Lecture 26 Supplement: Newtonian Cosmology

Note Title

Suppose we have an expanding cloud of galaxies which is spherically symmetric & roughly homogeneous.

Also suppose that as the universe expands, no galaxy passes another.
→ There is uniform scaling.

$$\frac{r_m(t)}{r_n(t)} = \frac{r_m(\text{today})}{r_n(\text{today})} = \frac{r_m^0}{r_n^0} \quad (1)$$

^{0} ← value today

Eqn (1) implies that the radial position as a fnc of time can be written as a constant times a scale factor **a(t)**.

$$\boxed{r_n(t) = r_n^0\, a(t)} \quad (2)$$

$a(t)$ ← scale factor
$a(0) = 0$ → b.c.

Problem: Suppose that galaxies travel radially, under influence of Newtonian gravity subject to condition (2).

↳ Solve for $r(t)$ or $a(t)$.

Newton's 2nd Law $\quad m_n \ddot{r}_n = -\dfrac{G M m_n}{r_n^2} \quad (3)$

w/ $\dot{} \equiv \dfrac{d}{dt}$, $\quad m_n$ - mass of galaxy

M - total mass inside r_n

a constant!

$$(3) \Rightarrow r_n^\circ \ddot{a} = -\frac{GM}{r_n^\circ a^2} \Rightarrow \boxed{\ddot{a} + \frac{GM}{(r_n^\circ)^3}\frac{1}{a^2} = 0} \quad (4)$$

Uck! Eqn 4 is hard looking! Fortunately, the energy is conserved.

$$E = \frac{1}{2} m_n \dot{r}_n^2 - \frac{G M m_n}{r_n} = \frac{1}{2} m_n (r_n^\circ)^2 \dot{a}^2 - \frac{G M m_n}{r_n^\circ a}$$

define $\mathcal{E} = \dfrac{E}{m_n r_n^{\circ 2}} \Rightarrow \boxed{\mathcal{E} = \dfrac{1}{2}\dot{a}^2 - \dfrac{GM}{(r_n^\circ)^3}\dfrac{1}{a}} \quad (5)$

Note: $\dfrac{d\mathcal{E}}{dt} = \dot{a}\ddot{a} + \dfrac{GM}{(r_n^\circ)^3}\dfrac{\dot{a}}{a^2} = \dot{a}\left(\ddot{a} + \dfrac{GM}{(r_n^\circ)^3}\dfrac{1}{a^2}\right) = 0$!

let $\alpha = \dfrac{GM}{r_n^{\circ 3}} \rightarrow (\dot{a})^2 = \dfrac{2\alpha}{a}$

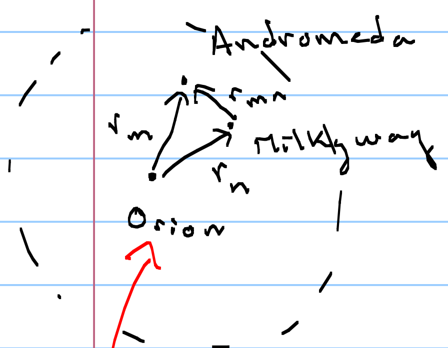
$$\frac{da}{dt} = \sqrt{2\alpha}\, a^{-1/2} \Rightarrow \sqrt{a}\, da = \sqrt{2\alpha}\, dt \Rightarrow \frac{2}{3} a^{3/2} = \sqrt{2\alpha}\, t + C$$

For $a(0) = 0 \Rightarrow C = 0 \Rightarrow \boxed{a = \left(\frac{3}{2}\sqrt{2\alpha}\, t\right)^{2/3}}$

or $$\boxed{a = \left(\frac{9}{2} \frac{GM}{(r_w^0)^3} \right)^{1/3} t^{2/3}}$$ ← This solution is one of the allowed solutions for cosmology by GR!

Note: even though the Newtonian equation is allowed by GR, the picture of space is wrong. (Here the galaxies are moving through some formerly empty space.)

Question: What is the speed of a galaxy relative to a galaxy at the center which never moves?



Galaxy at the center of the expanding cloud

$$v = \frac{dr_n}{dt} = r_n^0 \dot{a}$$ define $H = \frac{\dot{a}}{a} = \frac{2}{3} T^{-1}$ ← Hubble parameter

$$\dot{a} = a H$$

$$\boxed{v = H r_n^0 a = H r_n}$$ Hubble expansion Law!

← Speed of Milky Way as observed by a person on Orion.

<u>Question</u>: What velocity does an observer on the Milky Way observe for Andromeda?

$$\vec{v}_{mn} = \dot{\vec{r}}_{mn} = \frac{d}{dt}\left(\vec{r}_m - \vec{r}_n\right) = \dot{\vec{r}}_m - \dot{\vec{r}}_n = \left(\vec{r}_m^{\,o} - \vec{r}_n^{\,o}\right)\dot{a}$$

$$\boxed{\vec{v}_{mn} = \vec{r}_{mn}^{\,o}\, a H = \vec{r}_{mn} H}$$

A person not at the center sees exactly the same thing, Every galaxy recedes according to the Hubble law!

$\Rightarrow$ <u>Note</u>: If you can't see the boundaries, there is no way to tell that Orion is the center and Andromeda is not!