Lecture 4 Math 1 — Manifolds and coordinates

Note Title 1/24/2006

HW: Numbers 4 + 5

Reading: Wald Appendix A, Hartle 2.6, ch. 6, 7.1, 20.1

- The central mathematical object for our studies is the manifold.

  ⟹ Manifolds are very general objects — there is a precise definition, but we will try to get a feel through some examples.

Working Definition: An N-dimensional manifold is a collection of points which can be labeled by a system of N real coordinates in such a way that the points and labels are in 1 to 1 correspondence.

Index notation

⟹ Coordinates $x^a \equiv \{x^1, x^2, \ldots, x^N\}$

Note! For a coordinate, always use a superscript on the label.

x^a can be thought of as the coordinates of a position vector

$$\vec{x} = (x^1, x^2, \ldots x^N) \iff x^a = (x^1, x^2, \ldots x^N)$$

$\Longrightarrow$ An important point: the manifold may be covered by several sets of coord. in patches; there may not be 1 patch which covers everything.

$\Longrightarrow$ Another important point: In general, the coordinates are not special – they can be assigned in any manner.

Examples

① a) $\mathbb{R}^1 \leftarrow$ the real line $-\infty < x < \infty$

(covered by 1 patch)

b) an open interval in $\mathbb{R}^1$ $a < x < b$

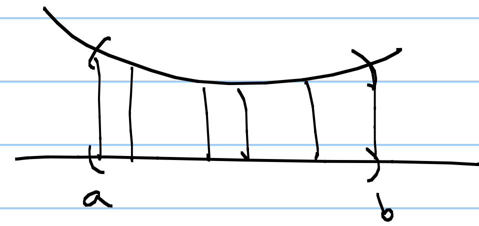
(covered by own patch)

or if $a = -\frac{\pi}{2}$, $b = \frac{\pi}{2}$

let $y = \tan x$

maps $a < x < b$ into entire real line!

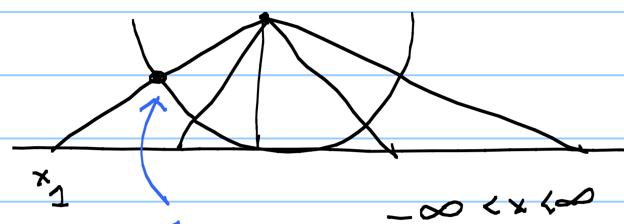
c) an open part of a continuous curve



→ Can label every pt on the curve uniquely by projecting to the real line

d) a semi circle

do a different projection.

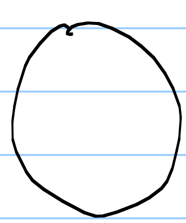


This point is labeled by x_1

⟹ $\boxed{\text{a) - d) are all equivalent}}$

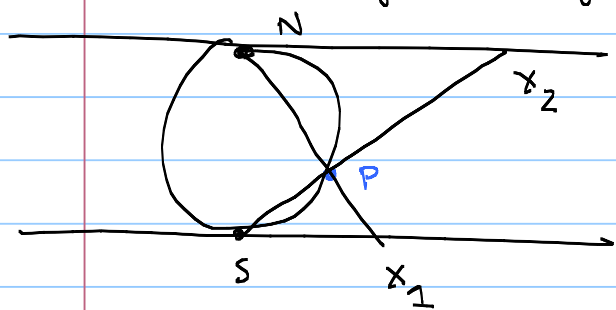
↳ All of these manifolds are topologically $\mathbb{R}^1$. (They are indistinguisible in a manifold sense.)

(2) What about the unit circle?



← Not equivalent to $\mathbb{R}^1$!

How can you assign coordinates to S^1?



From N, draw a line through P to real line tangent at S. Call this intersection x_1

$\Longrightarrow$ The real line will capture all the points on S^1 <u>except</u> N!

So there are more point on a circle than on the real line!

To capture N, make a 2nd coordinate system by projecting from S.

It take 2 coordinate systems to cover the S^1!

<u>Very Important:</u> The point P has representation in 2 coordinate systems.

If we relate x_1 to x_2, we get the <u>overlap fnc</u>:

$$\boxed{x_1 = \frac{4}{x_2}}$$

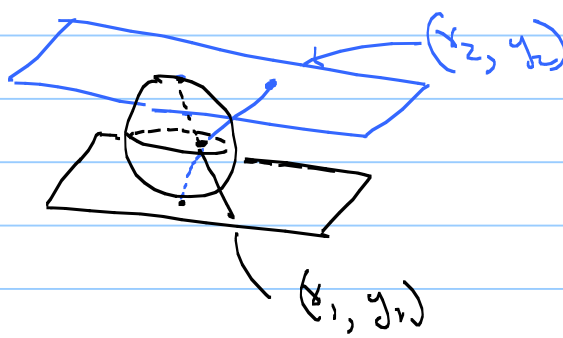
It is critical that the overlap fnc be C^∞ in the neighborhood in which it is defined. This means that the overlap fnc can be inverted.

C^∞ means that the fnc and all its derivatives have defined values.

③ $\mathbb{R}^2$ ← covered by itself ← 1 patch.

④ S^2 (the sphere) ← again requires at least two patches

1 way



if $\eta_1 = x_1 + i y_1$
$\eta_2 = x_2 + i y_2$

$$\boxed{\eta_1 = \frac{4}{\eta_2}} \; !$$

In general, there will never be a prefered system of coords!

↳ What is important is that if the manifold is covered by a set of patches, that the overlap functions are differentiable and have inverses.

$\Rightarrow$ Let x^a and y^μ be two coordinate systems where we know how to express

$$x^a = x^a(y^\mu) \quad \& \quad y^\mu = y^\mu(x^a)$$

$a, \mu = 1, \ldots N \leftarrow$ N fnc of N variables

Then $\Lambda^a{}_\mu = \dfrac{\partial x^a}{\partial y^\mu}$ & $\Lambda^\mu{}_a = \dfrac{\partial y^\mu}{\partial x^a}$

are matrices of partial derivatives — Jacobian matrix

$$\Lambda^a{}_\mu = \begin{pmatrix} \frac{\partial x^1}{\partial y^1} & \frac{\partial x^1}{\partial y^2} & \cdots & \frac{\partial x^1}{\partial y^N} \\ \frac{\partial x^2}{\partial y^1} & & & \\ \vdots & & \ddots & \end{pmatrix}$$

For the change of coordinates to go both ways, the $\det(\Lambda^a{}_\mu)$ can not be zero where the overlap fnc is defined.

$\Lambda^a{}_\mu$ & $\Lambda_a{}^\mu$ are inverses! $\displaystyle\sum_\mu \Lambda^a{}_\mu \Lambda^\mu{}_b = \delta^a{}_b$

Matrix multiplication requires sum over repeated index.

$$\delta^a{}_b = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

Kronecker delta fnc