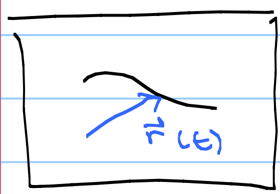
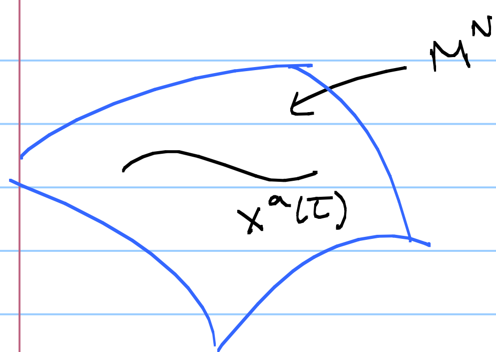
Lecture 5 – Math 2: Vectors & Coordinate Transformations

Note Title 1/29/2006

<u>Math you know</u>: Let $\vec{r}(t)$ describe the position of a particle at all times – it describes a curve in our manifold.



The velocity $\vec{v} = \frac{d\vec{r}}{dt}$ is tangent to the curve.



$v^a = \frac{dx^a}{d\tau}$ ← tangent vector to a curve

Notice that the index placement is naturally upstairs.

$\Rightarrow$ We usually write $\vec{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z}$

$\hat{x}, \hat{y}, \hat{z}$ are <u>basis</u> <u>vectors</u>

Now $v^a = (v^1, v^2, v^3)$ are components in the $(\hat{x}, \hat{y}, \hat{z})$ basis

Rename basis vectors:

$$\left.\begin{aligned} \vec{e}_x &= \hat{x} \\ \vec{e}_y &= \hat{y} \\ \vec{e}_z &= \hat{z} \end{aligned}\right\}$$

Call $\vec{e}_\mu$ the <u>collection</u> of basis vectors

Suppose we have a vector $\vec{v}$ w/ components v^a in the $(\vec{e}_x, \vec{e}_y, \vec{e}_z) = \vec{e}_a$ basis.

How do the components of $\vec{v}$ change if we change coordinates?

Einstein summation convention.

$$\vec{v} = v^1 \vec{e}_1 + v^2 \vec{e}_2 + v^3 \vec{e}_3 = \sum_a v^a \vec{e}_a = v^a \vec{e}_a$$

repeated indices means sum.

In new coordinates, the same vector $\vec{v}$ has new components and new basis vectors.

$$\vec{v} = v^a \vec{e}_a = v'^\mu \vec{e}\,'_\mu$$

How are v^a related to v'^μ and $\vec{e}_a$ related to $\vec{e}\,'_\mu$?

Take an easy example: 2-dim plane: coord (x,y) + (r,θ)

$x = r\cos\theta$ $\quad$ $r = \sqrt{x^2+y^2}$ $\quad$ $\vec{e}_1 = \vec{e}_x = (1,0)$

$y = r\sin\theta$ $\quad$ $\theta = \arctan(y/x)$ $\quad$ $\vec{e}_2 = \vec{e}_y = (0,1)$

? How is $\vec{e}_x$ defined? Let $\vec{r} = x\vec{e}_x + y\vec{e}_y = x^a\vec{e}_a$

$$\vec{e}_x = \frac{\partial \vec{r}}{\partial x} \implies \boxed{\vec{e}_a = \frac{\partial \vec{r}}{\partial x^a}}$$

now $\vec{r} = x\vec{e}_x + y\vec{e}_y = r\cos\theta\,\vec{e}_x + r\sin\theta\,\vec{e}_y$

Then
$$\vec{e}_r = \frac{\partial \vec{r}}{\partial r} = \cos\theta\,\vec{e}_x + \sin\theta\,\vec{e}_y$$
$$\vec{e}_\theta = \frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\,\vec{e}_x + r\cos\theta\,\vec{e}_y$$

These are the basis vectors in (r,θ) coord.

Note! The new basis vectors are orthogonal, but they are not normalized.

What did we just do? Call (x,y) the x^a coord and (r,θ) the x'^a coordinates.

We took $\vec{r}(x^a) = x\vec{e}_x + y\vec{e}_y$ & said $\vec{e}\,'_a = \dfrac{\partial \vec{r}(x^a)}{\partial x'^a}$

New basis in x'^a

• but we had $x^a = x^a(x'^a)$, so $\vec{e}\,'_a = \dfrac{\partial \vec{r}}{\partial x^b}\dfrac{\partial x^b}{\partial x'^a}$

or $$\boxed{\vec{e}\,'_a = \vec{e}_b \frac{\partial x^b}{\partial x'^a}}$$ ← How basis vectors transform.

$$\boxed{\vec{e}\,'_a = \vec{e}_b \frac{\partial x^b}{\partial x'^a}}$$ ← write out:

$$\vec{e}\,'_r = \vec{e}_x \frac{\partial x}{\partial r} + \vec{e}_y \frac{\partial y}{\partial r} = \cos\theta\,\vec{e}_x + \sin\theta\,\vec{e}_y$$

$$\vec{e}\,'_\theta = \vec{e}_x \frac{\partial x}{\partial \theta} + \vec{e}_y \frac{\partial y}{\partial \theta} = -r\sin\theta\,\vec{e}_x + r\cos\theta\,\vec{e}_y$$

Same as above

$$\boxed{\boxed{\vec{e}\,'_a = \vec{e}_b \frac{\partial x^b}{\partial x'^a}}}$$ ← General transformation law for basis vectors.

Now look at $v = v^a \vec{e}_a = v'^a \vec{e}\,'_a$

Two representations of the same vector in terms of different bases.

$$v^a \vec{e}_a = v'^b \vec{e}\,'_b = v'^b \vec{e}_a \frac{\partial x^a}{\partial x'^b}$$

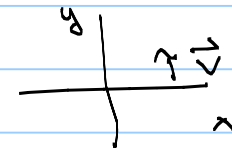
How must v'^b be related to v^a?

$$\boxed{v^a \frac{\partial x'^b}{\partial x^a} = v'^b}$$ Vector components transform opposite basis.

$$v'^b \vec{e}\,'_b = v^a \frac{\partial x'^b}{\partial x^a} \frac{\partial x^c}{\partial x'^b} \vec{e}_c = v^a \delta^c_a \vec{e}_c = v^a \vec{e}_a$$

A Kronecker delta fnc.

Example: Take a vector $(0,1) = V^a$ in the (e_x, e_y) basis located at $\vec{r} = 1\,\vec{e}_x$



What is this vector in (r, θ) coordinates?

$$V'^a = \frac{\partial x'^a}{\partial x^b} V^b$$

$$V'^r = \frac{\partial r}{\partial x}(0) + \frac{\partial r}{\partial y}(1) = \frac{y}{\sqrt{x^2+y^2}} = \frac{0}{\sqrt{1^2+0^2}} = 0.$$

$$V'^\theta = \frac{\partial \theta}{\partial x}(0) + \frac{\partial \theta}{\partial y}(1) = \frac{\partial \arctan(y/x)}{\partial y} = \frac{1}{x^2+y^2} = \underline{1}$$

$$\vec{V} = \vec{e}_\theta \quad \text{at} \quad \vec{r} = 1\,\vec{e}_x.$$