CN 37: <u>A Bessel eq. w/ $n=0$</u> is

$$r^2B'' + rB' + \lambda r^2 B = 0.$$

Seek a solution in the form of a power series.

$$B(r) = \sum_{m=0}^{\infty} c_m r^m.$$

Substitute into the eq., & determine the c_m values.

$$B'(r) = \sum_{m=0}^{\infty} c_m m r^{m-1}$$

$$B''(r) = \sum_{m=0}^{\infty} c_m m(m-1) r^{m-2}$$

$$\Rightarrow r^2 \left[\sum_{m=0}^{\infty} c_m m(m-1) r^{m-2} \right] + r \left[\sum_{m=0}^{\infty} c_m m r^{m-1} \right]$$

$$+ \lambda r^2 \left[\sum_{m=0}^{\infty} c_m r^m \right] = 0.$$

$J_0(r), \lambda$

$$\sum_{m=0}^{\infty} r^2 c_m m(m-1) r^{m-2} + \sum_{m=0}^{\infty} r c_m m r^{m-1} + \sum_{m=0}^{\infty} \lambda r^2 c_m r^m = 0.$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m m(m-1) r^m + \sum_{m=0}^{\infty} c_m m r^m + \sum_{m=0}^{\infty} \lambda c_m r^{m+2} = 0.$$

On the last sum, let's make the power j instead of $m+2$.

$$j = m+2 \rightarrow m = j-2.$$

$$\sum_{j=2}^{\infty} \lambda c_{j-2} r^j \overset{?}{=} \sum_{m=2}^{\infty} \lambda c_{m-2} r^m.$$

$$\sum_{m=0}^{\infty} c_m m(m-1) r^m + \sum_{m=0}^{\infty} c_m m r^m + \sum_{m=2}^{\infty} \lambda c_{m-2} r^m = 0.$$

Equate coefficients of like terms r^m:

$m=0$: $0+0=0$ ✓

$m=1$: $0+c_1=0 \rightarrow c_1=0$

$m \geq 2$: $c_m m(m-1) + c_m m + \lambda c_{m-2} = 0$

We can solve for c_m in terms of c_{m-2}:

$$c_m m^2 - \cancel{c_m m} + \cancel{c_m m} + \lambda c_{m-2} = 0$$

$$c_m m^2 = -\lambda c_{m-2} \Rightarrow c_m = \frac{-\lambda}{m^2} c_{m-2}, \quad m = 2, 3, \ldots$$

$m=2$: $c_2 = \frac{-\lambda}{2^2} c_0$ $\qquad$ $m=3$: $c_3 = \frac{-\lambda}{3^2} c_1 = 0$

$m=4$: $c_4 = -\frac{\lambda}{4^2} c_2 = \frac{-\lambda}{4^2} \frac{-\lambda}{2^2} c_0$ $\qquad$ $m=5$: $c_5 = \frac{-\lambda}{5^2} \cdot c_3 = 0$

$m=6$: $c_6 = -\frac{\lambda}{6^2} \cdot \underbrace{c_4} = \frac{-\lambda}{6^2} \cdot \underbrace{\frac{-\lambda}{4^2} \frac{-\lambda}{2^2} \cdot c_0}$

Recall $B(r) = \sum_{m=0}^{\infty} c_m r^m$

$$= c_0 + \overset{=0}{\cancel{c_1 r}} + c_2 r^2 + \overset{=0}{\cancel{c_3 r^3}} + \cdots + c_k r^k + \cdots$$

Let $c_0 = 1$. $B(r) = 1 - \frac{\lambda r^2}{2^2} + \frac{\lambda^2 r^4}{4^2 \cdot 2^2} - \frac{\lambda^3 r^6}{6^2 \cdot 4^2 \cdot 2^2} + \cdots$

$B(r)$ is a <u>Bessel</u> function of order 0.

If $\lambda = 1$, the standard <u>notation</u> is $J_0(r)$.

With λ present, B is denoted $J_0(\sqrt{\lambda}\, r)$.

The subscript n means that the frequency n in the θ direction is 0.

