Lecture 2: Special Relativity for Spacetime

Note Title 1/23/2011

→ In introductory physics and popular culture, discussions of special relativity center on time dilation, length contraction, the twin paradox, the speed of light, $E = mc^2$, etc.

(Sometime illuminating)

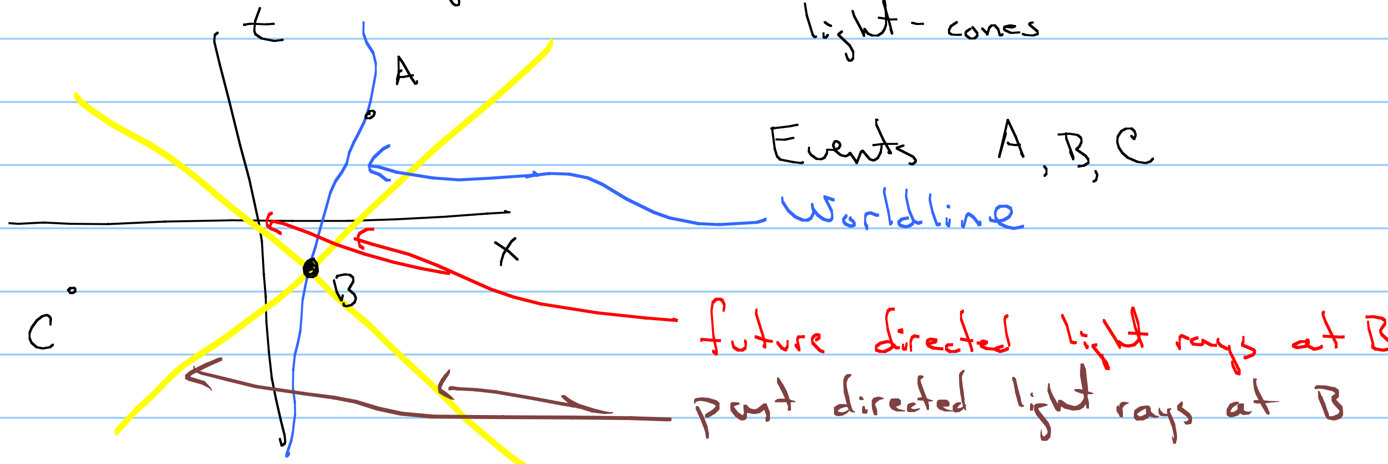
* In particle physics, we learn that a metric is present

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

and one uses it to make inner products of momentum 4-vectors, which help you do all sorts of scattering problems.

But these are not the important things for us.

① Space-time diagrams: events, world-lines, and light-cones



In an important sense, the light rays (always 45°, slope ±1 lines) are the most important thing because they determine the causal structure of the space.

(2) Special relativity has embedded in it a special mathematical structure → it is an example of a metric space.

Metric space: a manifold w/ a defined metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = \eta_{ab}\, dx^a dx^b$$

This is a psuedo-Euclidean metric space, and bears the name of the person who first figured this all out: Minkowski.

$$\eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$ in (t, x, y, z) cartesian coordinates.

Special properties of Minkowski space

1. The "space" part is "flat"
2. η_{ab} - the Minkowski metric - has this simple form at every place and time in the universe in cartesian coord.

③ If Joe is an inertial observer, and Paul is moving with constant velocity relative to Joe, Paul will also believe the metric looks like η_{ab}.

Aside: First discussion of coordinates

Coordinates are a way to label the points in the manifold. They do not inherently mean anything.

You may be given a metric that looks strange... If the spacetime is Minkowski spacetime, there is always a coordinate transformation that makes the metric look like $\eta_{ab} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ globally.

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③ Minkowski space is asymptotically flat, or flat in the limit as $r \to \infty$. (Trivial, since the entire spacetime is flat!)

Asymptotically flat spaces can be compactified - you can do a series of coordinate transformations that make the place "at infinity" take a concrete value.

There is a fairly involved calculation to do — see Brian Keith's thesis + the "B" appendices for details.

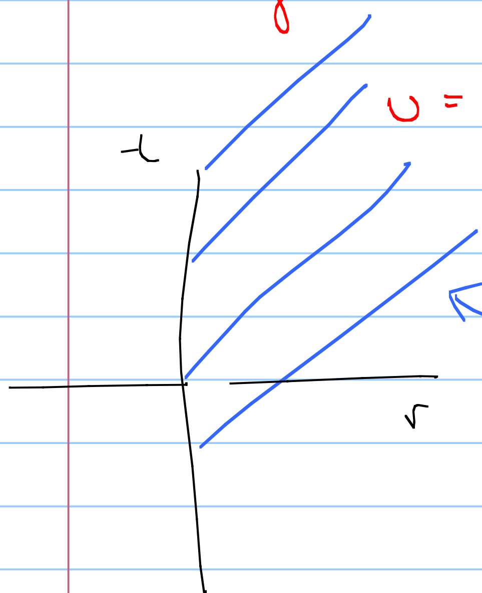
From $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$, go to (t, r, θ, ϕ)

$$\hookrightarrow \boxed{ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2}$$

Should recognize this as Minkowski in spherical coord.

