# Lecture 10: Supplemental Notes

Note Title 3/1/2006

**Two calculations:**

(1) Definition of $\Gamma^{\mu}_{\nu\rho}$

(2) Relation of geodesic equation to the Euler-Lagrange Equations

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1. Definition of $\Gamma^{\mu}_{\nu\rho}$

Given $\frac{\partial \xi^\alpha}{\partial x^\mu} = e^\alpha_{\ \mu}$, $\quad d\xi^\alpha = e^\alpha_{\ \mu}\, dx^\mu$,

$$g_{\mu\nu} = \eta_{\alpha\beta}\, e^\alpha_{\ \mu}\, e^\beta_{\ \nu}, \quad \& \quad \boxed{\partial_\mu e^\alpha_{\ \nu} = \partial_\nu e^\alpha_{\ \mu}}$$

This is symmetry of partial derivatives. i.e. $\frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}$

Want to show

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left\{ \partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\mu\nu} \right\}$$

$$\partial_\rho g_{\nu\sigma} = \eta_{\alpha\beta} \left[ (\partial_\rho e^\alpha_{\ \nu})\, e^\beta_{\ \sigma} + e^\alpha_{\ \nu}\, \partial_\rho e^\beta_{\ \sigma} \right]$$

$$\partial_\nu g_{\rho\sigma} = \eta_{\alpha\beta} \left[ (\partial_\nu e^\alpha_{\ \rho})\, e^\beta_{\ \sigma} + e^\alpha_{\ \rho}\, \partial_\nu e^\beta_{\ \sigma} \right]$$

$$\partial_\sigma g_{\nu\rho} = \eta_{\alpha\beta} \left[ (\partial_\sigma e^\alpha_{\ \nu})\, e^\beta_{\ \rho} + e^\alpha_{\ \nu}\, \partial_\sigma e^\beta_{\ \rho} \right]$$

$$\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho} = \eta_{\alpha\beta}\Big[ (\partial_\rho e^\alpha_\nu) e^\beta_\sigma + \cancel{e^\alpha_\nu \partial_\rho e^\beta_\sigma}$$
$$+ (\partial_\nu e^\alpha_\rho) e^\beta_\sigma + \cancel{e^\alpha_\rho \partial_\nu e^\beta_\sigma}$$
$$- \cancel{(\partial_\sigma e^\alpha_\nu) e^\beta_\rho} - \cancel{e^\alpha_\nu \partial_\sigma e^\beta_\rho} \Big]$$

$\alpha \Leftrightarrow \beta$ sym
plus red box
above

$$= 2\eta_{\alpha\beta} (\partial_\rho e^\alpha_\nu) e^\beta_\sigma = 2\eta_{\alpha\beta} e^\beta_\sigma \partial_\rho e^\alpha_\nu$$

Now $\quad g_{\mu\sigma} = \eta_{\alpha\beta} e^\alpha_\mu e^\beta_\sigma \quad$ implies

$$g_{\mu\sigma} e^\mu_\alpha = \eta_{\alpha\beta} e^\beta_\sigma \text{ , so}$$

$$2\eta_{\alpha\beta} e^\beta_\sigma \partial_\rho e^\alpha_\nu = 2 g_{\mu\sigma} e^\mu_\alpha \partial_\rho e^\alpha_\nu$$

$$\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho} = 2 g_{\mu\sigma} e^\mu_\alpha \partial_\rho e^\alpha_\nu$$

$$\boxed{\frac{1}{2} g^{\mu\sigma}\left\{ \partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho} \right\} = e^\mu_\alpha \partial_\rho e^\alpha_\nu = \Gamma^\mu_{\nu\rho}}$$

as we wanted to show.

## 2. Geodesics and Lagrangians

- From the coordinate transformation b/w a freely falling coordinate system $\xi^\mu$ and a general coordinate $x^a$, we saw that a freely falling particle on a world line $\xi^\mu(\tau)$ w/ $\ddot{\xi}(\tau) = 0$ + $g_{ab}\dot{\xi}^a\dot{\xi}^b > 0$ (time like worldline), was observed to move according to

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0 \qquad \text{- the geodesic eqn.}$$

$$\text{w/} \quad \Gamma^a_{bc} = \frac{1}{2} g^{ad}\{g_{bd,c} + g_{cd,b} - g_{bc,d}\}$$

- Here, we want to show that the paths given by the geodesic equation extremize the "length" of the path.

$$S = \int_{\tau_0}^{\tau_1} d\tau \, g_{ab}\dot{x}^a\dot{x}^b \implies \mathcal{L} = \frac{1}{2} g_{ab}\dot{x}^a\dot{x}^b$$

The Euler-Lagrange Eqn is

$$\frac{\partial \mathcal{L}}{\partial x^a} = \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^a}$$

$$\frac{1}{2} g_{ab,c} \dot{x}^a \dot{x}^b = \frac{d}{d\tau}\left(g_{ab} \dot{x}^a \delta^b_c\right) = \frac{d}{d\tau}\left(g_{ac} \dot{x}^a\right)$$

$$\frac{1}{2} g_{ab,c} \dot{x}^a \dot{x}^b = g_{ac} \ddot{x}^a + g_{ac,d} \dot{x}^a \dot{x}^d$$

Now $$g_{ac,d} \dot{x}^a \dot{x}^c = \frac{1}{2} g_{ac,d} \dot{x}^a \dot{x}^d + \frac{1}{2} g_{dc,a} \dot{x}^d \dot{x}^a$$

(just by "dummy" relabeling)

So

$$g_{ac} \ddot{x}^a + \frac{1}{2}\left(g_{ac,d} \dot{x}^a \dot{x}^d + g_{dc,a} \dot{x}^d \dot{x}^a - g_{ad,c} \dot{x}^a \dot{x}^c\right) = 0$$

Multiply by $g^{ce}$

$$g^{ce} g_{ac} = \delta^e_a \Rightarrow g^{ce} g_{ac} \ddot{x}^a = \ddot{x}^e$$

$$\ddot{x}^e + \frac{1}{2} g^{ec} \left\{ g_{ac,d} + g_{dc,a} - g_{ad,c} \right\} \dot{x}^a \dot{x}^d = 0$$

$$\boxed{\ddot{x}^e + \Gamma^e_{ad} \dot{x}^a \dot{x}^d = 0}$$

# Important Points

So the application of the E-L equation to $\mathcal{L} = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b$ yields the geodesic equation.

Since we have extremized the square metric interval using the E-L equation, this calculation formally proves that objects follow the extremal curve of space time as their natural motion.