Lecture 13 General Covariance

Note Title

In this lecture, we think more about the covariant derivative, general covariance and physics. This lecture motivates the next lecture, where we introduce the Einstein Field Equation.

→ We have introduced a derivative operator which is covariant, or coordinate independent. This means that the derivative operator is a tensor.

$\frac{D}{D\tau}$ has different forms when applied to different objects.

$$\frac{Df}{D\tau} \quad - f \text{ is a scalar} - \quad \frac{Df}{D\tau} = \frac{df}{d\tau}$$

$$\frac{Dq^\mu}{D\tau} = \frac{dq^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma} \dot{x}^\rho q^\sigma \quad \text{(applied to a vector)}$$

$$\frac{Dq_\mu}{D\tau} = \frac{dq_\mu}{d\tau} - \Gamma^\nu_{\mu\rho} q_\nu \dot{x}^\rho \quad \text{(applied to a dual)}$$

$$\boxed{\frac{D A_{\mu\nu}}{D\tau} = \frac{d A_{\mu\nu}}{d\tau} - \Gamma^{\tau}_{\mu\rho} A_{\tau\nu} \dot{x}^{\rho} - \Gamma^{\tau}_{\nu\rho} A_{\tau\mu} \dot{x}^{\rho}}$$

(applied to a $\binom{0}{2}$ index tensor)

If: $$\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} \left\{ g_{\nu\rho,\sigma} + g_{\nu\sigma,\rho} - g_{\rho\sigma,\nu} \right\}$$

Then: $$\boxed{\frac{D}{D\tau} g_{\mu\nu} = 0}$$ **Important!**

We view the covariant derivative as associated with a metric connection.

$$\begin{tikzcd}
 & g_{\mu\nu} \arrow[dl, leftrightarrow] \arrow[dr] & \\
\frac{D}{D\tau} \arrow[rr, leftrightarrow, bend right] & & \Gamma^{\mu}_{\nu\rho}
\end{tikzcd}$$

3 associated quantities.

Typically we consider the metric as primary and the connection and covariant derivative as secondary.

This is convention.

Einstein Field Eqn is an eqn for $g_{\mu\nu}$

Can treat any of the 3 as primary, or in fact other quantities.

$\rightarrow$ Take $\dfrac{Dq^\mu}{D\tau} = \dfrac{dq^\mu}{d\tau} + \Gamma^\mu_{\sigma\rho}\, q^\sigma \dot{x}^\rho$, multiply by $\dfrac{\partial\tau}{\partial x^\nu}$

$$\frac{Dq^\mu}{Dx^\nu} = \nabla_\nu q^\mu = \frac{\partial q^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\rho}\, q^\rho$$

or $$\boxed{q^\mu{}_{;\nu} = q^\mu{}_{,\nu} + \Gamma^\mu_{\sigma\nu}\, q^\sigma}$$

Notation: $\left.\begin{array}{l} , \text{ (comma)} \\ ; \text{ (semi colon)} \\ \nabla_\nu \end{array}\right\}$ all very common.

General Covariance:

In SR, $F^\mu = \dfrac{dp^\mu}{d\tau}$ is Newton's 2nd Law

with $F^\mu = \left(\gamma \dfrac{dE}{d\tau}, \gamma \dfrac{dp^i}{d\tau}\right)$

But $F^\mu = \dfrac{dp^\mu}{d\tau}$ is not a good physical law.

Under coord. transformation to a general frame, fictitious forces are introduced.

The reason is that $\frac{dp^\mu}{d\tau}$ is not a tensor!

$$\boxed{F^\mu = \frac{Dp^\mu}{D\tau}}$$ ← A good physical law!

$\frac{Dp^\mu}{D\tau}$ is a tensor, so

$F^\mu = \frac{Dp^\mu}{D\tau}$ is true for <u>any</u> observer.

<u>General Covariance</u>: (Important)

(1) Physical Laws must be expressed as tensor quantities so that they are valid in accelerating frames.

(2) In a free fall frame, all laws reduce to SR.

If you $\boxed{\text{replace , w/ ;}}$ in SR laws, you introduce general covariance.

(Part of the path to GR.)

In GR, we need generally covariant laws of nature. We also need an equation that determines either g_{ab}, $\frac{D}{D\tau}$, or Γ^a_{bc}.

So GR is contained in 2 basic insights.

(1) All physical laws must be generally covariant.

(2) The Einstein Field Equation will determine either g_{ab}, $\frac{D}{D\tau}$, Γ^a_{bc}, which in turn tells us how to write generally covariant laws.

Path to Einstein Field Equation (EFE)

→ We have learned that you can, at any pt, transform to the free fall frame, so that at that point in spacetime (x)

$$\left.\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} \\ g_{\mu\nu,\rho} &= 0 \end{aligned}\right\} \text{at } x$$

Then, $g_{\mu\nu}(x+\Delta x) = \eta_{\mu\nu} + \frac{1}{2} g_{\mu\nu,\rho\sigma} \Delta x^\rho \Delta x^\sigma$

$$g_{\mu\nu,\rho}(x+\Delta x) = g_{\mu\nu,\rho\sigma} \Delta x^\sigma$$

2nd derivatives!
in Taylor series.

$\Longrightarrow$ So 2nd derivatives embody the curvature!

Riemann Tensor: $R_{\alpha\beta\gamma\delta} = g_{\alpha\mu} R^{\mu}{}_{\beta\gamma\delta}$

$$\boxed{R_{\alpha\beta\gamma\delta} = 2\left(g_{\alpha\delta,\beta\gamma} - g_{\beta\delta,\alpha\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta}\right)}$$

Important: There is a theorem which says that the only tensor which is generally covariant that is made of 2nd derivatives of the metric is $R_{\alpha\beta\gamma\delta}$.

o We seek a differential equation for $g_{\mu\nu}$.

We want to see how gravity evolves.

(1) Must have 2 derivatives.
(2) Must be generally covariant.

(1) + (2) $\Longrightarrow$ We must use $R^{\mu}{}_{\nu\rho\sigma}$ or tensors derived from it to make the simplest EFE.

From the Riemann tensor, we can derive two other tensors.

<u>Ricci tensor</u>: $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$

order matters.

<u>Ricci Scalar</u>: $R = g^{\mu\nu} R_{\mu\nu}$

The EFE for $g_{\mu\nu}$ <u>most</u> include only R, $R_{\mu\nu}$, $R^{\mu}{}_{\nu\rho\sigma}$.

Ocham's Razor....

? What about $R^{\mu}{}_{\nu\rho\sigma} = 0$?

Only Solution is Flat Space, or

$g_{\mu\nu} = \eta_{\mu\nu}$

$R^{\mu}{}_{\nu\rho\sigma} = 0$ is too restrictive – the only solution has no curvature.

A note about Quantum Mechanics.

There are 2 versions of QM

- Ordinary QM based on the Schrödinger Eqn.
- QFT or relativistic QM based on the
  Dirac Eqn, Feynman diagrams, etc.

From a GR standpoint, both of these
must be wrong b/c they are not
generally covariant.

- QFT is based on SR

- So... if you take all the $,$ & replace w/ $;$
  in QFT, you should get a good
  theory of Quantum Gravity, right?

Well, no.

(1) If you assume that the background spacetime is fixed, you can get a good theory for QFT in a fixed curved background.

→ Book by Wald — very hard stuff.

(2) The problem is that the tricks used in Feynman diagrams no longer work, especially when applied to gravity particles.

So

The "replace , w/ ;" prescription for forming a generally covariant physical theory doesn't work for QFT.

No one knows why.