Lecture 12: Riemann Tensor, Parallel Transport

Note Title

Fundamental Question: How can we tell if a spacetime is flat?

To answer: Introduce mathematical definition of parallel transport, Jacobi vector fields, covariant derivatives, Riemann tensor

Many important GR mathematical ideas

$\Rightarrow$ As we saw in previous lecture, different observers, using different coordinates, see the metric in different ways.

Minkowski (SR) spacetime has a metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

for the family of inertial observers.

- Suppose an observer comes along and says the metric is $ds^2 = 4\,du\,dv - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2$

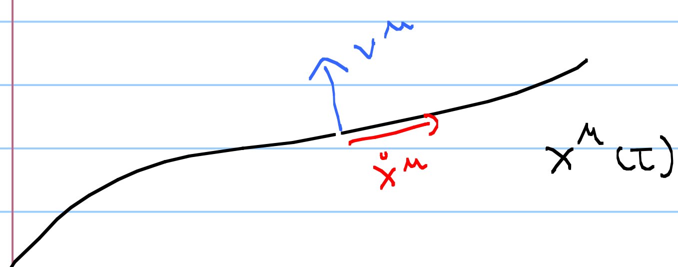
How can you tell if this metric is flat?

Side note

Note: Different notations exist for the covariant derivative − $\frac{D}{D\tau}$ & ∇_τ are common ones.

Fundamental Question → How do you know whether a metric is really Minkowski spacetime in wacky coordinates?

→ Define parallel transport along a curve (mathematically)



$x^\mu(\tau)$ is a curve which is geodesic

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\, \dot{x}^\nu \dot{x}^\rho = 0$$

⇒ To parallel transport v^μ along $x^\mu(\tau)$, we keep the angle b/w v^μ and $\dot{x}^\mu$ the same

$$\frac{d}{d\tau}\left(g_{\mu\nu}\, v^\mu \dot{x}^\nu \right) = 0 \quad \text{—} \quad \cos\theta = \frac{g_{\mu\nu}\dot{x}^\mu v^\nu}{|\dot{x}^\mu|\,|v^\mu|}$$

evaluated along $x^\mu(\tau)$

A long calculation (in the supplemental notes) yields

$$\frac{d}{d\tau}\left(g_{\mu\nu} v^\mu \dot{x}^\nu\right) = 0 \iff \boxed{\frac{d}{d\tau} v^\mu = -\Gamma^\mu_{\nu\rho}\, \dot{x}^\nu v^\rho}$$

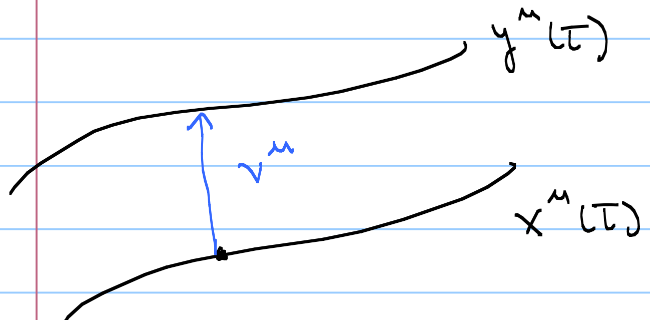
So we define $\boxed{\dfrac{D v^\mu}{D\tau} = \dfrac{d v^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}\, \dot{x}^\nu v^\rho}$

$\dfrac{D v^\mu}{D\tau}$ is the covariant derivative of v^μ along the curve $x^\mu(\tau)$

$\Rightarrow$ Eqn for parallel transport is $\boxed{\dfrac{D}{D\tau} v^\mu = 0}$

(parallel transport equation)

Now take 2 curves



$v^\mu \leftarrow$ connecting or Jacobi vector

$$y^\mu = x^\mu + v^\mu$$

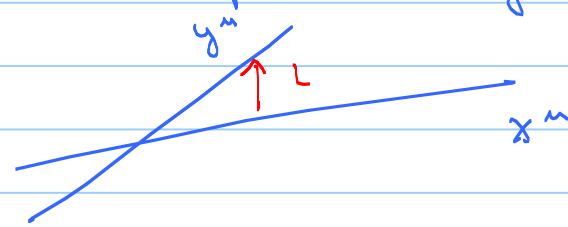
Both x^μ + y^μ will be geodesics, and v^μ will be parallel transported along $x^\mu(\tau)$

If L is the distance b/w curves, assume $\dot{L} \neq 0$.