Recall we had $u(r,\theta) = \cos(n\theta) \cdot B(r)$ or $\sin(n\theta) \cdot B(r)$, $n = 0$.

So $u(r,\theta) = u(r) = B(r)$

We want the perimeter of the circular drum of radius 1 to be anchored. The displacement u should be 0 at $r=1$.

We need $B(1)=0$ for our IVP. ($u(1,\theta)=1$ is the IC.)

This equation determines the λ values.

We can't find the λ values directly from

$$1-\frac{\lambda}{2^2}+\frac{\lambda^2}{2^2 4^2}-\frac{\lambda^3}{2^2 4^2 6^2}+\cdots=0.$$

However, they are obtainable.

$J_0(r)$

(33)

r on θ. Therefore both
$= -n^2 A$. This gives the
establishes that n is an
$= 2\pi$.

(34)

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(35)

$= 1$. Then for $m = 2$
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in the power series

(36)

l notation would be
t the frequency n in

It remains to determine the eigenvalues λ. They come from the boundary condition $B = 0$ at $r = 1$. The eigenfunction is $J_0(\sqrt{\lambda} r)$ and the requirement is $J_0(\sqrt{\lambda}) = 0$. The best way to appreciate these functions is by comparison with the cosine, whose series we know and whose behavior we know:

$$\cos(\sqrt{\lambda} r) = 1 - \frac{\lambda r^2}{2!} + \frac{\lambda^2 r^4}{4!} - \frac{\lambda^3 r^6}{6!} + \cdots. \tag{37}$$

Again we set $r = 1$ and pick out the values at which $\cos\sqrt{\lambda} = 0$. The zeros of the cosine, although you couldn't tell it from the series, are at $\sqrt{\lambda} = \pi/2, 3\pi/2, 5\pi/2, \ldots$. They occur at regular intervals with constant spacing π. The zeros of the Bessel function are almost that regular (fortunately for our ears). They occur at

$$\sqrt{\lambda} \approx 2.4, 5.5, 8.65, 11.8, 14.9, \ldots$$

and their spacing converges rapidly to π. In fact the Bessel function $J_0(r)$ approaches $\sqrt{2/\pi r}\cos(r - \pi/4)$, which looks like the cosine with its graph shifted by $\pi/4$ and its amplitude slowly decreasing. Figure 4.5 shows this function up to its third zero, at $\sqrt{\lambda_3}$. To get that far requires the r^{22} term in the series (36); the results of stopping at earlier terms are displayed.

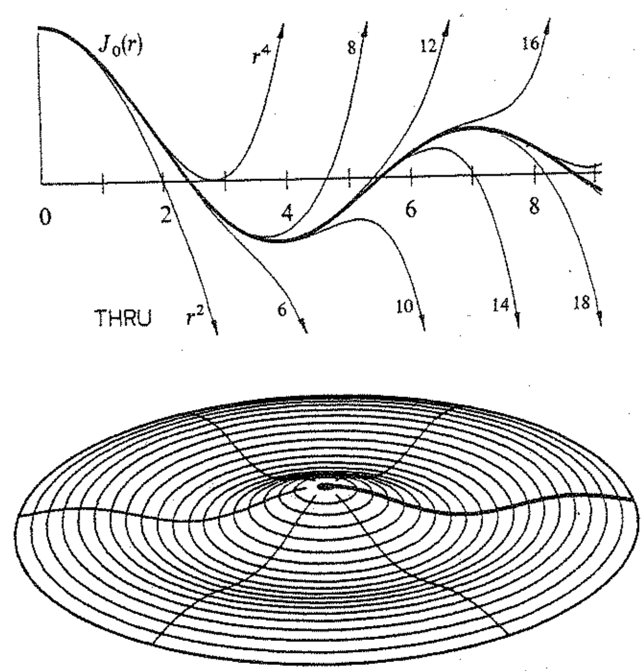

Fig. 4.5. The Bessel function $J_0(r)$ and a drum at frequency λ_3.

