_z \\
\vec{e}_v &= \vec{e}_x - \vec{e}_y + 2u\,\vec{e}_z \\
\vec{e}_w &= \vec{e}_z
\end{aligned}$$

③ $\vec{e}_u \cdot \vec{e}_v = 4uv, \quad \vec{e}_u \cdot \vec{e}_w = 2v, \quad \vec{e}_v \cdot \vec{e}_w = 2u$

(They are not orthogonal.)

④ $t'^a = \frac{\partial x'^a}{\partial x^b} t^b = \begin{pmatrix} \frac{\partial u}{\partial x} t^1 + \frac{\partial u}{\partial y} t^2 + \frac{\partial u}{\partial z} t^3 \\ \frac{\partial v}{\partial x} t^1 + \frac{\partial v}{\partial y} t^2 + \frac{\partial v}{\partial z} t^3 \\ \frac{\partial w}{\partial x} t^1 + \frac{\partial w}{\partial y} t^2 + \frac{\partial w}{\partial z} t^3 \end{pmatrix}$

$t^b = (2, 1, 5)$

$$t'^a = \left(\frac{3}{2}, \frac{1}{2}, -u - 3v + 5\right)$$

(5) $q_a = (-1, 0, 6)$ $\qquad q'_a = \frac{\partial x^b}{\partial x'^a} q_b = (12v-1, 12u-1, 6)$

(6) $t^a q_a = t^1 q_1 + t^2 q_2 + t^3 q_3 = 28$

$t'^a q'_a = \left(\frac{3}{2}\right)(12v-1) + \frac{1}{2}(12u-1) + (6u-3v+5)(6) = 28$!

<u>Note</u> $t^a q_a$ is equal at all points in <u>any</u> coordinate system.

Another way to do (4) above.

If you have $\quad \vec{e}_u = \vec{e}_x + \vec{e}_y + 2v\,\vec{e}_z$
$\vec{e}_v = \vec{e}_x - \vec{e}_y + 2u\,\vec{e}_z$
$\vec{e}_w = \vec{e}_z$

you can invert, finding

$\vec{e}_x = \frac{1}{2}(\vec{e}_u + \vec{e}_v) - (u+v)\vec{e}_w$
$\vec{e}_y = \frac{1}{2}(\vec{e}_u - \vec{e}_v) - (v-u)\vec{e}_w$
$\vec{e}_z = \vec{e}_w$

Then $\vec{z} = 2\vec{e}_x + 1\vec{e}_y + 5\vec{e}_z$

$$= \vec{e}_u + \vec{e}_v - 2(u+v)\vec{e}_w + \tfrac{1}{2}\vec{e}_u - \tfrac{1}{2}\vec{e}_v - (v-u)\vec{e}_w + 5\vec{e}_w$$

$$\vec{z} = \tfrac{3}{2}\vec{e}_u + \tfrac{1}{2}\vec{e}_v + (5-u-3v)\vec{e}_w$$

Same as we had before