Lecture 12 Supplemental Notes

Note Title 3/10/2006

In these supplemental notes, we consider the mathematical definition of parallel transport.

We want to show

$$\frac{d}{d\tau}\left(g_{\mu\nu}\dot{x}^{\mu}v^{\nu}\right) = 0 \iff \frac{Dv^{\mu}}{D\tau} = \frac{dv^{\mu}}{d\tau} + \Gamma^{\mu}_{\rho\sigma}\dot{x}^{\rho}v^{\sigma} = 0$$

$$\frac{d}{dt}\left(g_{\mu\nu}\dot{x}^{\mu}v^{\nu}\right) = 0 = g_{\mu\nu}\dot{v}^{\mu}\dot{x}^{\nu} + g_{\mu\nu}v^{\mu}\ddot{x}^{\nu} + g_{\mu\nu,\rho}v^{\mu}\dot{x}^{\nu}\dot{x}^{\rho}$$

but $\ddot{x}^{\nu} = -\Gamma^{\nu}_{\rho\sigma}\dot{x}^{\rho}\dot{x}^{\sigma}$ — $\boxed{\text{b/c } x^{\mu}(\tau) \text{ is geodetic}}$

$$0 = g_{\mu\nu}\dot{v}^{\mu}\dot{x}^{\nu} - g_{\mu\nu}v^{\mu}\Gamma^{\nu}_{\rho\sigma}\dot{x}^{\rho}\dot{x}^{\sigma} + g_{\mu\nu,\rho}v^{\mu}\dot{x}^{\nu}\dot{x}^{\rho}$$

Use $\Gamma^{\nu}_{\rho\sigma} = \frac{1}{2}g^{\nu\tau}\left(g_{\rho\tau,\sigma} + g_{\sigma\tau,\rho} - g_{\rho\sigma,\tau}\right)$

$$0 = g_{\mu\nu}\dot{v}^{\mu}\dot{x}^{\nu} - \frac{1}{2}\underline{g_{\mu\nu}g^{\nu\tau}}\left(g_{\rho\tau,\sigma} + g_{\sigma\tau,\rho} - g_{\rho\sigma,\tau}\right)v^{\mu}\dot{x}^{\rho}\dot{x}^{\sigma}$$

$$+ g_{\mu\nu,\rho}v^{\mu}\dot{x}^{\nu}\dot{x}^{\rho}$$

$g_{\mu\nu}g^{\nu\tau} = \delta_{\mu}^{\tau}$

$$0 = g_{\mu\nu}\dot{v}^{\mu}\dot{x}^{\nu} - \cancel{\frac{1}{2} g_{\rho\tau,\sigma} v^{\tau}\dot{x}^{\rho}\dot{x}^{\sigma}} \cancel{- \frac{1}{2} g_{\sigma\tau,\rho} v^{\tau}\dot{x}^{\rho}\dot{x}^{\sigma}}$$

$$+ \frac{1}{2} g_{\rho\sigma,\tau} v^{\tau}\dot{x}^{\rho}\dot{x}^{\sigma} + \cancel{g_{\mu\nu,\rho} v^{\mu}\dot{x}^{\nu}\dot{x}^{\rho}}$$

Cancels w/ relabeling

Use

$$g_{\rho\sigma,\tau} = \Gamma^{\alpha}_{\rho\tau} g_{\sigma\alpha} + \Gamma^{\alpha}_{\sigma\tau} g_{\alpha\rho}$$

(Kenyon p. 61)

$$0 = g_{\mu\nu}\dot{v}^{\mu}\dot{x}^{\nu} + \frac{1}{2}\Gamma^{\alpha}_{\rho\tau} g_{\sigma\alpha} v^{\tau}\dot{x}^{\rho}\dot{x}^{\sigma}$$

$$+ \frac{1}{2}\Gamma^{\alpha}_{\sigma\tau} g_{\alpha\rho} v^{\tau}\dot{x}^{\sigma}\dot{x}^{\rho}$$

Same term!

$$0 = g_{\mu\nu}\dot{v}^{\mu}\dot{x}^{\nu} + \underbrace{\frac{1}{2}\Gamma^{\alpha}_{\rho\tau} g_{\sigma\alpha} v^{\tau}\dot{x}^{\rho}\dot{x}^{\sigma} + \frac{1}{2}\Gamma^{\alpha}_{\cancel{\sigma}\rho\,\tau} g_{\alpha\cancel{\rho}\sigma} v^{\tau}\dot{x}^{\cancel{\sigma}\rho}\dot{x}^{\cancel{\rho}\sigma}}_{\Gamma^{\alpha}_{\rho\tau} g_{\sigma\alpha} v^{\tau}\dot{x}^{\rho}\dot{x}^{\sigma}}$$

$$0 = g_{\mu\nu}\dot{v}^{\mu}\dot{x}^{\nu} + \Gamma^{\alpha}_{\rho\tau} g_{\sigma\alpha} v^{\tau}\dot{x}^{\rho}\dot{x}^{\sigma}$$

Relabel $\Gamma^{\alpha}_{\rho\tau} g_{\sigma\alpha} v^{\tau}\dot{x}^{\rho}\dot{x}^{\sigma} = g_{\mu\nu}\Gamma^{\mu}_{\rho\tau} v^{\tau}\dot{x}^{\nu}\dot{x}^{\rho}$

$$0 = g_{\mu\nu}\, \dot{x}^{\nu} \left(\dot{v}^{\mu} + \Gamma^{\mu}_{\rho\tau}\, v^{\tau}\, \dot{x}^{\rho} \right)$$

So $\boxed{\dot{v}^{\mu} + \Gamma^{\mu}_{\rho\tau}\, v^{\tau}\, \dot{x}^{\rho} = 0}$

Define

$$\frac{Dv^{\mu}}{D\tau} = \dot{v}^{\mu} + \Gamma^{\mu}_{\rho\tau}\, v^{\tau}\, \dot{x}^{\rho}$$

$\frac{Dv^{\mu}}{D\tau} = 0$ is the eqn for parallel transport.