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The Bessel functions $B_k(r) = J_0(\sqrt{\lambda_k}\, r)$ are orthogonal over a circle:

$$\int_0^{2\pi}\int_0^1 B_k(r)\, B_\ell(r)\, r\, dr\, d\theta = 0 \quad \text{if } k \neq \ell.$$

The θ integral doesn't have an impact.

$$\underbrace{\int_0^1 B_k(r)\, B_\ell(r)\, r\, dr}_{} = 0 \quad \text{if } k \neq \ell.$$

weighted inner product with a weight fnc. r.

Our Bessel fnc came from the radially symmetric case $n=0$. There are other Bessel fncs that appear when the oscillations depend on θ.

Separating variables as $A(\theta)\,B(r)$ has other possibilities besides $A = 1$.

Then the eq. for B was

$$r^2 B'' + r B' + \lambda r^2 B = n^2 B.$$

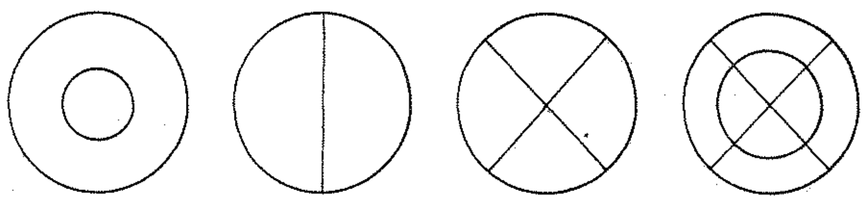
For $\lambda = 1$, this has a solution $J_n(r)$, which is finite at $r=0$. This is the Bessel func of order n.

integer n. Then the equation for B was (34):

$$rB'' + B' + \left(\lambda r - \frac{n^2}{r}\right) B = 0. \tag{40}$$

For $\lambda = 1$ this has a solution $J_n(r)$ which is finite at $r = 0$. That is the *Bessel function of order n*. (All other solutions blow up at $r = 0$; they involve Bessel functions of the second kind.) For every positive λ the solution is just rescaled to $J_n(\sqrt{\lambda}\, r)$. At $r = 1$ the boundary condition requires $J_n(\sqrt{\lambda}) = 0$; that picks out the eigenvalues. The products $A(\theta)B(r) = \cos n\theta\, J_n(\sqrt{\lambda_k}\, r)$ and $\sin n\theta\, J_n(\sqrt{\lambda_k}\, r)$ are the eigenfunctions. They give the shape of the drum in its pure oscillations, and Fig. 4.6 indicates roughly what they look like.

The simplest guide is the nodal lines along which the drum does not move. They are like the zeros of the sine function, where a violin string is still. For the drum we are in two dimensions and the eigenfunctions are $A(\theta)B(r)$. There is a nodal line from the center whenever $A = 0$ and a nodal circle whenever $B = 0$. For different values of n (the frequency in $\cos n\theta$) and k (the oscillation number in the r direction), the figure shows where the drumhead is motionless. The oscillations themselves are functions of time—they are solutions $A(\theta)B(r)e^{i\sqrt{\lambda}\, t}$ of the wave equation in a circle.



$n=0$, $k=2$ | $n=1$, $k=1$ | $n=2$, $k=1$ | $n=2$, $k=2$

Fig. 4.6. Nodal lines of drum = zero lines of $A(\theta)B(r)$.

Finally we mention a problem that is unsolved as of Christmas 1984. ***Can you hear the shape of a drum***? If you know the eigenvalues λ, does that determine the boundary of the drumhead? I think the eigenvalues above, for a circle, do not occur for any other shape. But whether two different drums could sound the same, no one knows.

EXERCISES

4.1.1 Find the Fourier series on $-\pi < x < \pi$ for
(a) $f(x) = \sin^3 x$, an odd function
(b) $f(x) = |\sin x|$, an even function
(c) $f(x) = x^2$, integrating either $x^2 \cos kx$ or the sine series for $f = x$
(d) $f(x) = e^x$, using the complex form of the series.
What are the even and odd parts of $f(x) = e^x$ and $f(x) = e^{ix}$?

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