CN 38. Fourier Integral

Consider $f(x)$ defined for all x, $f(x)$ not periodic. Suppose $f(x)$ is absolutely integrable: $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges, & suppose $f(x)$ is piecewise-smooth on $[-L,L]$ for every L. On $(-L,L)$ the F.S. for $f(x)$ is

$$a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$= \left[\frac{1}{2L}\int_{-L}^{L} f(\xi)\,d\xi\right] + \sum_{n=1}^{\infty}\left\{\left[\frac{1}{L}\int_{-L}^{L} f(\xi)\cos\frac{n\pi\xi}{L}\,d\xi\right]\cos\frac{n\pi x}{L} + \left[\frac{1}{L}\int_{-L}^{L} f(\xi)\sin\frac{n\pi\xi}{L}\,d\xi\right]\sin\frac{n\pi x}{L}\right\}$$

Denote $\frac{n\pi}{L} = \omega_n$, $n = 0, 1, 2, 3, \dots$

$$\underset{0}{[} \quad \omega_1 \quad \omega_2 \quad \omega_3 \quad \cdots \longrightarrow \omega^+ \text{ axis}$$

Mult. the F.S. by $\frac{\pi}{\pi}$

$$= \frac{1}{\pi}\left[\int_{-L}^{L} f(\xi)\,d\xi\right]\frac{\pi}{2L} + \sum_{n=1}^{\infty}\left\{\underline{\frac{1}{\pi}\left[\int_{-L}^{L} f(\xi)\cos\frac{n\pi\xi}{L}\,d\xi\right]\left(\cos\frac{n\pi x}{L}\right)\frac{\pi}{L}} + \frac{1}{\pi}\left[\int_{-L}^{L} f(\xi)\sin\frac{n\pi\xi}{L}\,d\xi\right]\left(\sin\frac{n\pi x}{L}\right)\left(\frac{\pi}{L}\right)\right\}$$

Replace $\left(\frac{\pi}{L}\right)$ by $\frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \omega_{n+1} - \omega_n = \Delta\omega$.

Consider the <u>terms</u>

$$\sum_{n=1}^{\infty}\frac{1}{\pi}\left[\int_{-L}^{L} f(\xi)\cos\omega_n\xi\,d\xi\right]\cos\omega_n x\,\Delta\omega \quad (*)$$

Let $L\to\infty$; replace your idea of discrete values ω_n with the continuum of ω values on the ω^+ axis.

(over)

As $\Delta\omega \to 0$, you get the series $\xrightarrow{(*)}$ an integral

$$\int_0^\infty \underbrace{\frac{1}{\pi}\left[\int_{-\infty}^{\infty} f(\xi)\cos\omega\xi\, d\xi\right]}_{a(\omega)} \cos\omega x\, d\omega .$$

The analogous thing happens for the sine terms.
The F.S. becomes the <u>Fourier</u> <u>integral</u>

$$\boxed{\begin{gathered}\int_0^\infty [a(\omega)\cos\omega x + b(\omega)\sin\omega x]\, d\omega, \\ a(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos\omega\xi\, d\xi, \quad b(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin\omega\xi\, d\xi\end{gathered}}$$

(Note $\lim\limits_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} f(\xi)\, d\xi = 0$.)

$\frac{1}{2L} \downarrow 0$

finite
because our $f(x)$ is
absolutely integrable