Lecture 15: Schwarzschild Spacetime & Black Holes 1

Note Title

Review of Perspective: (Important conceptual stocktaking)

(A) **Starting pt:** Manifold w/ local flat (free fall) coordinates, global coordinate systems, and a coordinate transformation b/w the two.

(B) **Next Issues:** – derive $\boxed{g_{\mu\nu}}$ by coordinate trans. on physical distances

(connections, metrics, parallel transport)

– derive $\boxed{\Gamma^{\alpha}_{\rho\sigma}}$ via the geodesic equation by transforming the equations of motion

– define parallel transport by keeping angles of vector field & tangent vector to curve fixed. This defines a covariant derivative $\boxed{\frac{D}{D\lambda}}$

(C) Curvature, curvature tensor

- Ask when $g_{\mu\nu}$ is Minkowski, and answer by asking when the connecting vector b/w 2 geodesics has vanishing acceleration.

- From $\frac{D^2 v^\mu}{D\tau^2} = 0$, get $R^\mu{}_{\nu\rho\sigma} = 0$ for Minkowski Spacetime

(D) Einstein Field Equation

- Based on need to use tensors & have a 2nd order differential equation, we get the Einstein Eqn.

(E) Solutions to EFE

(1) Minkowski space

$g_{\mu\nu} = \eta_{\mu\nu}$

(Trivial, flat + empty)

(2) -----

(3) ----- $\Big\}$ what we study now.

The solution to the EFE is a metric $g_{\mu\nu}$ that indicates the curvature + physical properties of the spacetime.

Schwarzschild Solution

to $R_{\mu\nu} = 0$ (Empty space - $T_{\mu\nu} = 0$)

$G = c = 1$

$$\boxed{ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)}$$

At $r = 2m$, there is a problem: ds^2 is not defined!

So, what do the coordinates mean?

(1) Hold t, r constant ($dr = dt = 0$)

$$ds^2 \Rightarrow dL^2 = r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

- This is the line element of a sphere, so θ, ϕ are ordinary angles.

- The surface area of such a sphere would be $\underline{4\pi r^2}$.

So, is $\underline{r}$ the "ordinary" radial coordinate?

(2) Hold t, θ, ϕ constant.

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2$$

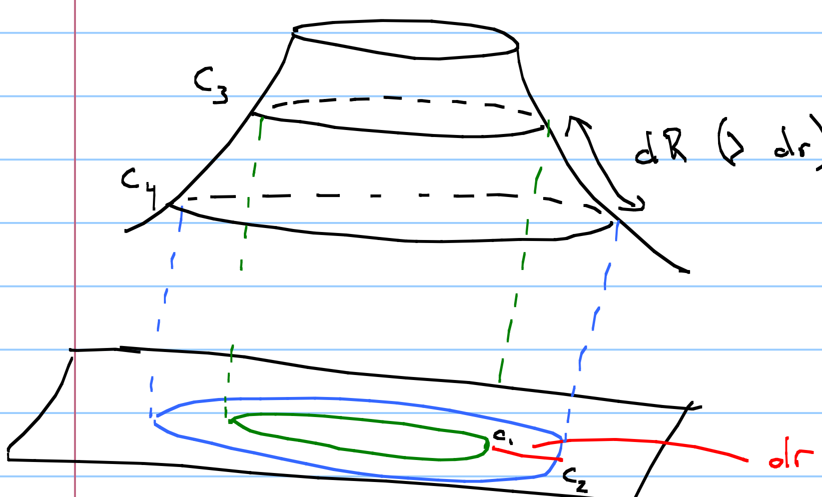
$$\Rightarrow \quad dR = \left(1 - \frac{2m}{r}\right)^{-1/2} dr$$

Physical radial distance

radial coordinate distance

$$\boxed{dR > dr} \ !$$

So r no longer measures the physical radial distance b/w cocentric spheres b/c of the spacetime curvature.



Spheres of radius r_1 & r_2 in Schwarzschild & flat space. The distance b/w the spheres is longer in Schwarzschild b/c of spacetime curvature.

(3) Hold r, θ, ϕ constant.

Proper time $d\tau$ — measured on clocks — is not equal to the coordinate time dt.

