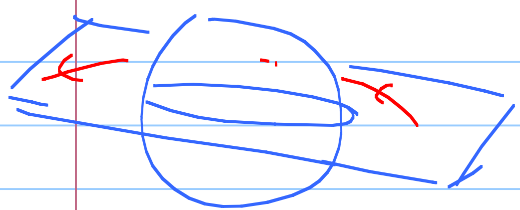
Lecture 18: Orbits near black holes

Note Title 4/5/2011

→ A few general things to start...

- $\vec{L} = \vec{r} \times \vec{p}$ as a vector. Since $\vec{L}$ is conserved in a central force problem, the motion takes place in a <u>plane</u>
- Another way to look at it... we have spherical symmetry, so motion must take place in 1 plane.



- Easiest plane is $\theta = \pi/2$ is constant, the "equitorial plane."

- From standard Mechanics – pt mass $\Phi = \frac{-GM}{r}$

$$E = \frac{1}{2} m v^2 - \frac{GMm}{r} = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 - \frac{GMm}{r}$$

$$\mathcal{L} = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 + \frac{GMm}{r}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0! \qquad \frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \ell$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m r^2 \dot{\phi} = \ell \implies \dot{\phi} = \frac{\ell}{m r^2}$$

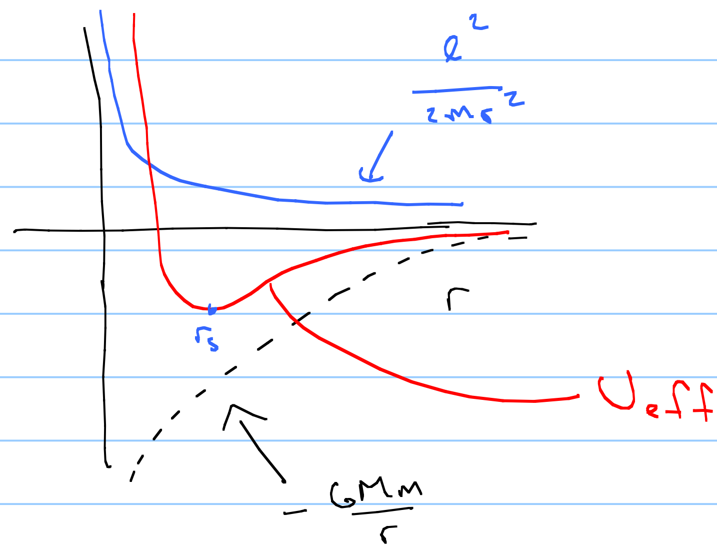
Go back to energy...

$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 - \frac{GMm}{r}$$

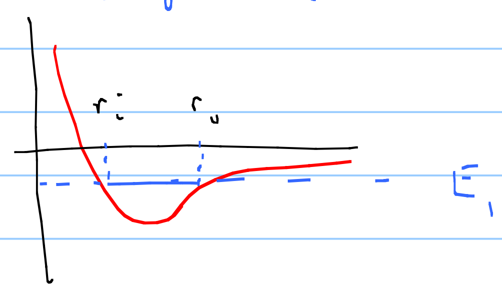
$$\dot{\phi}^2 = \frac{\ell^2}{m^2 r^4}$$

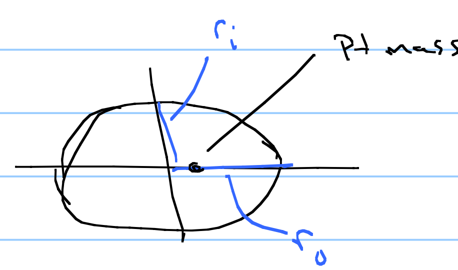
$$E = \frac{1}{2} m \dot{r}^2 + \underbrace{\boxed{\frac{1}{2} \frac{\ell^2}{m r^2} - \frac{GMm}{r}}}_{U_{effective}}$$

$$U_{eff} = \frac{1}{2} \frac{\ell^2}{m r^2} - \frac{GMm}{r}$$



At r_s there is a stable circular orbit. At other energies $E < 0$, get bound elliptical orbits. At $E \geq 0$, get open orbits.





Any perturbation of U_{eff} (due to other planets, a non-spheroidal sun, etc) cause the ellipse to precess — the orbit does not close.

At the time GR came about, it was known that the precession of the orbit of Mercury could not be accounted for by the known planets, to the rate of $\sim 43''$ per century.

Massive particle orbits: let $u^a = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}) = \dot{x}^a$ be the velocity vector

$$\mathcal{L} = \frac{1}{2} g_{ab} u^a u^b = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b = -\frac{1}{2}$$

my convenience later

$\mathcal{L}$ = constant because no friction-like terms
- value of $\mathcal{L}$ is negative, b/c massive particles follow timelike curves
- Any negative constant will do ... $-\frac{1}{2}$ works w/ added $\frac{1}{2}$ in other side.

