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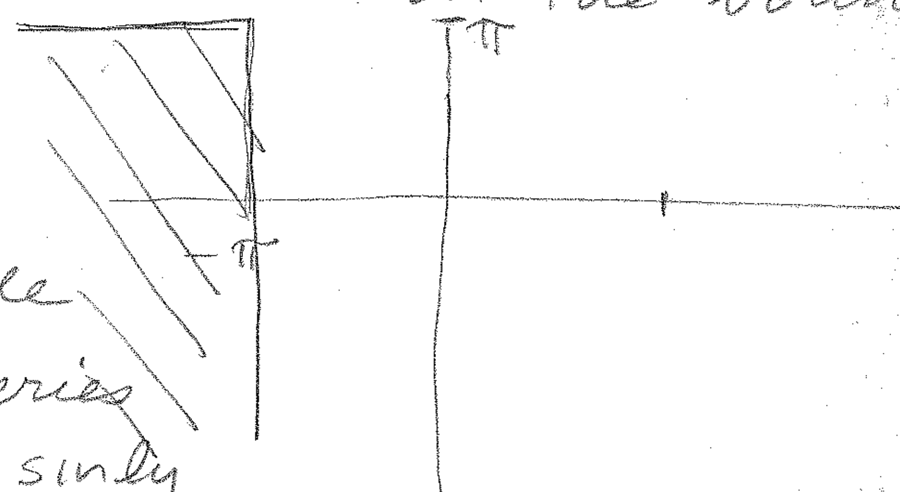
Bessel functions : a valuable ______

set of functions with weighted orthogonality.

Recall a double Fourier series can solve Poisson's eq. in a square.

For ex., Poisson's eq. $-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f$ is solved by $\sum\sum \frac{b_{kl}}{k^2+l^2} \sin kx \sin ly$ in the square $x < -\pi, y < \pi$ with $u=0$ on the boundary of the square,



where f has the double Fourier sine series $\sum\sum b_{kl} \sin kx \sin ly$.

Virtually any func. is a combination of sines & cosines of kx & ly or $e^{ikx} e^{ily}$.

But on a circle, these functions are not as good. If f depends only on the angle θ, periodic functions are still okay. But what if f depends also on r? Or if it depends only on r? ← "radial symmetry"

Then we need other orthogonal funcs.

Consider the content of a circular drum. If you strike it, it oscillates. Its motion contains a mixture of oscillations that go at different frequencies. What are the frequencies (λ) & the corresponding shapes of the drum (eigenfunctions)?

The problem is governed by Laplace's eq. in polar coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\lambda u. \qquad (*).$$

$u(r,\theta)$ is the displacement of the oscillating drum. The drum is anchored along the perimeter (the circle $r=1$), so the boundary condition (BC) is

$\underline{u=0}$ at $r=1$.

Let's seek eigenfunctions u of the form

$$u(r,\theta) = A(\theta)B(r)$$

to satisfy $(*)$.

This is called the method of separation of variables.

Substitute $u = A(\theta)B(r)$ into $(*)$.

Need $$\frac{\partial^2 u}{\partial r^2} = \frac{\partial^2}{\partial r^2}\left[A(\theta)B(r)\right] = A B''(r)$$

$$\frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left[A(\theta)B(r)\right] = \frac{1}{r} A B'(r)$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2}{\partial \theta^2}\left[A(\theta)B(r)\right] = B A''$$

$$\frac{A B'' + \frac{1}{r} A B' + \frac{1}{r^2} A'' B = -\lambda A B}{A B}$$

$$r^2\left(\frac{B''}{B} + \frac{1}{r}\frac{B'}{B} + \frac{1}{r^2}\frac{A''}{A} = -\lambda\right)$$

There are various ways to separate A & B onto opposite sides of the eq.

$$\frac{r^2 B''}{B} + r\frac{B'}{B} + \frac{A''}{A} = -\lambda r^2$$

$$-\frac{A''}{A} + \lambda r^2 \qquad -\frac{A''}{A} + \lambda r^2$$

$$\frac{r^2B''}{B} + \frac{rB'}{B} + \lambda r^2 \cdot \frac{B}{B} = -\frac{A''}{A}.$$

$$\frac{r^2B'' + rB' + \lambda r^2 B}{B} = -\frac{A''}{A}.$$

The left side only depends on r.
The right side only depends on θ.
Both sides must be constant.
What if the constant is n^2?

$$\frac{r^2B'' + rB' + \lambda r^2 B}{B} = n^2 \qquad -\frac{A''}{A} = n^2.$$

$$\boxed{r^2B'' + rB' + \lambda r^2 B = n^2 B.} \qquad A'' = -An^2.$$

↑ This is one form of Bessel's eq.

$A = e^{in\theta}$ or $e^{-in\theta}$.

or $A = \cos n\theta$ or $\sin n\theta$.

We are interested (for each n) in solutions that are 0 on the boundary of the circle. The eigenfunctions $u = AB$ of the Laplacian will be $\cos n\theta \cdot B$ and $\sin n\theta \cdot B$. We have not established the eigenvalues λ, which affect B.

so solve Bessel's eq, consider the case $n=0$. (the radially symmetric case where u depends only on r).

The displacement u of the drum is angle-independent. It depends only on radial distance.