If there are no forces $\Leftrightarrow$ $\ddot{L} = 0$ $\Leftrightarrow$ Mink. space

$\ddot{L} \neq 0$, you are in a curved space.

Geodesics in Minkowski space are "straight" lines. Two non-parallel straight lines in Minkowski space will separate linearly in time



$\dot{L} = c$ in flat space w/ c constant

Study $\dfrac{D^2 v^\mu}{D\tau^2}$ to 1st order in v^μ.

Jacobi fields are small.

y^μ is geodetic $\Rightarrow$ $\ddot{y}^\mu + \Gamma^\mu_{\rho\sigma}\, \dot{y}^\rho\, \dot{y}^\sigma = 0$

but $y^\mu = x^\mu + v^\mu$

Use $\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\, \dot{x}^\rho\, \dot{x}^\sigma = 0$ & $\dot{v}^\mu = -\Gamma^\mu_{\rho\sigma}\, \dot{x}^\rho\, v^\sigma$

$\boxed{1}$ x^μ is a geodesic $\boxed{2}$ v^μ is $\parallel$ transported

$$\left(\ddot{x}^\mu + \ddot{v}^\mu\right) + \Gamma^\mu_{\rho\sigma}(x+v)\left(\dot{x}^\rho + \dot{v}^\rho\right)\left(\dot{x}^\sigma + \dot{v}^\sigma\right) = 0$$

$$\underbrace{-\Gamma^\mu_{\rho\sigma}(x)\,\dot{x}^\rho\,\dot{x}^\sigma}_{\boxed{1}} + \ddot{v}^\mu + \left(\Gamma^\mu_{\rho\sigma}(x) + v^\alpha\partial_\alpha\Gamma^\mu_{\rho\sigma}(x)\right)\left(\dot{x}^\rho + \dot{v}^\rho\right)\left(\dot{x}^\sigma + \dot{v}^\sigma\right) = 0$$

Taylor expanding

$$\Gamma(x+v) = \Gamma(x) + v^\alpha\partial_\alpha\Gamma(x)$$

Cancels

$$\ddot{v}^\mu + \underbrace{\cancel{\Gamma^\mu_{\rho\sigma}\dot{v}^\rho\dot{v}^\sigma}}_{\text{2nd order}} + 2\Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{v}^\sigma + v^\alpha\partial_\alpha\Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma + \underbrace{\cancel{\mathcal{O}(v^2)}}_{\text{2nd order}} = 0$$

$$\frac{d}{d\tau}\left(\frac{d}{d\tau}v^\mu\right) + 2\Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{v}^\sigma + v^\alpha\partial_\alpha\Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = 0$$

$$\boxed{2} \rightarrow \frac{d}{d\tau}v^\mu = \frac{Dv^\mu}{D\tau} - \Gamma^\mu_{\rho\sigma}\dot{x}^\rho v^\sigma \qquad \boxed{3}$$

$$\boxed{\frac{d}{d\tau}\left(\frac{Dv^\mu}{d\tau} - \Gamma^\mu_{\rho\sigma}\dot{x}^\rho v^\sigma\right)} + 2\Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{v}^\sigma + v^\alpha\partial_\alpha\Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = 0$$

$$\frac{d}{d\tau}\frac{Dv^\mu}{D\tau} - \frac{d}{d\tau}\left(\Gamma^\mu_{\rho\sigma}\dot{x}^\rho v^\sigma\right)$$

product rule

$$\boxed{3}\quad \frac{D^2v^\mu}{D\tau^2} - \Gamma^\mu_{\rho\sigma}\dot{x}^\rho v^\sigma\,\underbrace{\cancel{\frac{Dv^\mu}{D\tau}}}_{=0}\,\boxed{2} - \Gamma^\mu_{\rho\sigma}\ddot{x}^\rho v^\sigma - \Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{v}^\sigma - \dot{x}^\alpha\left(\partial_\alpha\Gamma^\mu_{\rho\sigma}\right)\dot{x}^\rho v^\sigma$$

chain

+2 − 1 cancellation.

