CN 40: <u>Partial Differential Equations (PDEs).</u>

$\frac{d}{dx}\left(c\frac{du}{dx}\right) = -f$ is an ordinary differential eq. (ODE)

Unknown fnc. $u(x)$ is the displacement of point x on an elastic rod, subject to an external force per unit length f, in equilibrium.

Or $u(x)$ is the equilibrium temperature at point x on a rod, subject to a heat source $f(x)$.

etc.

$\underline{\nabla} \cdot \left(c \underline{\nabla} u(x,y) \right) = -f$ is Poisson's eq, a PDE.

$$\underline{\nabla} \cdot \left(c \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \right) = -f$$

$$\underline{\nabla} \cdot \left(c \frac{\partial u}{\partial x}, c \frac{\partial u}{\partial y} \right) = -f$$

$$\frac{\partial}{\partial x} \left(c \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(c \frac{\partial u}{\partial y} \right) = -f$$

Unknown func $u(x,y)$ is the displacement of point (x,y) on an elastic membrane (sheet), subject to an external force per unit area f in equilibrium.

or $u(x,y)$ is the equilibrium temp at point (x,y) on a membrane, subject to a heat source $f(x,y)$. etc

PDEs can have additional independent variables with physical interpretations, like z, t, ...

a <u>second-order</u> equation is a D.E. where the largest number of derivatives on the unknown func. is 2.

a <u>nonhomogeneous</u> DE has term(s) that do not involve the unknown func.

a <u>linear</u> DE contains no terms that are nonlinear in the unknown func.

If our equations are linear, then we can write them in operator notation

$$Lu = g,$$

where L is a linear operator.

(1) $Lu = g$, where $Lu = \frac{d}{dx}\left(c\frac{du}{dx}\right)$, $g = -f$

(2) $Lu = g$, where $Lu = \underline{\nabla}\cdot(c\underline{\nabla}u)$, $g = -f$.

__def__. an operator L is linear means

$$L(\alpha u + \beta v) = \alpha Lu + \beta Lv,$$

α, β arbitrary constants.

Consider c = constant

(1) $Lu = g$, where $Lu = c \dfrac{d^2u}{dx^2}$

(2) $Lu = g$, where $Lu = c\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$

Show these operators are linear.

(1) Show $L(\alpha u + \beta v) = \alpha Lu + \beta Lv$.

$\Leftrightarrow$ Show $c\left(\dfrac{d^2}{dx^2}(\alpha u + \beta v)\right) = \alpha\left(c\dfrac{d^2u}{dx^2}\right) + \beta\left(c\dfrac{d^2v}{dx^2}\right)$

$$c\left(\frac{d^2}{dx^2}(\alpha u + \beta v)\right) = c\left(\frac{d^2}{dx^2}(\alpha u) + \frac{d^2}{dx^2}(\beta v)\right)$$

$$= c\left(\alpha\frac{d^2u}{dx^2} + \beta\frac{d^2v}{dx^2}\right)$$

$$= c\alpha\frac{d^2u}{dx^2} + c\beta\frac{d^2v}{dx^2}$$

$$= \alpha\left(c\frac{d^2u}{dx^2}\right) + \beta\left(c\frac{d^2v}{dx^2}\right) \checkmark$$

(2) Show $L(\alpha u+\beta v) \stackrel{?}{=} \alpha Lu + \beta Lv$

$\Longleftrightarrow$ Show $c\left(\frac{\partial^2}{\partial x^2}(\alpha u+\beta v) + \frac{\partial^2}{\partial y^2}(\alpha u+\beta v)\right)$

$$= \alpha\left(c\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right) + \beta\left(c\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right)$$

$$c\left(\frac{\partial^2}{\partial x^2}(\alpha u+\beta v) + \frac{\partial^2}{\partial y^2}(\alpha u+\beta v)\right)$$

$$= c\left(\underline{\alpha\frac{\partial^2 u}{\partial x^2}} + \beta\frac{\partial^2 v}{\partial x^2} + \underline{\alpha\frac{\partial^2 u}{\partial y^2}} + \beta\frac{\partial^2 v}{\partial y^2}\right)$$

$$= c\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) + c\beta\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$= \alpha\left(c\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right) + \beta\left(c\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right) \checkmark$$

a <u>system of DEs</u> consists of multiple equations in multiple unknown functions.

ex. $$\left.\begin{aligned} x'(t) &= F_1(x(t), y(t), t). \\ y'(t) &= F_2(x(t), y(t), t). \end{aligned}\right\}$$ an ODE system in the unknown fncs $x(t), y(t)$.