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 \Rightarrow \boxed{d\tau = \left(1 - \frac{2m}{r}\right)^{1/2} dt}$$

$(\underline{r > 2m})$, r fixed

The closer r is to $2m$, the longer dt is for a given $d\tau$.

(Deeper in field, clocks run slower.)

Let us consider the paths of light rays traveling radially between r_1 + r_2 $(r_2 > r_1)$

$$ds^2 = 0 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2$$

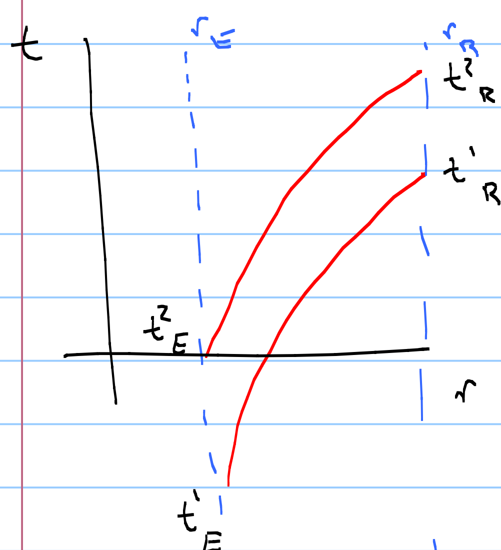
property of light rays

$$dt^2 = \frac{dr^2}{\left(1 - \frac{2m}{r}\right)^2}$$

$$\int_{t_1}^{t_2} dt = \int_{r_1}^{r_2} \frac{dr}{1 - \frac{2m}{r}}$$

$$\boxed{t_2 - t_1 = \left(r + 2m \ln(r - 2m)\right)\Big|_{r_1}^{r_2}}$$

Let a person at r_E send 2 light signals to a person at r_R. The sender emits the light at t_E^1 & t_E^2. How much time elapses at r_R b/w these 2 signals?



Now $t_R^1 - t_E^1 = t_R^2 - t_E^2$

so $t_E^2 - t_E^1 = t_R^2 - t_R^1$

or $\Delta t_E = \Delta t_R$

the coordinate times b/w pulses are the same.

Coordinate time is not physical time

$$\Delta \tau = \left(1 - \frac{2m}{r}\right)^{1/2} \Delta t$$

Let $r_R = \infty$, then $\Delta \tau_R = \Delta t_R = \Delta t_E$

But at emission location $\Delta\tau_E = \left(1 - \frac{2m}{r_E}\right)^{1/2} \Delta t_E$

$$\boxed{\Delta\tau_E = \left(1 - \frac{2m}{r_E}\right)^{1/2} \Delta\tau_R}$$ ← Relation b/w physical times

$\Delta\tau_E < \Delta\tau_R$!

So if 1s elapses b/w emission events, more than 1s will elapse b/w reception events.

In general
$$\frac{\Delta\tau_R}{\Delta\tau_E} = \left[\frac{1 - \frac{2m}{R_R}}{1 - \frac{2m}{R_E}}\right]^{1/2} = \left(\frac{g_{00}(r_R)}{g_{00}(r_E)}\right)^{1/2}$$

<u>First note</u>: Suppose we are at reception pt watching a pulsating atom whose <u>local</u> frequency is $\nu = \frac{n}{\Delta t}$

If the atom is at r_E, an observer at r_E observers
$$\nu_E = \frac{n}{\Delta t_E}.$$

<u>But</u> we observer at r_R $\nu_R = \frac{n}{f\,\Delta\tau_E} = \frac{n}{\Delta\tau_R}$

where $$f = \left(\frac{g_{00}(r_R)}{g_{00}(r_E)} \right)^{1/2},$$

The ratio of these 2 frequencies is

$$\frac{\nu_R}{\nu_E} = \left(\frac{g_{00}(r_E)}{g_{00}(r_R)} \right)^{1/2} < 1 \quad \text{if} \quad r_E < r_R$$

The observed frequency (further out) is shorter,
so the wavelength is longer, & the photon
is redshifted!

Second note: At $r = 2m$, it takes an infinitely long physical time for the light signal to reach you at large r. Also, photons are infinitely redshifted and can not be observed.

So, what is physically happening at $r = 2m$?