$$\frac{D^2 v^\mu}{D\tau^2} - \Gamma^{\mu}_{\rho\sigma}\, \underbrace{\ddot{x}^{\rho}}_{\boxed{1}}\, v^{\sigma} + \Gamma^{\mu}_{\rho\sigma}\, \dot{x}^{\rho}\, \overset{\boxed{2}}{\dot{v}^{\sigma}} + v^{\alpha}\left(\partial_\alpha \Gamma^{\mu}_{\rho\sigma}\right)\dot{x}^{\rho}\dot{x}^{\sigma}$$
$$- \dot{x}^{\alpha}\left(\partial_\alpha \Gamma^{\mu}_{\rho\sigma}\right)\dot{x}^{\rho} v^{\sigma} = 0$$

$$\frac{D^2 v^\mu}{D\tau^2} - \Gamma^{\mu}_{\rho\sigma}\left(-\Gamma^{\rho}_{\alpha\beta}\right)\dot{x}^{\alpha}\dot{x}^{\beta} v^{\sigma} + \Gamma^{\mu}_{\rho\sigma}\,\dot{x}^{\rho}\left(-\Gamma^{\sigma}_{\alpha\beta}\,\dot{x}^{\alpha} v^{\beta}\right)$$
$$+ \left(\partial_\alpha \Gamma^{\mu}_{\rho\sigma}\right)\dot{x}^{\rho}\dot{x}^{\sigma} v^{\alpha} - \left(\partial_\alpha \Gamma^{\mu}_{\rho\sigma}\right)\dot{x}^{\alpha}\dot{x}^{\rho} v^{\sigma}$$
$$= 0$$

$$\frac{D^2 v^\mu}{D\tau^2} - \left(\partial_\alpha \Gamma^{\mu}_{\rho\sigma}\right)\dot{x}^{\alpha}\dot{x}^{\rho} v^{\sigma} + \left(\partial_\alpha \Gamma^{\mu}_{\rho\sigma}\right)\dot{x}^{\rho}\dot{x}^{\sigma} v^{\alpha}$$
$$+ \Gamma^{\mu}_{\rho\sigma}\Gamma^{\rho}_{\alpha\beta}\,\dot{x}^{\alpha}\dot{x}^{\beta} v^{\sigma} - \Gamma^{\mu}_{\rho\sigma}\Gamma^{\sigma}_{\alpha\beta}\,\dot{x}^{\rho}\dot{x}^{\alpha} v^{\beta}$$
$$= 0$$

The indices on $\dot{x}$, v are all contracted, so you can relabel them. Write all terms as

$$\dot{x}^{\gamma}\dot{x}^{\delta} v^{\beta}$$

$$\frac{D^2 v^\alpha}{D\tau^2} = \partial_\delta \Gamma^{\alpha}_{\gamma\beta}\,\dot{x}^{\delta}\dot{x}^{\gamma} v^{\beta} - \partial_\beta \Gamma^{\alpha}_{\delta\gamma}\,\dot{x}^{\delta}\dot{x}^{\gamma} v^{\beta}$$
$$- \Gamma^{\alpha}_{\sigma\beta}\Gamma^{\sigma}_{\delta\gamma}\,\dot{x}^{\delta}\dot{x}^{\gamma} v^{\beta} + \Gamma^{\alpha}_{\delta\sigma}\Gamma^{\sigma}_{\gamma\beta}\,\dot{x}^{\delta}\dot{x}^{\gamma} v^{\beta}$$

$$\frac{D^2 v^\alpha}{D\tau^2} = -R^{\alpha}{}_{\beta\gamma\delta}\,\dot{x}^{\gamma}\dot{x}^{\delta} v^{\beta}$$

where

$$\boxed{R^{\alpha}{}_{\beta\gamma\delta} = \partial_{\beta}\Gamma^{\alpha}_{\gamma\delta} - \partial_{\gamma}\Gamma^{\alpha}_{\delta\beta} + \Gamma^{\alpha}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\delta} - \Gamma^{\alpha}_{\sigma\delta}\Gamma^{\sigma}_{\beta\gamma}} \quad (*)$$

So that $\dfrac{D^2 v^{\mu}}{D\tau^2} = 0 \iff R^{\alpha}{}_{\beta\gamma\delta} = 0$

Condition for metric to represent Minkowski space.

$R^{a}{}_{bcd}$ is the Riemann curvature tensor. Very Important.

has 2nd derivatives of the metric.

In this relabeling, we used also

(*)

$$\Gamma^{\alpha}_{\beta\gamma}\dot{x}^{\beta}v^{\gamma} = \Gamma^{\alpha}_{\beta\gamma}\dot{x}^{\gamma}v^{\beta}$$

which is true b/c $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$.