# Lecture 14: The Einstein Field Equation

Note Title

We will avoid details – see Hartle, Kenyon and other books.

**Step 1:** We want to somehow capture the matter (particles + fields other than gravity) in a tensor to use in the EFE.

(We want to determine the stress-energy tensor.)

- The idea here is that any type of matter should be a source for curvature.

⇒ In SR, Energy and momentum are related

$$\boxed{E^2 - c^2 p^2 = m^2 c^4}$$

Consider a cloud of dust w/ density $\rho_0$ in a "rest frame." This cloud has an energy density

$$\rho_0 c^2 = m_0 n_0 c^2$$

where $n_0 \equiv$ # grains/Vol, $m_0$ - mass of each

$\hookrightarrow$ In a frame which is boosted,

$$m_0 \to \gamma m_0$$
$$n_0 \to \gamma n_0$$

(b/c of Lorentz contraction shrinks the volume)

So $\rho_0 \Rightarrow \rho = \gamma^2 \rho_0$ $\leftarrow$ not like a vector or a scalar

$\hookrightarrow$ but like a rank 2 Tensor

$$T^{\mu\nu} = \rho_0 V^\mu V^\nu \quad \text{where}$$

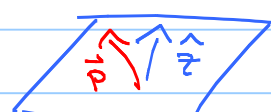
$$V^\mu = (\gamma c, \gamma v_x, \gamma v_y, \gamma v_z)$$

Scalar ; $\rho \to \rho$
Vector ; $\rho \to \gamma \rho$

$T^{\mu\nu}$ contains information about the energy and momentum.

## Properties

(1) $T^{00}$ is the energy density
(2) $cT^{0i}$ is energy flow per unit area in $i$ dir
(3) $T^{ii}$ is momentum component $i$ per unit area in the $i$ dir.
( How much momentum is flowing through a surface .... )




$\sim T^{zz}$

(4) $T^{ij}$ $(i \neq j)$ flow of $i$ comp. of momentum in $j$ direction.
( How much momentum is flowing parallel to surface )
(5) $cT^{i0}$ is the density of the $i$ component of the momentum

Note: Under a Lorentz boost, the spatial components will mix w/ the time components. All things, matter and energy, warp spacetime.

Mathematical properties

$T^{\alpha\beta} = T^{\beta\alpha}$

$T^{0\alpha}{}_{;\alpha} = 0 \leftarrow$ general cons. of Energy

$T^{i\alpha}{}_{;\alpha} = 0 \leftarrow$ general cons. of momentum

if $T^{\alpha\beta}{}_{;\alpha} = 0 \leftarrow$ "divergence free"

if $T^{\alpha\beta} = 0 \leftarrow$ no matter of any kind.

Step 2: We want a tensor eqn that relates Riemann, Ricci tensor, Ricci Scalar and Stress energy tensor.

As discussed last class

$R^{\alpha}{}_{\beta\gamma\delta} = 0$ is too restrictive (only sol. is flat space).

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}}$$

The Einstein Field Eqn.

There are 10 independent, coupled, non-linear 2nd order ODE's for the 10 components of $g_{\mu\nu}$.

Ocham's Razor: The EFE is the simplest possible equation that has at least 2 der. & is tensorial (generally covariant).

Some people study "higher order" or "higher derivative" theories.

To date, there is no reason to believe in higher derivative theories.

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Other cases, issues

→ Cosmological Constant

Define $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

The Einstein Tensor

Then $$\boxed{G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}}$$

w/ $\Lambda$ a constant

EFE w/ cosmological constant

is allowed. It is now in vogue to believe that $\Lambda \neq 0$ in the universe.

→ No matter, or $T_{\mu\nu} = 0$

then $G_{\mu\nu} = R_{\mu\nu} = 0 \quad (\Lambda = 0)$

a nice simplification

$R_{\mu\nu} = -\Lambda g_{\mu\nu} \quad (\Lambda \neq 0)$

$R_{\mu\nu} = 0$ gives many, and some of the most interesting, solutions.

- Schwarzschild Black Hole
- Kerr-Newman Black Hole.

<u>Note</u>: Black hole solutions are curved spacetimes where no matter is present anywhere.

→ You can have curvature w/ no sources.