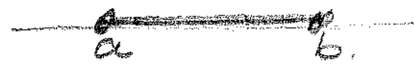
ex. $$\left.\begin{aligned} \frac{\partial T(x,t)}{\partial t} &= \frac{\partial^2 T(x,t)}{\partial x^2} + F(T(x,t), C(x,t)) \\ \frac{\partial C(x,t)}{\partial t} &= -F(T(x,t), C(x,t)), \end{aligned}\right\}$$ a PDE system in the unknown fncs $T(x,t)$, $C(x,t)$.

e.g. $$F = C(x,t)\exp\left(\frac{-E}{R_g T(x,t)}\right)$$

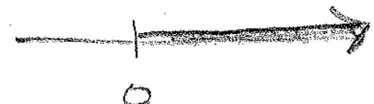
E, R_g constants

Domains

ODE : Ex. $a < x < b$ or $x > 0$

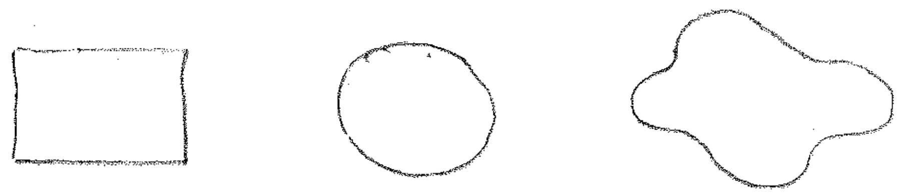


bounded domain

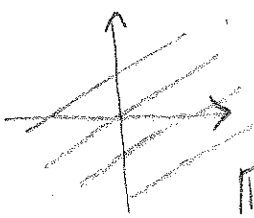


unbounded domain.

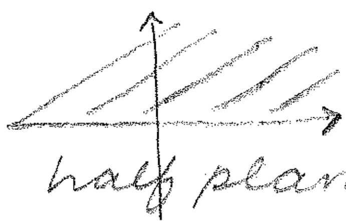
PDE : I. Bounded domain



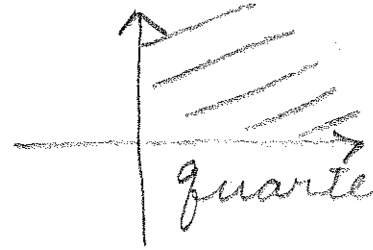
II Unbounded domain



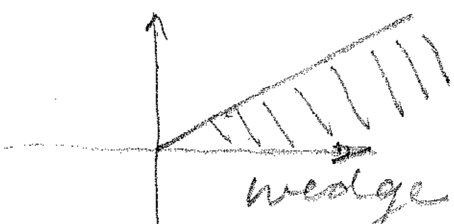
$\mathbb{R}^2$



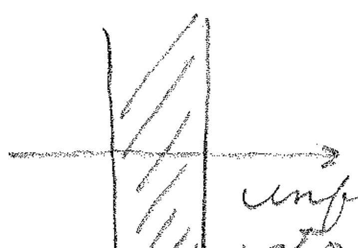
half plane



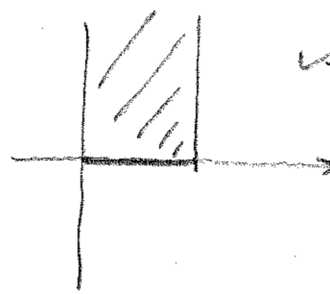
quarter plane



wedge



infinite strip



semi-infinite strip

III moving boundary:

prescribed motion of the boundary

IV free boundary:

boundary is at an unknown location.
Solve for it!

Three Classifications for $\underbrace{\text{linear 2nd order PDEs}}$.

$$\boxed{a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2}} +$$

$$+ d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)\,u = g(x,y).$$

$a-g$ are prescribed.

$\boxed{\text{principal part}}$ of the operator L for this eq. $L(u) = g$.

① If $b^2 - ac > 0$, the eq. is hyperbolic.

prototype: wave eq. $u_{tt} - u_{xx} = 0$.

$a=1,\ b=0,\ c=-1 \rightarrow b^2 - ac = 1 > 0$.

② If $b^2 - ac = 0$, the eq. is parabolic.

prototype: heat eq. $u_t - u_{xx} = 0$.

$a = 0,\ b = 0,\ c = -1 \rightarrow b^2 - ac = 0$.

③ If $b^2 - ac < 0$, the eq. is elliptic

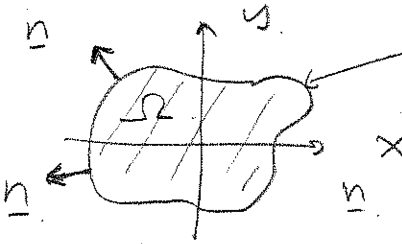
prototype: Laplace's eq. $u_{xx} + u_{yy} = 0$

$a = c = 1,\ b = 0 \rightarrow b^2 - ac = -1 < 0$.

Problems involving parameters could change type as parameter values are changed.

Boundary-value problem (BVP)

Ex. PDE : $L(u(x,y,t)) = 0$ on Ω.



boundary of Ω is $\partial\Omega$.

$\underline{n}$ is unit outward normal

BC$^{\underline{s}}$: $B(u(x,y,t)) = f(t)$ on $\partial\Omega$.

Types of B:

(1). Prescribe u on the boundary

$$u(x,y,t)\Big|_{\partial\Omega} = f(t)$$

Dirichlet condition or Condition of $1^{\underline{st}}$ kind

(2). Prescribe flux across boundary

$$D_{\underline{n}}(u(x,y,t))\Big|_{\partial\Omega} = \frac{\partial u}{\partial \underline{n}}\Big|_{\partial\Omega} = D_{\underline{n}} u\Big|_{\partial\Omega} = \underline{\nabla} u \cdot \underline{n}\Big|_{\partial\Omega} = f(t)$$

Neumann Condition or condition of $2^{\underline{nd}}$ kind

(3). Robin condition or mixed condition

$$u\Big|_{\partial\Omega} + \beta \frac{\partial u}{\partial \underline{n}}\Big|_{\partial\Omega} = f(t).$$