Take $\theta = \frac{\pi}{2}$, $\sin\theta = 1$, $\dot{\theta} = 0$

$$\mathcal{L} = \frac{1}{2}\left(-\left(1-\frac{2m}{r}\right)\dot{t}^2 + \left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2\right) = -\frac{1}{2}$$

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{t}} = \text{Constant} = -e$$

These eqns show why the $\frac{1}{2}$ is in $\mathcal{L}$ definition!

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = -\left(1-\frac{2m}{r}\right)\dot{t} = -e$$

$$\boxed{\left(1-\frac{2m}{r}\right)\dot{t} = e} \qquad \text{eqn } 9.21$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \text{Constant} = r^2\dot{\phi} = \ell$$

Now, what to do for r... you could do

$$\frac{\partial \mathcal{L}}{\partial r} = -\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{r}} \leftarrow \text{get a 2nd order eqn for } \ddot{r} = \underline{\qquad}$$

But, it is easier to solve $\mathcal{L} = -\frac{1}{2}$ for $\dot{r}$. Then you get a 1st order eqn for $\dot{r}$.

$$-\left(1-\frac{2m}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1-\frac{2m}{r}} + r^2\dot{\phi}^2 = -1$$

$$\dot{r}^2 = \left(1-\frac{2m}{r}\right)\left[-1 + \left(1-\frac{2m}{r}\right)\dot{t}^2 - r^2\dot{\phi}^2\right]$$

$$\dot{r}^2 = \left(1-\frac{2m}{r}\right)\left[-1 + \left(1-\frac{2m}{r}\right)\frac{e^2}{\left(1-\frac{2m}{r}\right)^2} - r^2\frac{\ell^2}{r^4}\right]$$

$$\boxed{\dot{r}^2 = \left(1-\frac{2m}{r}\right)\left[-1 + \frac{e^2}{1-\frac{2m}{r}} - \frac{\ell^2}{r^2}\right]}$$

Work this another way

$$\mathcal{L} = \frac{1}{2}\left[-\left(1-\frac{2m}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1-\frac{2m}{r}} + r^2\dot{\phi}^2\right] = -\frac{1}{2}$$

$$-\left(1-\frac{2m}{r}\right)\frac{e^2}{\left(1-\frac{2m}{r}\right)^2} + \frac{\dot{r}^2}{1-\frac{2m}{r}} + \frac{r^2 \ell^2}{r^4} = -1$$

$$\frac{-e^2}{1-\frac{2m}{r}} + \frac{\dot{r}^2}{1-\frac{2m}{r}} + \frac{\ell^2}{r^2} = -1$$

$$-e^2 + \dot{r}^2 + \frac{1}{r^2}\left(\ell^2\right)\left(1-\frac{2m}{r}\right) = -1 + \frac{2m}{r}$$

$$\boxed{\dot{r}^2 - \frac{2M}{r} + \frac{\ell^2}{r^2}\left(1-\frac{2m}{r}\right) = e^2 - 1}$$

Now multiply by $\frac{1}{2}m_t \ldots$ m_t = test mass

define $\frac{1}{2}m_t\left(e^2-1\right) = E$

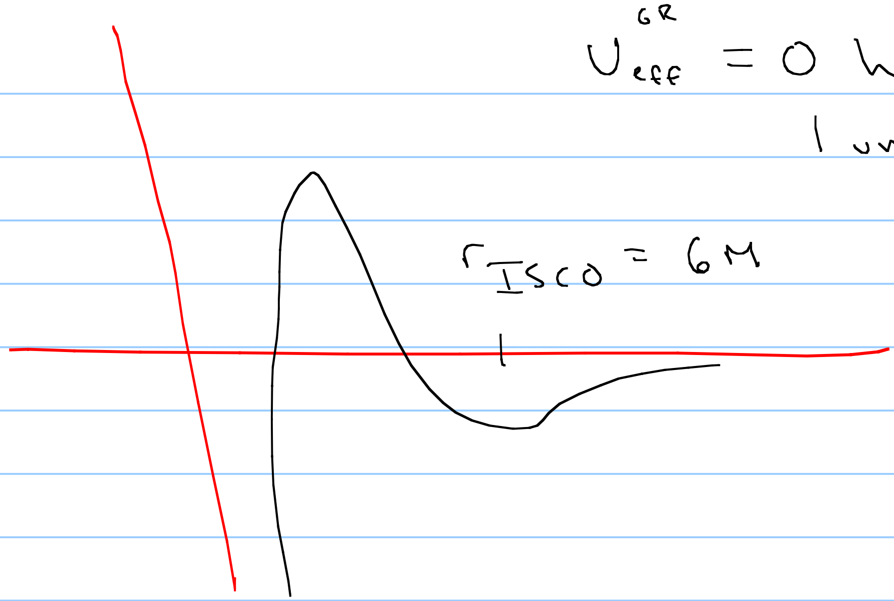
$$\boxed{\frac{1}{2}m_t\dot{r}^2 - \frac{Mm_t}{r} + \frac{m_t\ell^2}{2r^2}\left(1-\frac{2m}{r}\right) = E}$$

