Lecture 10: Curves

Note Title 2/20/2011

In general, we explain "gravity" as geometry by saying spacetime is curved, and the paths taken are the shortest (actually <u>extremal</u>) paths through the curved space-time.

This is well understood as part of the <u>Calculus of Variations</u> – a standard subject in classical mechanics.

Define the <u>functional</u> J to be

$$J = \int_{x_1}^{x_2} f\{y(x), y'(x); x\}\, dx$$

where f is a function that depends on 2 other functions (y & $y' = \frac{dy}{dx}$)

→ Goal is to pick the fnc $y(x)$ such that J is extremal - local max or min compared to other choices of $y(x)$.

To find $y(x)$, use the Euler-Lagrange Eqn.

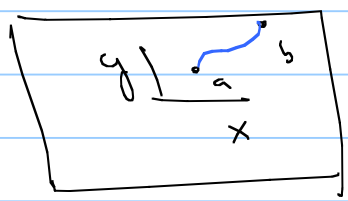
$$\boxed{\frac{\partial f}{\partial y} = -\frac{d}{dx}\frac{\partial f}{\partial y'}}$$

→ One context is to define $J = S$, the "action" and $f \equiv \mathcal{L} = T - V$ ← difference of kinetic and potential energies.

↳ Then the curve you find is solution to $\vec{F} = m\vec{a}$ ...

→ Another application is to take J to be the metric length in some (possibly) curved space.

<u>Example</u>



$\mathbb{R}^2$ – flat plane

– What is shortest curve from a to b?

$$ds^2 = dx^2 + dy^2 \Rightarrow L = \int_a^b \sqrt{dy^2 + dx^2}$$

↑ length of curve

$$L = \int_a^b dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \int_a^b dx \sqrt{1 + \dot{y}^2} \qquad \dot{y} \equiv \frac{dy}{dx}$$

Then $f = \sqrt{1 + \dot{y}^2}$ $\qquad \frac{\partial f}{\partial y} = 0$

$$0 = \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) \Rightarrow \frac{\partial f}{\partial \dot{y}} = C$$

$$\frac{\partial f}{\partial \dot{y}} = \frac{2\dot{y}}{\sqrt{1 + \dot{y}^2}} = C$$

$$\left(\frac{2}{c}\right)^2 \dot{y}^2 = 1 + \dot{y}^2 \Rightarrow \dot{y}^2\left(\left(\frac{2}{c}\right)^2 - 1\right) = 1$$

$$\dot{y} = \sqrt{\frac{1}{\left(\frac{2}{c}\right)^2 - 1}} = \alpha \leftarrow \text{some constant}$$

$$\dot{y} = \frac{dy}{dx} = \alpha \Rightarrow dy = \alpha x \Rightarrow \boxed{y = \alpha x + y_0}$$

(A straight line)

Aside: Can do the curve parametrically...

Also easier - define Lagrangian as

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right)$$ for plane...

$$\frac{\partial \mathcal{L}}{\partial x} = 0 = \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}} \Rightarrow \dot{x} = v_{0x}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0 = \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{y}} \Rightarrow \dot{y} = v_{0y}$$

$$\boxed{\begin{aligned} x &= v_{0x}\lambda + x_0 \\ y &= v_{0y}\lambda + y_0 \end{aligned}}$$

Parametric form of straight line

$$x - x_0 = v_{0x}\lambda \Rightarrow \lambda = \frac{(x - x_0)}{v_{0x}}$$

$$\& \quad \boxed{y - y_0 = v_{0y}\lambda = \left(\frac{v_{0y}}{v_{0x}}\right)(x - x_0)}$$

Straight line

So in General Relativity...

$$\mathcal{L} = \sqrt{-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta} \quad - \text{eqn } 8.10$$

or $\mathcal{L} = \frac{1}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta$ – simpler to use

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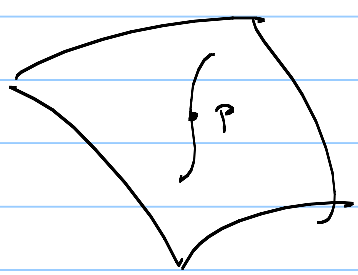
But, why does this work? Can we connect the "shortest path" idea with our idea of equivalence?

↳ Return to our 2 observers... freely falling observer w/ coordinates ξ^α, other w/ coordinates x^μ

Both observers watch a particle move through space... particle is in free fall too...

Now let the particle move:



$\xi^\alpha(\tau)$ describes the particle in the free fall frame.

What is the eqn of motion?

$\Rightarrow$ assume $\dot{\xi}^\alpha(\tau)\,\dot{\xi}^\beta(\tau)\,\eta_{\alpha\beta} = -1$ (timelike)

$\ddot{\xi}^\alpha(\tau) = 0$

Not accelerating in free fall frame

What is the eqn of motion in the x^μ coordinates?

let $x^\mu(\tau)$, so that $\xi^\alpha(\tau) = \xi^\alpha(x^\mu(\tau))$ – using the coord. transformation

$$0 = \frac{d}{d\tau}\frac{d}{d\tau}\xi^\alpha(\tau) = \frac{d}{d\tau}\left(e^\alpha_{\ \mu}\dot{x}^\mu(\tau)\right)$$

or
$$\underline{e^\alpha_{\ \mu}\ddot{x}^\mu + \left(\frac{\partial}{\partial x^\nu}e^\alpha_{\ \mu}\right)\dot{x}^\mu\dot{x}^\nu = 0}$$

use $e^\alpha_{\ \rho}\, e^\mu_{\ \alpha} = \delta^\mu_\rho$ – inverses

$$(e^\rho_{\ \alpha})\quad e^\alpha_{\ \mu}\ddot{x}^\mu = -\left(\frac{\partial}{\partial x^\nu}e^\alpha_{\ \mu}\right)\dot{x}^\mu\dot{x}^\nu \quad (e^\rho_{\ \alpha})$$

multiply both sides

$$\ddot{x}^\rho = -e^\rho_{\ \alpha}\left(\frac{\partial}{\partial x^\nu}e^\alpha_{\ \mu}\right)\dot{x}^\mu\dot{x}^\nu$$

or
$$\boxed{\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0}$$
geodesic equation

VERY IMPORTANT

w/ $$\boxed{\Gamma^\mu_{\nu\rho} = e^\mu_{\ \alpha}\left(\frac{\partial}{\partial x^\nu}e^\alpha_{\ \rho}\right)}$$
– Levi-Civita Connection, or Christoffel symbol

– note that $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$

but also that $\Gamma^\mu_{\nu\rho}$ is not a tensor!

This is a general derivation of eqn 8.14 in Hartle. In the supplemental notes, we prove that $\Gamma^{\mu}_{\nu\rho}$ can be written as derivatives of g_{ab} (below) and that the geodesic equation is the same as the E-L eqns of

$$\mathcal{L} = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b .$$

Using tetrad + $\partial_\mu e^\alpha_\nu = \partial_\nu e^\alpha_\mu$, you can show

$$\boxed{\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}\left(\partial_\rho g_{\sigma\nu} + \partial_\nu g_{\sigma\rho} - \partial_\sigma g_{\nu\rho}\right)}$$