

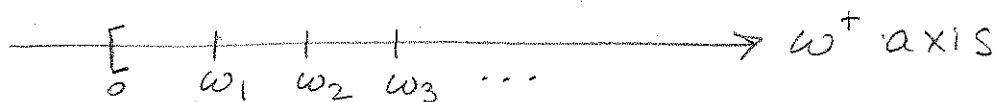
# CN38: Fourier Integral

Consider  $f(x)$  defined for all  $x$ ,  $f(x)$  not periodic. Suppose  $f(x)$  is absolutely integrable:  $\int_{-\infty}^{\infty} |f(x)| dx$  converges, & suppose  $f(x)$  is piecewise-smooth on  $[-L, L]$  for every  $L$ . On  $(-L, L)$  the F.S. for  $f(x)$  is

$$a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$= \left[ \frac{1}{2L} \int_{-L}^L f(\xi) d\xi \right] + \sum_{n=1}^{\infty} \left\{ \left[ \frac{1}{L} \int_{-L}^L f(\xi) \cos \frac{n\pi \xi}{L} d\xi \right] \cos \frac{n\pi x}{L} + \left[ \frac{1}{L} \int_{-L}^L f(\xi) \sin \frac{n\pi \xi}{L} d\xi \right] \sin \frac{n\pi x}{L} \right\}$$

Denote  $\frac{n\pi}{L} = \omega_n$ ,  $n = 0, 1, 2, 3, \dots$



mult. the F.S. by  $\frac{\pi}{L}$

$$= \frac{1}{\pi} \left[ \int_{-L}^L f(\xi) d\xi \right] \left( \frac{\pi}{L} \right) + \sum_{n=1}^{\infty} \left\{ \frac{1}{\pi} \left[ \int_{-L}^L f(\xi) \cos \frac{n\pi \xi}{L} d\xi \right] \left( \cos \frac{n\pi x}{L} \right) \left( \frac{\pi}{L} \right) + \frac{1}{\pi} \left[ \int_{-L}^L f(\xi) \sin \frac{n\pi \xi}{L} d\xi \right] \left( \sin \frac{n\pi x}{L} \right) \left( \frac{\pi}{L} \right) \right\}$$

Replace  $\left( \frac{\pi}{L} \right)$  by  $\frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \omega_{n+1} - \omega_n = \Delta\omega$ .

Consider the terms

$$\sum_{n=1}^{\infty} \frac{1}{\pi} \left[ \int_{-L}^L f(\xi) \cos \omega_n \xi d\xi \right] \cos \omega_n x \Delta\omega \quad (*)$$

Let  $L \rightarrow \infty$ ; replace your idea of discrete values  $\omega_n$  with the continuum of  $\omega$  values on the  $\omega^+$  axis.

Let

## Fourier int., 2

As  $\Delta\omega \rightarrow 0$ , you get the series <sup>(\*)</sup>  $\rightarrow$  an integral

$$\int_0^\infty \underbrace{\frac{1}{\pi} \left[ \int_{-\infty}^\infty f(\xi) \cos \omega \xi d\xi \right]}_{a(\omega)} \cos \omega x d\omega.$$

The analogous thing happens for the sine terms  
the F.S. becomes the Fourier integral

$$\int_0^\infty [a(\omega) \cos \omega x + b(\omega) \sin \omega x] d\omega,$$

$$a(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(\xi) \cos \omega \xi d\xi, \quad b(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(\xi) \sin \omega \xi d\xi$$

(Note  $\lim_{L \rightarrow \infty} \underbrace{\frac{1}{2L}}_{\downarrow 0} \underbrace{\int_{-L}^L f(\xi) d\xi}_{\text{finite}} = 0.$ )  
because our  $f(x)$  is absolutely integrable