$$KE + U_{\text{newton}} + U^{\text{Fake}}_{L\text{ cons.}} = \text{Energy}$$

$$KE + U^{GR}_{\text{Effective}} = E$$

But look at $U^{GR}_{\text{effective}} = \underbrace{-\frac{Mm_t}{r} + \frac{m_t\ell^2}{2r^2}}_{\text{same as Newton}} \underbrace{- \frac{Mm_t\ell^2}{r^3}}_{\text{a new GR part!}}$

$U_{\text{eff}}^{GR} = 0$ has 2 roots, 1 stable,
1 unstable



All sorts of different motion, but the $\dfrac{-Mm_f \ell^2}{r^3}$ part perturbs the closed orbit. For Mercury, Einstein computed this to be ~ 43"/century!

Light rays: $\mathcal{L} = \frac{1}{2} g_{ab} u^a u^b = 0$! Null

(By the way, I was certainly not the 1st person to do this, but what follows is pretty much the 1st 10 or so lines of my Ph.D. thesis.)

$$\mathcal{L} = \frac{1}{2}\left(-\left(1-\frac{2m}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1-\frac{2m}{r}} + r^2\dot{\phi}^2\right) = 0$$

$$-\dot{t}\left(1-\frac{2m}{r}\right) = -1 \qquad (\text{take } e = 1)$$

$$\dot{t} = \frac{1}{1-2m/r}$$

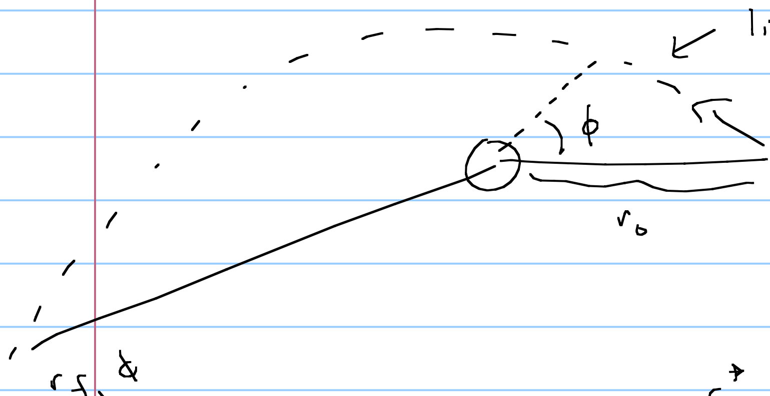
$$r^2 \dot{\phi} = b \Rightarrow \dot{\phi} = \frac{b}{r^2} = \frac{d\phi}{d\lambda}$$

$$\mathcal{L} = 0 \Rightarrow -\left(1-\frac{2m}{r}\right)\left(\frac{1}{1-\frac{2m}{r}}\right)^2 + \frac{\dot{r}^2}{1-\frac{2m}{r}} + r^2\frac{b^2}{r^4} = 0$$

$$\frac{\dot{r}^2}{1-\frac{2m}{r}} = \frac{1}{1-\frac{2m}{r}} - \frac{b^2}{r^2}$$

$$\dot{r} = \pm\sqrt{1 - \frac{b^2}{r^2}\left(1-\frac{2m}{r}\right)} = \frac{dr}{d\lambda}$$

Then $$\frac{d\phi}{dr} = \frac{d\phi}{d\lambda}\frac{d\lambda}{dr} = \pm\frac{b}{r^2}\frac{1}{\sqrt{1-\frac{b^2}{r^2}\left(1-\frac{2m}{r}\right)}}$$



$\frac{d\phi}{dr} < 0$ (− sign) on way in ($\phi \uparrow$ but $r \downarrow$)

$$\phi - \phi_0 = -\int_{r_0}^{r^*} \frac{b\,dr}{r^2\sqrt{1-\frac{b^2}{r^2}\left(1-\frac{2m}{r}\right)}} + \int_{r^*}^{r_f} \frac{b\,dr}{r^2\sqrt{1-\frac{b^2}{r^2}\left(1-\frac{2m}{r}\right)}}$$

where r_* is the root of $1-\frac{b^2}{r^2}\left(1-\frac{2m}{r}\right) = 0$

If you take $r_0 \to -\infty$, $r_f \to +\infty$ $\boxed{\phi - \phi_0 = \Delta\phi = \frac{4GM}{c^2 b}}$