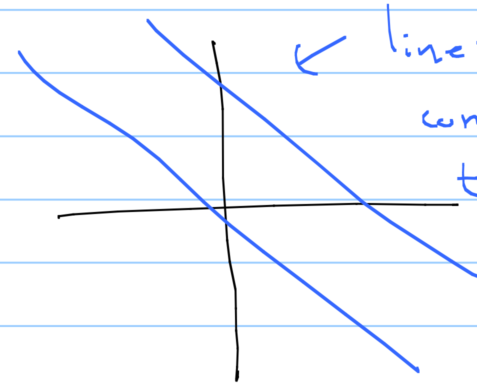
Now go to "double null" coordinates:

$u = t + r$, $v = t - r$



lines of const v

$t = v + r$
(lightrays!)



lines of const u

$t = u - r$

$$\left.\begin{aligned} du &= dt + dr \\ dv &= dt - dr \end{aligned}\right\} \rightarrow \begin{aligned} dt &= \tfrac{1}{2}(du + dv) \\ dr &= \tfrac{1}{2}(du - dv) \end{aligned}$$

Get

$$ds^2 = du\,dv - \tfrac{1}{2}(v^2 - u^2)\, d\theta^2 - \tfrac{1}{2}(v^2 - u^2) \sin^2\theta \, d\phi^2$$

Now multiply by a conformal factor — a positive fnc that changes the metric — a whole theory on this - see Wald Appendix C.

$$\Omega^2 = \frac{4}{(1+u^2)(1+v^2)}$$

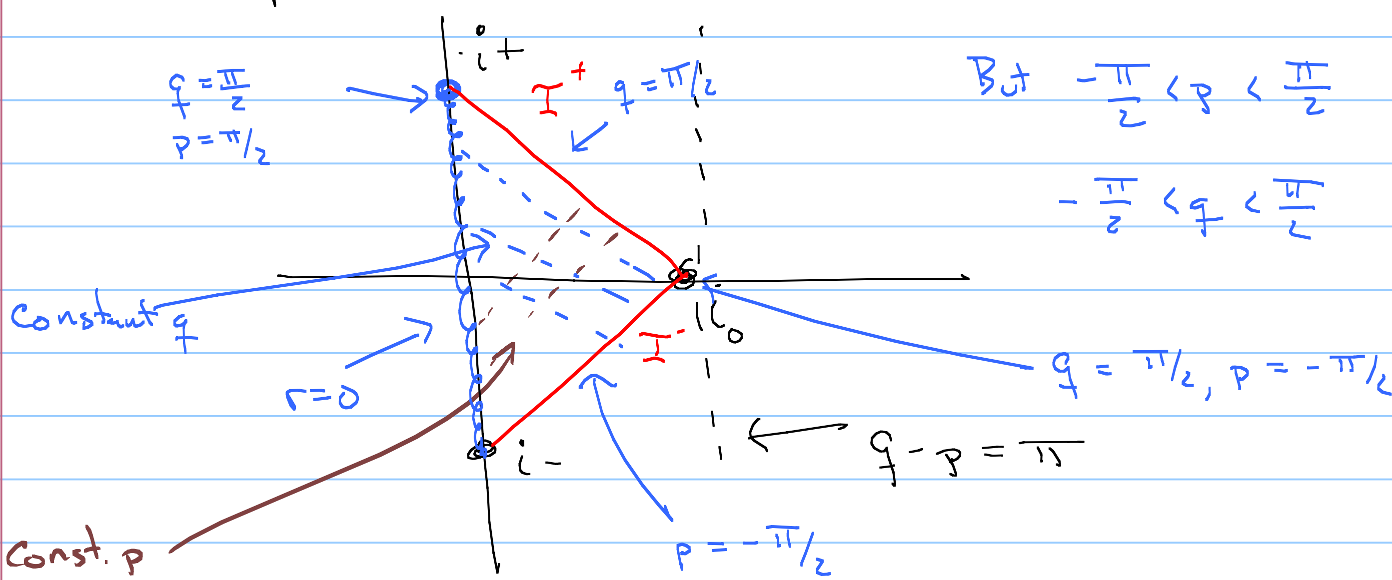
$$ds'^2 = \Omega^2 ds^2 = \frac{4\,du\,dv}{(1+u^2)(1+v^2)} - \frac{(v^2-u^2)(d\theta^2 + \sin^2\theta\, d\phi^2)}{(1+u^2)(1+v^2)}$$

let $u = \tan p \qquad v = \tan q$

$$ds'^2 = 4\,dp\,dq - \sin^2(q-p)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$\boxed{\text{What does it all mean?}}$

$$q - p = 0 \iff r = 0 \,!$$



$i^+ \leftarrow$ future timelike infinity
$i^- \leftarrow$ past timelike infinity
$i_0 \leftarrow$ spatial infinity
} each "1 place"

$\mathcal{I}^+$ ← full null infinity ← where all light goes

$\mathcal{I}^-$ ← past null infinity ← where all light comes from.

The entire Minkowski space is mapped into a region inside the half-diamond.

See Keith's thesis, pages 17 & 18 for full discussion.