CN 39:

Derive the Fourier Integral transform (F.T.)

Rewrite the Fourier integral as $f(x) \overset{?}{=}$

$$\frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(\xi)\underbrace{\left[\cos\omega\xi\cos\omega x + \sin\omega\xi\sin\omega x\right]}_{\cos(\omega\xi-\omega x)=\cos(\omega[\xi-x])} d\xi\, d\omega$$

$$= \frac{1}{2}\left[e^{i\omega[\xi-x]} + e^{-i\omega[\xi-x]}\right]$$

$$\frac{1}{2\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(\xi)\, e^{i\omega\xi}\, e^{-i\omega x}\, d\xi + \int_{-\infty}^{\infty} f(\xi)\, e^{-i\omega\xi}\, e^{+i\omega x}\, d\xi\right] d\omega$$

c.o.v. $\omega \to -\omega$ & change order of int$\underline{n}$.

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0}\int_{-\infty}^{\infty} f(\xi)\, e^{-i\omega\xi}\, e^{+i\omega x}\, d\xi\, d\omega + \int_0^\infty\int_{-\infty}^{\infty} f(\xi)\, e^{-i\omega\xi}\, e^{i\omega x}\, d\xi\, d\omega\right]$$

$\int_{-\infty}^{0} g + \int_0^\infty g = \int_{-\infty}^{\infty} g.$

$$= \boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\left[\underbrace{\int_{-\infty}^{\infty} f(\xi)\, e^{-i\omega\xi}\, d\xi}_{\mathcal{F}\{f(x)\} \equiv F(\omega) = \hat{f}(\omega)}\right] d\omega = f(x)}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} F(\omega)\, d\omega \longleftarrow \mathcal{F}^{-1}\{F(\omega)\} = f(x).$$

~~if f is smooth on [−L, L]~~ $\longrightarrow$ if $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges.

(old idea).

We found

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \left[\int_{-\infty}^{\infty} f(\xi) e^{-i\omega \xi} \, d\xi \right] d\omega.$$

Fourier Transform (F.T.) of $f(t)$: If $\int_{-\infty}^{\infty} |f(t)| \, dt$ converges, then the F.T. of $f(t)$ is

$$\mathcal{F}\{f(t)\} = F(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

Inverse Fourier Transform (I.F.T.) of $f(t)$ is

$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega.$$

def $|\hat{f}(\omega)|$ is the amplitude spectrum of $f(t)$.

Properties of F.T.

(1) The F.T. is a linear operator:

$$\mathcal{F}\{\alpha f(x) + \beta g(x)\} = \alpha \mathcal{F}\{f(x)\} + \beta \mathcal{F}\{g(x)\}.$$

(2) The inverse F.T. (I.F.T.) is a linear operator:

$$\mathcal{F}^{-1}\{\alpha \hat{f}(\omega) + \beta \hat{g}(\omega)\} = \alpha \mathcal{F}^{-1}\{\hat{f}(\omega)\} + \beta \mathcal{F}^{-1}\{\hat{g}(\omega)\}$$

(3) Derivative property:

$$\mathcal{F}\{f^{(n)}(x)\} = (i\omega)^n \hat{f}(\omega).$$

(4) Convolution property.

(a) $\mathcal{F}\{(f * g)(x)\} = \mathcal{F}\{f(x)\}\mathcal{F}\{g(x)\}$, i.e.

$$\mathcal{F}\left\{\int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi\right\} = \hat{f}(\omega)\,\hat{g}(\omega)$$

(b) $$(f * g)(x) = \mathcal{F}^{-1}\{\hat{f}(\omega)\,\hat{g}(\omega)\}.$$